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CONDITIONAL EXCURSION THEORY

by

David Williams

1. The purpose of this note is to draw attention to CMO formulae, that is, 'conditional' versions of the Motoo-Okabe formula (of Theorem VIII.1 of Maisonneuve [3]) in which the sample paths of the 'process on the boundary' are assumed known. Various CMO formulae have long been used - often with more intuition than formal 'rigour' - in constructing sample paths of Markov chains. The type of general CMO formulae for Markov processes which I have in mind will be intuitively obvious from the special case described here.

Walsh ([4]) has recently provided very illuminating explanations of the theorems of Knight and Ray on diffusion local time, and of other Markovian properties in the space variable. The CMO formulae establish Walsh's conjecture that, for diffusions with known initial and final values, martingales relative to the excursion fields can jump only at the process minimum M (whence any excursion-field stopping 'time' S with $P\{S=M\}=0$ is previsible).

This brief note is merely intended to generate interest. It is not very polished. I stick to my favourite diffusion, '3-dimensional' Bessel process, BES(3), so as to be able to provide cross-checks on some complicated formulae. I hope that a more complete version of some of these ideas (with much less cavalier use of the word 'obvious') will soon take shape at Swansea.

2. Let Ω be the space $C(\mathbf{R}^+, \mathbf{R}^+)$ of continuous paths with state-space $[0,\infty)$. Let X be the coordinate process:

$$X(t,\omega) \equiv X_t(\omega) \equiv \omega(t) ; \quad X(\infty,\omega) \equiv \partial .$$

Fix x in $(0,\infty)$, and make the following definitions:

$$F \equiv [0,x] ; \quad T \equiv \inf\{t>0 : X_t \in F\} ;$$

$$H(t) \equiv \text{measure}\{s \le t : X_s \in F\} ; \quad H \equiv H(\infty) ;$$

$$\rho(\tau) \equiv \inf\{t : H(t) > \tau\} ; \quad Y(\tau) \equiv X \circ \rho(\tau) .$$

(By the usual convention that $\inf \emptyset \equiv \infty$, $Y(\tau) = \partial$ for $\tau \ge H$.)

The σ-algebra $\mathcal{E}_x$ determined by excursions below x is defined as follows:

$$\mathcal{E}_x \equiv \sigma\{Y(\tau) : 0 \le \tau < \infty\} .$$

This agrees with Walsh's definition for the diffusion case.

In the situations which concern us, the local time:

$$L_x^Y(\tau) \equiv \lim_{h \downarrow 0} \frac{\text{measure}\{\sigma \le \tau : Y(\sigma) \in (x-h,x)\}}{2h}$$

exists.

We use the functional notation for measures, so that we do not require different symbols for probability and expectation. Define:

BES^y to be the law of '3-dimensional' Bessel process starting at y ;

BM^y to be the law of Brownian motion starting at y ;

ITO^x to be the Itô excursion law (characteristic measure) at boundary point x for reflecting Brownian motion on $[x,\infty)$, with the Itô-McKean normalisation of local time at x .

Fix $\lambda > 0$, and define

$$\gamma \equiv (2\lambda)^{\frac{1}{2}} .$$

The appearance of γ in the CMO formulae derives from the well-known fact that

$$ITO^x\{1 - \exp(-\lambda T)\} = \gamma .$$

3. FIRST-ORDER CMO FORMULAE FOR BES(3).

Let f be a bounded Borel function on $[0,\infty)$ and let ξ be an exponentially distributed variable of rate λ which is independent of X .

Then, on the set $\{T = \infty\}$, we have:

(1) $\quad BES^b\{f(X_\xi) \mid \mathcal{E}_x\} = BES^{b-x}\{f(x + X_\xi)\} .$

On the set $\{T < \infty\}$, we have:

(2) $$\mathrm{BES}^b\{f(X_\xi);\ T > \xi \mid \mathcal{G}_x\}$$
$$= \mathrm{BM}^b\{f(X_\xi);\ T > \xi\};$$

(3) $$\mathrm{BES}^b\{f(X_\xi);\ T < \xi;\ X_\xi \in F \mid \mathcal{G}_x\}$$
$$= \mathrm{BM}^b\{T < \xi\} \int_0^H \lambda \exp[-\lambda\tau - \gamma L_x^Y(\tau)]\, f(Y_\tau)\, d\tau$$

(4) $$\mathrm{BES}^b\{f(X_\xi);\ T < \xi;\ X_\xi \notin F;\ \xi < \rho(H-) \mid \mathcal{G}_x\}$$
$$= \mathrm{BM}^b\{T < \xi\}\ \mathrm{ITO}^x\{f(X_\xi);\ \xi < T\} \int_0^H \exp[-\lambda\tau - \gamma L_x^Y(\tau)]\, dL_x^Y(\tau);$$

(5) $$\mathrm{BES}^b\{f(X_\xi);\ T < \xi;\ X_\xi \notin F;\ \xi > \rho(H-) \mid \mathcal{G}_x\}$$
$$= \mathrm{BM}^b\{T < \xi\}\ \mathrm{BES}^0\{f(x + X_\xi)\} \exp[-\lambda H - \gamma L_x^Y(H)].$$

Proof. The above formulae seem to me to be obvious given Itô excursion theory ([2]) and Theorem 3.1 of Williams [5].

4. The reason that I believe the formulae is that they 'integrate out' correctly. If, for example, formula (3) is correct, then we must have, for $b \le x$,

(3*) $$\mathrm{BES}^b\{\int_0^H \exp[-\lambda\tau - \gamma L_x^Y(\tau)]\, f(Y_\tau)\, d\tau\} = h(b),$$

where

$$h(b) \equiv \mathrm{BES}^b\{\int_0^\infty e^{-\lambda t} f(X_t)\, I_F(X_t)\, dt\}.$$

Now (see, for example, [5]) it is well known that, for $b < x$,

$$bh(b) = \gamma^{-1} e^{-\gamma b} \int_0^b 2y \sinh(\gamma y)\, f(y)\, dy + \gamma^{-1} \sinh(\gamma b) \int_b^x 2y e^{-\gamma y} f(y)\, dy.$$

Thus, h is bounded near 0, h satisfies

$$\lambda h - \tfrac{1}{2} h'' - b^{-1} h' = 0 \quad \text{on} \quad (0,x),$$

and h obeys the elastic-barrier condition:

$$[xh(x)]' + \gamma x h(x) = 0.$$

These facts confirm (3*) (but not, of course, (3)!).

With a little effort, you can check that (4) integrates out properly, using (among other things) the results:

$$\text{ITO}^x\{f(X_\xi);\ \ \xi < T\} = 2\lambda\int_x^\infty e^{-\gamma(y-x)} f(y)\,dy\ ;$$

$$\text{BES}^b\{\int_0^H \exp[-\lambda\tau - \gamma L_x^Y(\tau)]\,dL_x^Y(\tau)\} = (\gamma b)^{-1}\sinh(\gamma b)xe^{-\gamma x}\ .$$

5. Formulae (1)-(5) determine the martingale Z, where

$$Z_x \equiv \text{BES}^b\{f(X_\xi)\,|\,\mathcal{G}_x\}\ ,$$

and show that Z is continuous in x except perhaps at $x = M$, where

$$M \equiv \inf\{X(t) : t \geq 0\}\ .$$

It is clear that what we must do now is to transfer the proof of Meyer's celebrated previsibility theorem for Markov processes 'from time to space'. If you consult pages 36-38 of Getoor [1], you will see that Walsh's conjecture will follow once we prove the following result.

THEOREM. <u>Let</u> $n \in \mathbb{N}$ <u>and let</u> $\xi_1, \xi_2, \ldots, \xi_n$ <u>be exponentially distributed variables</u> <u>(of rates</u> $\lambda_1, \lambda_2, \ldots, \lambda_n$) <u>such that</u> $\xi_1, \xi_2, \ldots, \xi_n$ <u>and</u> X <u>are independent</u>. <u>Let</u> $f_1, f_2, \ldots, f_n$ <u>be bounded continuous functions on</u> $[0,\infty]$. <u>Then the martingale</u>

$$x \to \text{BES}^b\{f_1(X_{\xi_1})f_2(X_{\xi_1+\xi_2})\cdots f_n(X_{\xi_1+\xi_2+\ldots+\xi_n})\,|\,\mathcal{G}_x\} \tag{6}$$

<u>is continuous except perhaps at</u> $x = M$.

(The fact that this theorem implies Walsh's conjecture is easier to establish than the corresponding result in Getoor [1] where the ξ variables are not added together.)

6. Instead of working through all the tedious details of a full proof of the theorem, let us look at one case which gives the key 'inductive' idea.

Let ξ and η be exponential variables (of rates λ, μ respectively) such that ξ, η and X are independent. Then we have the following extension of (4): if f and g are bounded Borel functions on $[0,\infty)$,

$$\begin{aligned}(7)\qquad &\text{BES}^b\{f(X_\xi)g(X_{\xi+\eta});\ T < \xi;\ X_\xi \notin F;\ \xi < \rho(H-)\,|\,\mathcal{G}_x\} \\ &= \text{BM}^b\{T < \xi\}\int_0^H \exp[-\lambda\tau - \gamma L_x^Y(\tau)]\,A_x\,dL_x^Y(\tau) \qquad \text{on } \{T<\infty\}\end{aligned}$$

where A_x is a shorthand for the expression (depending on many 'parameters'):

$$A_x \equiv 2\lambda \int_x^\infty e^{-\gamma(y-x)} f(y)\,[BES^y\{g(X_\eta)|Y\}] \circ \theta_\tau^Y \, dy\,.$$

Here, $\theta_\tau^Y = \theta_{\rho(\tau)}$ shifts Y through Y-time τ. Of course, equation (7) relies heavily on the Markovian property of the Itô excursion. Given Itô's results, equation (7) is at least intuitively obvious (and perhaps obvious). You can check, if you wish, that equation (7) integrates out properly.

The theorem for the case when $n = 2$ follows easily from second-order CMO formulae such as (7).

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