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<u>Random Fourier Series on Locally Compact Abelian Groups</u>

M. B. Marcus and G. Pisier

In 1930 Paley and Zygmund [9] introduced the problem of whether the random series

$$(1) \qquad \sum_{k=0}^{\infty} a_k \varepsilon_k \cos(kt + \alpha_k), \qquad t \in [0, 2\pi]$$

converges uniformly a.s., where $\{a_k\}$ and $\{\alpha_k\}$ are sequences of real numbers and $\{\varepsilon_k\}$ is a Rademacher sequence, that is, a sequence of independent, symmetric random variables taking on the values $\pm 1$. This problem was subsequently studied by Salem and Zygmund [11], Kahane [6] and others (see [8], [10]). In [8] we give a necessary and sufficient condition for the uniform convergence of the series in (1). An interesting aspect of this result is that the condition remains valid when the sequence $\{\varepsilon_k\}$ is replaced by other sequences of random variables, for example, independent gaussian random variables with mean zero and variance 1 (N(0,1)). Our results in [8] are a consequence of the Dudley-Fernique [2], [3] necessary and sufficient condition for the continuity of stationary Gaussian processes and a line of approach initiated in [4] (see also [7]). In this paper, by adding some technical modifications we show that the results in [8] extend directly to the more general class of random series mentioned in the title. The case of compact abelian groups is included in [10].

Let $G$ be a locally compact abelian group with identity element 0. Let $K \subset G$ be a compact symmetric neighborhood of 0. Let $\Gamma$ denote the characters of $G$ and let $A \subset \Gamma$ be countable. Therefore, $\{\gamma | \gamma \in A\}$ is a countable collection of characters of $G$. (We only consider Fourier series with spectrum in $A$. Therefore, in all that follows, we may as well assume that $\Gamma$ is separable, so that the compact subsets of $G$ are metrizable.) We also define the following sequences of random variables indexed by $\gamma \in A$: $\{\varepsilon_\gamma\}$ a Rademacher sequence, $\{g_\gamma\}$ independent $N(0,1)$ random variables and $\{\xi_\gamma\}$ complex valued random variables satisfying

$$\sup_{\gamma \in A} E|\xi_\gamma|^2 < \infty \text{ and } \liminf_{\gamma \in A} E|\xi_\gamma| > 0. \tag{2}$$

Let $\{a_\gamma\}$ be complex numbers satisfying $\sum_{\gamma \in A} |a_\gamma|^2 = 1$ and consider the random Fourier series

$$Z(x) = \sum_{\gamma \in A} a_\gamma \varepsilon_\gamma \xi_\gamma \gamma(x), \qquad x \in K. \tag{3}$$

For each fixed $x \in K$ the series converges a.s. so the sum is well defined. We will give a necessary and sufficient condition for the series (3) to converge uniformly a.s. on $K$.

Define $K \oplus K = \{x + y | x \in K, y \in K\}$ and in a similar fashion define $\bigoplus_{i=1}^{n} K_i$. Let $\tau(x)$ be a non-negative function on $K \oplus K$ and let

$$m_\tau(\varepsilon) = \mu(x \in K \oplus K | \tau(x) < \varepsilon) \tag{4}$$

where $\mu$ is the Haar measure on $G$. Define

$$\overline{\tau(u)} = \sup\{y | m_\tau(y) < u\} \tag{5}$$

and let $\mu_n = \mu(\bigoplus_{i=1}^{n} K_i)$. Therefore $0 \leq m_\tau(\varepsilon) \leq \mu_2$ so that the domain of $\bar{\tau}$ is the interval $[0,\mu_2]$. Note that $\bar{\tau}$ viewed as a random variable on $[0,\mu_2]$ has the same probability distribution with respect to normalized Lesbesgue measure on $[0,\mu_2]$ that $\tau(x)$ has with respect to normalized Haar measure on $K \oplus K$. In keeping with classical terminology we call $\bar{\tau}$ the non-decreasing rearrangement of $\tau$ (with respect to $K \oplus K$). In terms of $\mu$, $\tau$ and $K$ we define the integral

$$(6) \qquad I(K,\mu,\tau(s)) = I(\tau(s)) = I(\tau)$$

$$= \int_0^{\mu_2} \frac{\overline{\tau(s)}}{s(\log \frac{4\mu_4}{s})^{1/2}}\, ds.$$

Finally, we define a translation invariant pseudo-metric $\sigma$ on $G$ by

$$(7) \qquad \sigma(x-y) = \Big(\sum_{\gamma\in A} |a_\gamma|^2 |\gamma(x)-\gamma(y)|^2\Big)^{1/2}$$

$$= \Big(\sum_{\gamma\in A} |a_\gamma|^2 |\gamma(x-y)-1|^2\Big)^{1/2}.$$

To see the motivation for this note that when $E|\xi_\gamma|^2 = 1$ for all $\gamma\in A$ then $\sigma(x-y) = (E|Z(x)-Z(y)|^2)^{1/2}$. We can now state our result.

<u>Theorem 1</u>: Employing the notation and definitions given above let $\|Z\| = \sup_{x\in K}|Z(x)|$. If $I(\sigma) < \infty$ the series (3) converges uniformly a.s. and

$$(8) \qquad (E\|Z\|^2)^{1/2} \le C(\sup_\gamma E|\xi_\gamma|^2)^{1/2}\Big[\Big(\sum_{\gamma\in A} |a_\gamma|^2\Big)^{1/2} + I(\sigma)\Big]$$

where $C$ is a constant independent of $\{a_\gamma\}$ and $\sigma$. Let $\{\gamma_k, k = 1,2,\dots\}$ be an ordering of $\gamma \epsilon A$ and let $\{a_k\}$, $\{\epsilon_k\}$ and $\{\xi_k\}$ be the corresponding orderings of $\{a_\gamma\}$, $\{\epsilon_\gamma\}$ and $\{\xi_\gamma\}$. If $I(\sigma) = \infty$ then for all open sets $U \subset K$

$$(9) \qquad \sup_n \sup_{x \epsilon U} \left| \sum_{k=1}^{n} a_k \epsilon_k \xi_k \gamma_k(x) \right| = \infty$$

on a set of measure greater than zero. (Note that neither (2) nor (7) depend on the order of $\{\gamma_k\}$ so that the implications of $I(\sigma) < \infty$ are also valid for all orderings $\{\gamma_k\}$ of $\gamma \epsilon A$.)

Proof: The first step is a adaptation of Dudley's theorem on a sufficient condition for continuity of the sample paths of a Gaussian process. It is well known that this theorem is also valid for processes with sub-gaussian increments. Let $\{Y(t), t \epsilon T\}$, $T$ an arbitrary index set, be a real valued stochastic process. The process is said to have subgaussian increments if there exists a $\delta > 0$ such that for all $s,t \epsilon T$ and $\lambda > 0$

$$E\{\exp(\lambda(X(s)-X(t)))\} \epsilon \exp\{\lambda^2 \delta^2 E(X(t)-X(s))^2/2\}.$$

Let $(S,\rho)$ be a metric (or pseudo-metric) space. We denote by $N_\rho(S,\epsilon)$ the minimum number of balls in the metric (or pseudo-metric) $\rho$ that is necessary to cover $S$. The following theorem is an immediate consequence of Theorem 4.1 [7]; it is similar to a theorem of Fernique, [13].

Theorem 2: Let $\tilde{S} = \{\tilde{X}(t), t \epsilon T\}$, $T$ a compact topological space, be a stochastic process with subgaussian increments and let

$\rho(t,s) = (E(\tilde{X}(t)-\tilde{X}(s))^2)^{1/2}$ be continuous on $T \times T$. Define $\hat{\rho} = \sup_{s,t\in T} \rho(s,t)$ and assume that

(10) $$J(\tilde{S},\rho) = J(\rho) = \int_0^{\hat{\rho}} (\log N_\rho(\tilde{S},u))^{1/2} du < \infty.$$

Then there exists a version $S = \{X(t), t\in T\}$ of the process, with continuous sample paths, satisfying the inequality

(11) $$E[\sup_{t\in T} |X(t)|] \leq C'[E|X(t_0)| + \hat{\rho} + J(S,\rho)]$$

where $t_0 \in T$ and $C' = C'(\delta)$ is a constant independent of $\rho$. (Note that $N_\rho(S,u) = N_\rho(\tilde{S},u)$ so, in particular, $J(\tilde{S},\rho) = J(S,\rho)$.)

We will use this theorem in the special case in which $\rho$ is translation invariant. In this case we can relate the integrals defined in (6) and (10). In order to do this we need the following lemma which is a generalization of Lemma 2.1 [4].

Lemma 3. Let $\tau$ be a translation invariant pseudo-metric on $G$ then

(12) $$\frac{\mu_1}{m_\tau(\epsilon)} \leq N_\tau(K \oplus K, \epsilon) \leq \frac{\mu_4}{m_\tau(\epsilon/2)}.$$

Proof: Since this lemma is the only ingrediant in the proof of Theorem 1 that is not supplied in [8] or [10] we will sketch the proof. Note that when $G$ is compact we can take $K = G$. In this case the proof is elementary and (12) reduces to

$$\frac{\mu(G)}{m_\tau(\epsilon)} \leq N_\tau(G,\epsilon) \leq \frac{\mu(G)}{m_\tau(\epsilon/2)}.$$

Let $B(t,\epsilon) = \{x\epsilon G | \tau(x-t) < \epsilon\}$ and let $M_\tau(K \oplus K,\epsilon)$ denote the maximal number of balls of radius $\epsilon$ in the $\tau$ pseudo-metric centered in $K \oplus K$ and disjoint in $\bigoplus_{i=1}^{4} K_i$. Then for all $t\epsilon K \oplus K$ we have

$$\mu\{B(t,\epsilon) \cap \bigoplus_{i=1}^{4} K_i\} \geq \mu\{B(0,\epsilon) \cap K \oplus K\}$$

and

$$M_\tau(K \oplus K,\epsilon/2) \geq N_\tau(K \oplus K,\epsilon)$$

Denote the centers of the $M_\tau(K \oplus K,\epsilon/2)$ balls of radius $\epsilon/2$ centered in $K \oplus K$ and disjoint in $\bigoplus_{i=1}^{4} K_i$ by $\{t_j, j = 1,\ldots,M_\tau(K \oplus K,\epsilon)\}$ then

$$\mu_4 \geq \mu\left(\bigcup_{j=1}^{M_\tau(K\oplus K,\epsilon/2)} \{B(t_j,\epsilon/2) \cap \bigoplus_{i=1}^{4} K_i\}\right)$$

$$\geq M_\tau(K \oplus K,\epsilon/2)\mu(B(0,\epsilon/2) \cap K \oplus K)$$

$$\geq N_\tau(K \oplus K,\epsilon) m_\tau(\epsilon/2)$$

This proves the right side of (12); the proof of the left side is similar.

We note two other standard results

(13) $$N_\tau(K,\epsilon) \leq N_\tau(K \oplus K,\epsilon)$$

(14) $$N_\tau(K \oplus K,2\epsilon) \leq N_\tau^2(K,\epsilon)$$

and define the integral expression

(15) $$\tilde{I}(K,\mu,\tau(u)) = \tilde{I}(\tau(u)) = \tilde{I}(\tau)$$

$$= \int_0^{\mu_2} \frac{\int_0^s \overline{\tau}(u)du}{s^2(\log \frac{4\mu_4}{s})^{1/2}} ds$$

for $\overline{\tau}$ as defined in (5). The next lemma follows from (12), (13), (14) and integration by parts.

Lemma 5: Let $\hat{\tau} = \sup_{x \in K \oplus K} \tau(x)$ and assume that $J(K,\tau) = J(\tau) < \infty$, then the following inequalities hold:

$$(16) \qquad -C_1\hat{\tau} + I(\tau) \leq \tilde{I}(\tau) \leq 2I(\tau)$$

$$(17) \qquad -C_2\hat{\tau} + \frac{1}{2\sqrt{2}}I(\tau) \leq J(\tau) \leq C_2'\hat{\tau} + 2I(\tau)$$

$$(18) \qquad -C_3\hat{\tau} + \frac{1}{4\sqrt{2}}\tilde{I}(\tau) \leq J(\tau) \leq C_3'\hat{\tau} + 2\tilde{I}(\tau)$$

where $C_1, C_2, C_2', C_3, C_3'$ are all positive and finite.

The next step in the proof is a Jensen type inequality for the non-decreasing rearrangements of a family of random functions. Let $(\Omega, \mathscr{F}, P)$ be some probability space with expectation operator $E$ and let $\tau(x,\omega)$, $x \in K \oplus K$, $\omega \in \Omega$ be a family of random non-negative functions such that $E|\tau(x,\omega)|^2 < \infty$ for $x \in K \oplus K$. Following (4) and (5) we define the random families $m_{\tau(\cdot,\omega)}(\varepsilon)$ and $\overline{\tau(\cdot,\omega)}$. We have

Lemma 6: For $0 \leq h \leq \mu_2$

$$(19) \qquad (E|\int_0^h \overline{\tau(u,\omega)}du|^2)^{1/2} \leq \int_0^h \overline{(E|\tau(u,\omega)|^2)^{1/2}}du.$$

This lemma is a generalization of Lemma 1.1 [7]. The proof is essentially the same as the one given in [8].

We can now obtain the implications of $I(\sigma) < \infty$ in Theorem 1. Let $(\Omega_1, \mathscr{F}_1, P_1)$ denote the probability space of $\{\xi_\gamma\}$ and $(\Omega_2, \mathscr{F}_2, P_2)$ denote the probability space of $\{\epsilon_\gamma\}$ and denote the corresponding expectation operators by $E_1$ and $E_2$. The series (3) is defined on the probability space $(\Omega_1 \times \Omega_2, \mathscr{F}_1 \times \mathscr{F}_2, P_1 \times P_2)$. We shall refer to this space as $(\Omega, \mathscr{F}, P)$ and denote the corresponding expectation operator by $E$ (not to be confused with the space used to explain Lemma 6).

Without loss of generality we can assume $\sup_{\gamma \in A} E|\xi_\gamma|^2 \leq 1$; the second assumption of (2) is not used in this part of the proof. Fix $\omega_1 \in \Omega_1$ and consider

(20) $$Z(x,\omega_1) = \sum_{\gamma \in A} a_\gamma \epsilon_\gamma \xi_\gamma(\omega_1)\gamma(x), \qquad x \in K$$

as a random series on $(\Omega_2, \mathscr{F}_2, P_2)$. Note that

$Z_1(x,\omega_1) = \sum_{\gamma \in A} \epsilon_\gamma \mathrm{Re}[a_\gamma \xi_\gamma(\omega_1)\gamma(x)]$ and

$Z_2(x,\omega_1) = \sum_{\gamma \in A} \epsilon_\gamma \mathrm{Im}[a_\gamma \xi_\gamma(\omega_1)\gamma(x)]$ are both processes with sub-gaussian increments (see e.g. Chapter 2, Section 2 [5]) and both $(E_2|Z_1(x,\omega_1)-Z_1(y,\omega_1)|^2)^{1/2}$ and $(E_2|Z_2(x,\omega_1)-Z_2(y,\omega_1)|^2)^{1/2}$ are less than or equal to

(21) $$\sigma(x-y,\omega_1) = \Big(\sum_{\gamma \in A} |a_\gamma|^2 |\xi_\gamma(\omega_1)|^2 |\gamma(x-y)-1|^2\Big)^{1/2}.$$

By Theorem 2 with $t_0 = 0$ and (18) we have

(22) $$E_2[\sup_{x \in K} |Z(x,\omega_1)|] \leq D\Big[\Big(\sum_{\gamma \in A} |a_\gamma|^2 |\xi_\gamma(\omega_1)|^2\Big)^{1/2} + \tilde{I}(\sigma(u,\omega_1))\Big],$$

for some constant $D$, where we use the facts that

$$\hat{\sigma} = \sup_{x \in K \oplus K} \sigma(x) \leq 2\Big(\sum_{\gamma \in A} |a_\gamma|^2 |\xi_\gamma(\omega_1)|^2\Big)^{1/2}$$

and

$$E_2|Z(0,\omega_1)| \le (E_2|Z(0,\omega_1)|^2)^{1/2}$$
$$= (\sum_{\gamma\in A} |a_\gamma|^2|\xi_\gamma(\omega_1)|^2)^{1/2}.$$

The series (20) is a Rademacher series therefore by Kahane's inequality we have

$$(23) \qquad E_2[\sup_{x\in K}|Z(x,\omega_1)|^2]^{1/2} \le C\ E_2[\sup_{x\in K}|Z(x,\omega_1)|]$$

where $C$ is a constant independent of the values of $\{a_\gamma\xi_\gamma(\omega_1)|\gamma\in A\}$. By Lemma 6 we have

$$(24) \qquad (E_1|\tilde{I}(\sigma(u,\omega_1))|^2)^{1/2}$$
$$\le \int_0^{\mu_2} \frac{(E_1|\int_0^s \overline{\sigma(u,\omega_1)}du|^2)^{1/2}}{s^2(\log \frac{4\mu_4}{s})^{1/2}}ds \le \tilde{I}(\sigma)$$

where $\sigma$ is given in (7). Also

$$(25) \qquad (E_1 \sum_{\gamma\in A} |a_\gamma|^2|\xi_\gamma(\omega_1)|^2)^{1/2} \le (\sum_{\gamma\in A} |a_\gamma|^2)^{1/2}.$$

Using (23), (24), (25) and (16) in (22) we obtain (8).

We now show that the series (3) converges uniformly a.s. It follows from (24) and Lemma 5 that $I(\sigma) < \infty$ implies $J(K,\sigma(\cdot,\omega_1)) < \infty$ a.s. $(P_1)$. Therefore by Theorem 2 there exists a set $\bar{\Omega}_1 \subset \Omega_1, P(\bar{\Omega}_1) = 1$, such that for $\omega_1\in\bar{\Omega}_1$, $Z(x,\omega_1)$ has a version which is continuous a.s. $(P_2)$. Therefore by the Ito-Nisio theorem (Theorem 2.3.4 [5]) the series (20) converges uniformly a.s. $(P_2)$

for each $\omega_1 \epsilon \bar{\Omega}_1$. This implies, by Fubini's theorem, that the series (3) converges uniformly a.s. (P).

We now obtain the implications of $I(\sigma) = \infty$. The major result in this direction is Fernique's necessary condition for the continuity of stationary Gaussian processes. Consider

$$(26) \qquad G(x) = \sum_{\gamma \epsilon A} a_\gamma g_\gamma \gamma(x), \quad x \epsilon K.$$

We use the following version of Fernique's theorem.

Theorem 7. A necessary condition for the series (26) to converge uniformly a.s. is that $J(K,\sigma) < \infty$.

Proof: Fernique's theorem (Theorem 8.1.1 [3]) is proved for real valued processes on $R^n$ but only minor modifications are necessary to adapt the proof to the case considered here. Instead of G(x) it is sufficient to prove Theorem 7 for the real valued process

$$(27) \qquad Y(x) = \sum_{\gamma \epsilon A} g_\gamma \mathrm{Re}(a_\gamma \gamma(x)) + \sum_{\gamma \epsilon A} g'_\gamma \mathrm{Im}(a_\gamma \gamma(x)), x \epsilon K,$$

where $\{g'_\gamma | \gamma \epsilon A\}$ is an independent copy of $\{g_\gamma | \gamma \epsilon A\}$, since $E(G(x)-G(y))^2 = E(Y(x)-Y(y))^2 = \sigma^2(x-y)$ and the series (26) and (27) either both converge uniformly a.s. or neither does.

The only point in the proof of Theorem 8.1.1 [3] that needs to be extended is Lemma 8.1.2. Let $H = \{x \epsilon G | \sigma(x) = 0\}$ and form the quotient group $G' = G/H$. There exists a cannonical mapping of G onto G'; let K' be the image of K under this mapping. Denote by $\sigma'$ the metric on K' that corresponds to the pseudo-metric $\sigma$ on K.

Lemma 8: There exists a $\delta_0 > 0$ and a compact symmetric neighborhood of $0$ $S \subset K$ such that if $s,t \in \bigoplus_{i=1}^{4} S_i$ then $\sigma'(s-t) \leq \delta_0$ implies $s-t \in S$.

Proof: Let $S$ be a compact symmetric neighborhood of $0 \in K'$ such that $\bigoplus_{i=1}^{8} S \subset K'$. Let $\beta = \min\{\sigma'(x), x \in \bigoplus_{i=1}^{8} S_i / S\}$. Since $0$ is the unique zero of $\sigma'$ on $K'$ we have $\beta > 0$. Let $s,t \in \bigoplus_{i=1}^{4} S_i$ then $s-t \in \bigoplus_{i=1}^{8} S_i$. Set $\delta_0 = \beta/2$ then $\sigma'(s-t) \leq \delta_0$ implies $s-t \in S$.

Consider $S$ as given in Lemma 8 and let $T = \bigoplus_{i=1}^{4} S_i$. Following the notation of Theorem 8.1.1 [3] we define $B(S,\delta_0) = \bigcup_{s \in S} B(s,\delta_0)$ where $B(s,\delta)$ denotes an open ball of radius $\delta$ in $K'$ with respect to the $\sigma'$ metric. Let $s,t \in B(S,\delta_0) \cap T$, we show that for $\delta \leq \delta_0$, $B(s,\delta) \cap T = A_1$ and $B(t,\delta) \cap T = A_2$ are translates of each other, i.e. if $u \in A_1$ then $u + t - s \in A_2$. To do this we need only show that $u + t - s \in T$. Since $t \in B(S,\delta_0)$ there exists a $t' \in S$ such that $\sigma(t-t') < \delta_0$. Set

$$u + t - s = t' + (t-t') + (u-s).$$

Since $t,t' \in T = \bigoplus_{i=1}^{4} S_i$, by Lemma 8, $t-t' \in S$. Similarly $u-s \in S$ and since $t' \in S$ we have $u + t - s \in T$.

Consider the process

$$Y'(x) = \sum_{\gamma \in A} g_\gamma \operatorname{Re}(a_\gamma \gamma(x)) + \sum_{\gamma \in A} g'_\gamma \operatorname{Im}(a_\gamma \gamma(x)), \quad x \in K'. \tag{28}$$

This is a real valued stationary Gaussian process with $(E|Y'(x)-Y'(y)|^2)^{1/2} = \sigma'(x-y)$ and an equivalent of Lemma 8.1.2 [3] holds for this process.

Assume that the series (28) converges uniformly a.s. on $K'$. By the Landau, Shepp, Fernique theorem (Corollary 2.4.6 [5]) we have $E(\sup_{x\in K'} Y'(x)) < \infty$. We refer to the second paragraph of 8.1.4 [3] with $S$ and $T$ as given above. This shows that there exists a $\delta' > 0$ such that

$$\int_0^{\delta'} (\log N_{\sigma'}(S,u))^{1/2} du < \infty$$

and since $S$ is compact we also have $J(S,\sigma') < \infty$. Finally, since $K'$ is compact, there exists a constant $C > 0$ such that $N_{\sigma'}(S,u) \geq C\, N_{\sigma'}(K',u)$. Therefore $J(K',\sigma') < \infty$. To obtain Theorem 7 for $Y(x)$, $x\in K$ we note that the series (27) and (28) either both converge uniformly a.s. or neither does. Furthermore

$$E(\sup_{x\in K'} Y'(x)) = E(\sup_{x\in K} Y(x))$$

and $N_{\sigma'}(K',u) = N_{\sigma}(K,u)$. Therefore we obtain Theorem 7.

Let $\{\gamma_k,\ k = 1,2,\ldots\}$ be an ordering of $A \subset \Gamma$. Our main result on necessary conditions for the convergence of random Fourier series is contained in the following lemma.

<u>Lemma 9</u>: In the notation established above we have

$$(29) \qquad (E \sup_n \| \sum_{k=1}^{n} a_k \epsilon_k \xi_k \gamma_k \|^2)^{1/2}$$

$$\leq C(\sup_k E|\xi_k|^2)^{1/2} (E \sup_n \| \sum_{k=1}^{n} a_k g_k \gamma_k \|^2)^{1/2}$$

and

(30) $$\left(E \sup_n \left\| \sum_{k=1}^{n} a_k g_k \gamma_k \right\|^2\right)^{1/2} \le C' \left(E \sup_n \left\| \sum_{k=1}^{n} a_k \varepsilon_k \gamma_k \right\|^2\right)^{1/2}.$$

In particular, if $\sum_{\gamma \in A} a_\gamma \varepsilon_\gamma \gamma(x)$ converges uniformly a.s. then so does $\sum_{\gamma \in A} a_\gamma g_\gamma \gamma(x)$ and

(31) $$\left(E \left\| \sum_{\gamma \in A} a_\gamma g_\gamma \gamma \right\|^2\right)^{1/2} \le C'' \left(E \left\| \sum_{\gamma \in A} a_\gamma \varepsilon_\gamma \gamma \right\|^2\right)^{1/2}.$$

(Here $C, C'$ and $C''$ are finite constants independent of $\{a_k\}$).

<u>Proof</u>: Belyaev's dichotomy [1], states that a stationary Gaussian process on the real line either has continuous sample paths a.s. or else is unbounded on all intervals. This dichotomy also holds for $G(x)$ and is a consequence of results of Ito and Nisio. (A proof can be made using Theorems 3.4.7 and 3.4.9 [5].) Consequently, we have that either $\sum_{k=1}^{\infty} a_k g_k \gamma_k(x)$ converges uniformly a.s. on $K$ or else for all open sets $U \subset K$

(32) $$\sup_n \sup_{x \in U} \left| \sum_{k=1}^{n} a_k g_k \gamma_k(x) \right| = \infty \text{ a.s.}$$

We also note that by Levy's inequality and the Landau, Shepp, Fernique theorem, if $\sum_{k=1}^{\infty} a_k g_k \gamma_k$ converges uniformly a.s. then

(33) $$E\left(\sup_n \left\| \sum_{k=1}^{n} a_k g_k \gamma_k \right\|^2\right) < \infty.$$

Inequality (29) is a consequence of the closed graph theorem. Let $B_1$ be the Banach space of sequences of complex numbers $\{a\} = \{a_1, a_2, \ldots\}$ for which $\sum_{k=1}^{\infty} a_k g_k \gamma_k$ converges uniformly a.s. on $K$ and $\|\{a\}\|_1 = \left(E \sup_n \left\| \sum_{k=1}^{n} a_k g_k \gamma_k \right\|^2\right)^{1/2} < \infty$. Let $B_2$ denote the

Banach space of sequences of complex numbers $\{a\} = \{a_1, a_2, \ldots\}$ for which $\sum_{k=1}^{\infty} a_k \epsilon_k \xi_k \gamma_k$ converges uniformly a.s. on $K$ for all sequences of complex random variables $\{\xi_k\}$ satisfying $E|\xi_k|^2 \leq 1$ and $\|\{a\}\|_2 = \sup_{\{\xi_k\}} (E \sup_n \|\sum_{k=1}^{n} a_k \epsilon_k \xi_k \gamma_k\|^2)^{1/2} < \infty$. Then $\|\{a\}\|_1 < \infty$ implies $I(\sigma) < \infty$ by Theorem 7, (33) and (17) and $I(\sigma) < \infty$ implies $\|\{a\}\|_2 < \infty$ by (8) of Theorem 1 (which we have already proved) and Levy's inequality. Therefore (29) follows from the closed graph theorem applied to the identity mapping of $B_1$ onto $B_2$.

To obtain (30) we write $g_k = g_k' + g_k''$ where $g_k' = g_k I[|g_k| < N]$ ($I[A]$) is the indicator function of the set $A$) and $N$ is chosen so that $(E|g_k''|^2)^{1/2} = (2C)^{-1}$. Then, for all $j$

$$E(\sup_{n \leq j} \|\sum_{k=1}^{n} a_k g_k \gamma_k\|^2)^{1/2}$$

$$\leq (E \sup_{n \leq j} \|\sum_{k=1}^{n} a_k g_k' \gamma_k\|^2)^{1/2} + (E \sup_{n \leq j} \|\sum_{k=1}^{n} a_k g_k'' \gamma_k\|^2)^{1/2}.$$

By Theorem 5.3 [12]

$$(E \sup_{n \leq j} \|\sum_{k=1}^{n} a_k g_k' \gamma_k\|^2)^{1/2} \leq N(E \sup_{n \leq j} \|\sum_{k=1}^{n} a_k \epsilon_k \gamma_k\|^2)^{1/2}$$

and by (29)

$$(E \sup_{n \leq j} \|\sum_{k=1}^{n} a_k g_k'' \gamma_k\|^2)^{1/2} \leq \frac{1}{2}(E \sup_{n \leq j} \|\sum_{k=1}^{n} a_k g_k \gamma_k\|^2)^{1/2}.$$

Putting this together we have

$$(E \sup_{n\leq j} \| \sum_{k=1}^{n} a_k g_k \gamma_k \|^2)^{1/2} \leq 2N(E \sup_{n\leq j} \| \sum_{k=1}^{n} a_k \varepsilon_k \gamma_k \|^2)^{1/2}.$$

Passing to the limit as $j \to \infty$ we obtain (30) with $C' = 2N$.

If $\sum_{\gamma\in A} a_\gamma \varepsilon_\gamma \gamma(x)$ converges uniformly a.s. the right side of (30) is finite by Kahane's theorem; therefore $\sum_{\gamma\in A} a_\gamma g_\gamma \gamma(x)$ converges uniformly a.s. by (30) and the extended Belyaev dichotomy.

By Lemma 9, and what we have already proved, we have that $I(\sigma) < \infty$ is a necessary and sufficient condition for the uniform convergence a.s. of $\sum_{k=1}^{\infty} a_k \varepsilon_k \gamma_k$. This, essentially, is all we need to complete the proof of Theorem 1. For instance, if $\{\xi_k\}$ is also independent (besides satisfying (2)) then by Theorem 5.1 [12], $I(\sigma) < \infty$ is a necessary and sufficient condition for the uniform convergence a.s. of the series in (3). Also, one can easily show that $I(\sigma) = \infty$ implies $E \sup_n \| \sum_{k=1}^{n} a_k \varepsilon_k \xi_k \gamma_k \|^2 = \infty$. For the actual completion of the proof of Theorem 1 we refer the reader to Lemma 2.5 [8] and the brief "Proof of Theorem 1.1" on page 2.11 [8]. This material, although written for the case $G = R$, extends immediately to the case considered here.

All the results of [8] have versions for the more general class of random series considered in this paper. These include a central limit theorem for $Z(x)$ and the identification of the uniformly convergent series of the type given in (3) with a Banach space of cotype 2. An application of random Fourier series to a non-random problem in the study of lacunary series is given in [10].

Note that it is not necessary to assume that $\sup_\gamma E|\xi_\gamma|^2 < \infty$ in

(3). Let $\{\xi_\gamma\}$ be simply a sequence of complex valued random variables on the probability space $(\Omega_1, \mathscr{F}_1, P_1)$. Then a necessary and sufficient condition of the series (3) to converge uniformly a.s. is that

$$(34) \qquad I((\sum_\gamma |a_\gamma|^2 |\xi_\gamma(\omega_1)|^2 |\gamma(s)-1|^2)^{1/2}) < \infty \text{ a.s. } (P_1).$$

From (34) we can obtain results even when the $\{\xi_\gamma\}$ do not have second moments. For example, let $\{\xi_\gamma\}$ be independent copies of $\xi$ where $E[e^{it\xi}] = e^{-|t|^p}$. Then we have

$$I((\sum_\gamma |a_\gamma|^p |\gamma(s)-1|^p)^{1/p}) < \infty$$

implies $\sum_\gamma a_\gamma \varepsilon_\gamma \xi_\gamma \gamma(x)$ converges uniformly a.s. (see Theorem 2.9 [8]).

We plan to elaborate upon these remarks and to give a more detailed proof of Theorem 1 in a later paper.

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