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WEIGHTED NORM INEQUALITIES FOR MARTINGALES

M. Izumisawa and T. Sekiguchi

In this note we extend THEOREM 4 in M. Izumisawa and N. Kazamaki [1], to the case when the weight is not continuous as a martingale.

1. Theorem.

Let $(\Omega, \underline{F}, P)$ be a probability space with an increasing right continuous family $(\underline{F}_t)_{t \geq 0}$ of sub- σ -fields of \underline{F} such that $\underline{F} = \bigvee_{t \geq 0} \underline{F}_t$. We use the same notations $[X, Y]$, X^* and so on as in P. A. Meyer [2]. Let Z be a uniformly integrable martingale with $E[Z_\infty] = 1$ and $Z_\infty > 0$ a. s. . We put $\hat{P} = Z_\infty \cdot P$ and

$$(1) \quad \hat{M} = - \int_{]0, \cdot]} Z_{s-} d(Z_s^{-1}) .$$

Then \hat{M} is a local martingale with respect to \hat{P} as is shown later.

THEOREM. Let X be any local martingale with respect to P . Then we have the inequalities

$$(2) \quad (1/2)\{1 - (2\sqrt{2}+1)\|\hat{M}\|_{B(\hat{P})}\} \hat{E}[[X, X]_\infty^{1/2}] \leq \hat{E}[X^*]$$

$$\leq \sqrt{2}(4+5\|\hat{M}\|_{B(\hat{P})})\hat{E}[[X, X]_{\infty}^{1/2}],$$

where $\hat{E}[\]$ and $\|\cdot\|_{B(\hat{P})}$ denote the expectation and the BMO-norm with respect to the probability measure \hat{P} respectively.

By applying Garsia's lemma (see [2] V.24, p. 347) to the above theorem, we obtain the following corollary.

COROLLARY. Let ϕ be a continuous increasing convex function on $[0, \infty[$ with $\phi(0) = 0$ and satisfy the growth condition, that is, there exists a constant A such that $\phi(2t) \leq A\phi(t)$ for all t . Assume that $\|\hat{M}\|_{B(\hat{P})} < (2\sqrt{2} + 1)^{-1}$. Then for any local martingale X with respect to P .

$$(3) \quad c\hat{E}[\phi(X^*)] \leq \hat{E}[\phi[X, X]_{\infty}^{1/2}] \leq C\hat{E}[\phi(X^*)].$$

Here, the choice of c and C depends only on the growth parameter A of ϕ .

2. Proof of the Theorem.

For a local martingale X with respect to P we define

$$(4) \quad \hat{X} = X - \int_{]0, \cdot]} Z_s^{-1} d[X, Z]_s.$$

Then \hat{X} is a local martingale with respect to \hat{P} (see [2] VI. 22-26, p. 376). We put

$$(5) \quad M = \int_{]0, \cdot]} Z_{s-}^{-1} dZ_s.$$

We apply Ito's formula to $1 = Z_t Z_t^{-1}$ obtaining

$$\begin{aligned} M_t + \int_{]0, t]} Z_{s-} d(Z_s^{-1}) \\ = - \int_{]0, t]} d[Z, Z^{-1}]_s \\ = \int_{]0, t]} d\langle Z^c, \frac{1}{Z_{s-}^2} \cdot Z^c \rangle_s - \sum_{0 \leq s \leq t} \Delta Z_s \Delta (Z^{-1})_s \\ = \int_{]0, t]} Z_{s-}^{-1} d\langle M^c, Z^c \rangle_s + \sum_{0 \leq s \leq t} Z_{s-}^{-1} \Delta M_s \Delta Z_s \\ = \int_{]0, t]} Z_{s-}^{-1} d[M, Z]_s. \end{aligned}$$

Therefore \hat{M} defined in (1) is a local martingale with respect to \hat{P} .

We proceed to prove the following relations:

$$(6) \quad \hat{X} = X - [X, \hat{M}],$$

$$(7) \quad (1 - \|\hat{M}^d\|_{B(\hat{P})})[X, X]^{1/2} \leq [\hat{X}, \hat{X}]^{1/2} \leq (1 + \|\hat{M}^d\|_{B(\hat{P})})[X, X]^{1/2}$$

$$(8) \quad (1/2)\hat{E}[[\hat{X}, \hat{X}]^{1/2}] \leq \hat{E}[\hat{X}^*] \leq 4\sqrt{2}\hat{E}[[\hat{X}, \hat{X}]^{1/2}],$$

and

$$(9) \quad \hat{E} \left[\int_{]0, \infty[} |d[X, M]_s| \right] \leq \sqrt{2} \hat{E}[[X, X]_{\infty}^{1/2}] \|\hat{M}\|_{B(\hat{P})}.$$

From (4)

$$\hat{M}_t = \int_{]0, t]} Z_{s-}^{-1} dZ_s - \int_{]0, t]} Z_s^{-1} d[Z_s^{-1} \cdot Z, Z]_s,$$

we have

$$\begin{aligned} [X, \hat{M}]_t &= \int_{]0, t]} Z_{s-}^{-1} d\langle X^c, Z^c \rangle_s - \sum_{0 < s \leq t} \Delta X_s \{ \Delta Z_s / Z_{s-} - Z_s^{-1} (\Delta Z_s / Z_{s-}) \Delta Z_s \} \\ &= \int_{]0, t]} Z_s^{-1} d\langle X^c, Z^c \rangle_s + \int_{]0, t]} Z_s^{-1} d[X^d, Z^d]_s \\ &= \int_{]0, t]} Z_s^{-1} d[X, Z]_s \end{aligned}$$

and so we obtain the equality (6). According to the equalities $\langle \hat{X}^c \rangle = \langle X^c \rangle$ (see [2] VI. 25, p. 378) and $\Delta \hat{X}_t = \Delta X_t - \Delta X_t \Delta \hat{M}_t = \Delta X_t (1 - \Delta \hat{M}_t)$, we get easily the inequality (7). The inequality (8) is nothing but Davis' inequality (see [2] V. 29, P. 349).

The last inequality (9) is of Feffermann's type.

The proof of [2] V. 9, p. 337 is still valid in our case where X is a semi-martingale with respect to \hat{P} .

Now it follows from the above equation and inequalities (6) - (9) that

$$\begin{aligned} \hat{E}[X^*] &= \hat{E}[(\hat{X} + [X, \hat{M}])^*] \\ &\geq \hat{E}[\hat{X}^*] - \hat{E}[\int_0^\infty |d[X, \hat{M}]_s|] \end{aligned}$$

$$\begin{aligned}
&\geq (1/2)\hat{E}[[\hat{X}, \hat{X}]_{\infty}^{1/2}] - \sqrt{2}\|\hat{M}\|_{B(\hat{P})}\hat{E}[[X, X]_{\infty}^{1/2}] \\
&\geq (1/2)(1 - \|\hat{M}^d\|_{B(\hat{P})})\hat{E}[[X, X]_{\infty}^{1/2}] - \sqrt{2}\|\hat{M}\|_{B(\hat{P})}\hat{E}[[X, X]_{\infty}^{1/2}] \\
&= (1/2)\{1 - (2\sqrt{2}+1)\|\hat{M}\|_{B(\hat{P})}\}\hat{E}[[X, X]_{\infty}^{1/2}]
\end{aligned}$$

and

$$\begin{aligned}
\hat{E}[X^*] &= \hat{E}[(\hat{X} + [X, \hat{M}])^*] \\
&\leq \hat{E}[\hat{X}^*] + \hat{E}[\int_0^{\infty} |d[X, \hat{M}]_s|] \\
&\leq 4\sqrt{2}\hat{E}[[\hat{X}, \hat{X}]_{\infty}^{1/2}] + \sqrt{2}\|\hat{M}\|_{B(\hat{P})}\hat{E}[[X, X]_{\infty}^{1/2}] \\
&\leq 4\sqrt{2}(1 + \|\hat{M}^d\|_{B(\hat{P})})\hat{E}[[X, X]_{\infty}^{1/2}] + \sqrt{2}\|\hat{M}\|_{B(\hat{P})}\hat{E}[[X, X]_{\infty}^{1/2}] \\
&= \sqrt{2}(4 + 5\|\hat{M}\|_{B(\hat{P})})\hat{E}[[X, X]_{\infty}^{1/2}]
\end{aligned}$$

Finally we remark that, even though M is not continuous, for each continuous local martingale X with respect to P the constants of the inequalities (2) $(1/2)\{1 - (2\sqrt{2}+1)\|\hat{M}\|_{B(\hat{P})}\}$ and $\sqrt{2}(4 + 5\|\hat{M}\|_{B(\hat{P})})$ can be replaced by $(1/2)(1 - 2\sqrt{2}\|\hat{M}\|_{B(\hat{P})})$ and $\sqrt{2}(4 + \|\hat{M}\|_{B(\hat{P})})$ respectively, which are the same constants as in M. Izumisawa and N. Kazamaki [1].

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- [2] P. A. Meyer, Un cours sur les intégrales stochastiques,
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