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Domains of attraction in Banach spaces

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1. Introduction. In what Banach spaces can we obtain results on domains of attraction to stable measures that resemble those in the line or in R^n? This seems to be the first question upon which to start the theory of domains of attraction in Banach spaces. It is indeed a natural question given that Hoffmann-Jorgensen and Pisier [12] and Jain [13], Aldous [5] and Chobanian and Tarieladze [9] solved very neatly the same question for the domain of normal attraction to any Gaussian law. The problem has been recently studied by Araujo and Giné [7], Mandrekar and Zinn [4], Marcus and Woyczynski [17], [18], and Woyczynski [33]. In this note I will try to give a unified account of this theory (showing that apparently different formulations are aquivalent); for the sake of completeness there will be a non void intersection with some of the mentioned papers, but several proofs as well as some examples are new.

The present state of the theory of domains of attraction to non-Gaussian laws in Banach spaces, except for some preliminary results in C(S), is roughly as follows: 1) the "natural or classical" conditions for X to be in the domain of attraction of a stable law of order $\alpha\in(0,2)$ are necessary in general; 2) they are sufficient in type p-Rademacher spaces for $p>\alpha$ (thus in type α-stable spaces by a theorem of Maurey and Pisier [19]); 3) if they are sufficient in B then B is of type α-stable; 4) several sets of "natural" conditions used by different authors are equivalent; and 5) examples. I have chosen to base the proof of 1) and 2) on the general limit theorems in de Acosta, Araujo and Giné [4] (which contain other interesting less general theorems [16], [23], and are not too difficult to prove) and the proofs of 3) and 5) on some very nice work on series by Marcus and Woyczynski [17]. The proofs of 4) are essentially standard. Part of the work in [17] about series is generalized to the case of not necessarily normal attraction, thus providing new examples.

The theory on domains of normal attraction is presented separately

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and before the general case because it is somewhat neater, does not require slowly varying functions and may be more appropriate to teach in a course, eventually. Moreover, the general case, aside from the use of slowly varying functions, is very similar to the normal, both in results and techniques. The general domain of attraction to Gaussian laws is not treated; for a result in this direction the reader is referred to [7]. We will also give some preliminary results in C(S).

The notation is as follows. B will be a separable Banach space and $\mathcal{B}$ its σ-algebra; a measure on B will always mean a Borel measure. The notation $X\in DNA(\rho)$ (or DNA(Y)) (domain of normal attraction of ρ (or Y)) where X is a B-valued rv, ρ a stable p.m. (Y a stable rv) of order α on B, will mean that if X_i are independent copies of X, then there exist $b_n\in B$ such that $L(\Sigma_{i=1}^n X_i/n^{1/\alpha}-b_n)\to_{w*}\rho(L(Y))$. We will write $X\in DA(\rho)$ (DA(Y)) (X is in the domain of attraction of ρ (of Y)) if there exist $a_n\in R_+$ and $b_n\in B$ such that the above limit holds with a_n instead of $n^{1/\alpha}$. If $\{a_n\}$ is known, we write $X\in DA_{\{a_n\}}(\rho)$. We will denote $S_B=\{x\in B:\|x\|=1\}$ and $B_\delta=\{x\in B:\|x\|\le\delta\}$. σ will usually be a finite measure on S_B, and $\mu=\mu(\alpha,\sigma)$ will denote the measure on B defined by $\mu\{0\}=0$, $d\mu(r,s)=d\sigma(s)dr/r^{1+\alpha}$, for all $s\in S$ and $r>0$. $d(x,F)$ will denote the distance from $x\in B$ to the set $F\subset B$; $\partial(W)$ will be the boundary of $W\subset B$, and $\mu|C$ will be the restriction to C of the measure μ. If X is a B-valued rv, then $\dot{X}=XI_{\{\|X\|\le\delta\}}$, and if $S_n=\Sigma_j X_{nj}$, then $S_{n,\delta}=\Sigma_j X_{nj\delta}$.

If μ is a σ-finite measure on B with $\mu\{0\}=0$ and such that there exist $\mu_n\uparrow\mu$, μ_n finite, for which the sequence $\{\exp(\mu_n-|\mu_n|\delta_0)\}$ is shift tight, then we say that μ is a Lévy measure; in this case we denote by $c_\tau\text{Pois}\mu$ (or cPoisμ if τ is not relevant, or Poisμ if μ is symmetric) the centered Poisson p.m. with Lévy measure μ, i.e. the measure $c_\tau\text{Pois}\mu=w^*\text{-}\lim_n\delta_{c_n}*\exp(\mu_n-|\mu_n|\delta_0)$ where $c_n=-\int_{\|x\|\le\tau}x\,d\mu_n(x)$; this limit exists for every $\tau>0$. See e.g. [4]. In this connection it is interesting to recall the following theorem ([2], [6], [7], [14], [20], [21], [22]).

1.1 Theorem. 1) If ρ is stable of order α in B, $\alpha\in(0,2)$, then there exists a finite measure σ on S_B and $x\in B$ such that

$$(1.1)\qquad \rho=\delta_x*c\text{Pois}\mu(\alpha,\sigma),$$

2) B is of type α-stable if and only if every finite measure σ on S_B defines a Lévy measure $\mu(\alpha,\sigma)$, and therefore a stable p.m. by equation (1.1).

A triangular system of rv's $\{X_{nj}:j=1,\dots,k_n,\ n\in N\}$ is an infinitesi-

<u>mal array</u> if for each n, the X_{nj} are independent, and if $\max_j P\{\|X_{nj}\|>\varepsilon\}\to 0$ for every $\varepsilon>0$. The result in [4] which we will need is as follows:

<u>1.2. Theorem</u>. Let B be a separable Banach space, X_{nj} a triangular array of B-valued rv's and $S_n=\Sigma_j X_{nj}$. Then $\{\mathcal{L}(S_n)\}$ is shift convergent if and only if

(1) there exists a σ-finite measure μ on B, $\mu\{0\}=0$, such that

$$\Sigma_j \mathcal{L}(X_{nj})|B_\delta^c \to_{w*} \mu|B_\delta^c \text{ for every } \delta>0 \text{ with } \mu(\partial B_\delta)=0;$$

(2) $\lim_{\delta\downarrow 0}\begin{Bmatrix}\limsup\\ \liminf\end{Bmatrix}_n \Sigma_j Ef^2(X_{nj\delta}-EX_{nj\delta})=\psi(f,f)<\infty$ for every $f\in B'$;

(3) there exists (for all) a sequence of finite dimensional subspaces of B, $F_m\uparrow$, $\overline{\cup F_n}=B$, such that $\lim_m \limsup_n Ed^p(S_{n,\delta}-ES_{n,\delta},F_m)=0$ for some (all) $p>0$.

In this case, ψ defines the covariance of a centered Gaussian p.m. γ, μ is a Lévy measure and $\mathcal{L}(S_n-ES_{n,\delta})\to_{w*}\gamma * c_\delta \mathrm{Pois}\,\mu$ for every $\delta>0$ such that $\mu(\partial B_\delta)=0$. If B is of type p-Rademacher, condition (3) as a sufficient condition for shift convergence of $\{\mathcal{L}(S_n)\}$ can be replaced by

(3)' $\lim_m \limsup_n \Sigma_j Ed^p(X_{nj\delta}-EX_{nj\delta},F_m)=0$

((3)' is also necessary in some cotype p spaces).

For the definition and useful theorems on slowly and regularly varying functions, the reader is referred to Feller [10], Ch. VIII, Sections 8 and 9.

2. <u>Domains of normal attraction</u>. The theorem in the line is as follows: if ρ is a stable measure on R of order $\alpha\in(0,2)$ with associated Lévy measure $\mu=\mu(c_1,c_2,\alpha)$ defined as

$$d\mu(x) = \begin{cases} c_1 dx/x^{1+\alpha} & \text{for } x>0 \\ c_2 dx/|x|^{1+\alpha} & \text{for } x<0,\ \mu\{0\}=0, \end{cases}$$

then a random variable ξ belongs to the domain of normal attraction of ρ if and only if

$$\text{(2.1)}\qquad \begin{cases} xP\{\xi>x^{1/\alpha}\}\to c_1/\alpha \\ xP\{\xi<-x^{1/\alpha}\}\to c_2/\alpha \end{cases}$$

as $x\to\infty$. Condition (2.1) is obviously equivalent to

$$\text{(2.2)}\qquad n\mathcal{L}(\xi/n^{1/\alpha})|\{|x|>\delta\}\to_{w*}\mu|\{|x|>\delta\}$$

for every $\delta>0$. One of the several ways to prove this theorem is the following: by the general CLT in R, (2.2) is necessary for $\xi\in\mathrm{DNA}(\rho)$, and (2.2) together with the condition

(2.3) $$\lim_{\delta\downarrow 0}\sup_n n^{1-2/\alpha}\int_{|\xi|\le\delta n^{1/\alpha}}\xi^2 dP = 0$$

is sufficient; but (2.3) is contained in (2.2) as the following simple computation shows: if $\nu=L(|\xi|)$, we have

(2.4) $$n^{1-2/\alpha}\int_{|\xi|\le\delta n^{1/\alpha}}\xi^2 dP=2n^{1-2/\alpha}\int_0^{\delta n^{1/\alpha}}\int_0^x u\,du\,d\nu(x)$$

$$=2n^{1-2/\alpha}\int_0^{\delta n^{1/\alpha}}\int_u^{\delta n^{1/\alpha}}u\,d\nu(x)\,du\le 2cn^{1-2/\alpha}\int_0^{\delta n^{1/\alpha}}u^{1-\alpha}du$$

$$=2c(2-\alpha)^{-1}\delta^{2-\alpha}\to 0 \quad \text{uniformly in } n, \text{ as } \delta\to 0,$$

where $c=\sup_{u>0}u^{\alpha}P\{|\xi|>u\}<\infty$ by (2.2). This is certainly a well known compute; it is written down here only because some analogous computations will appear along this exposition.

Condition (2.2), with Euclidean norm instead of absolute value, is also necessary and sufficient in R^n, and the same proof does it, however (2.2) cannot be expressed is such a nice way as (2.1). We thus have two main questions: (i) in what Banach spaces B is the condition

(2.2)' $$nL(X/n^{1/\alpha})|B_\delta^c\to_{w*}\mu|B_\delta^c \quad \text{for all } \delta>0,$$

necessary and in what sufficient for a B-valued rv X to belong to the domain of normal attraction of a stable p.m. ρ with Lévy measure μ? And (ii): is it possible to replace (2.2)' by simpler equivalent conditions?

Question (i) can be completely answered, and this is the subject of the main result in this section. Question (ii) is easier, but it is impossible to get conditions as simple as in R (even if $B=R^n$, $n>1$). These are the equivalences;

<u>2.1. Proposition</u>. If B is a separable Banach space, X a B-valued rv, σ a finite Borel measure on S_B, and $\mu=\mu(\alpha,\sigma)$; then the following are equivalent:

(1) X satisfies condition (2.2)',

(2) for each Borel set $W\subset S_B$ such that $\sigma(\partial W)=0$, and $r>0$,
$nP\{X/\|X\|\in W, \|X\|>rn^{1/\alpha}\}\to_{n\to\infty}\sigma(W)/\alpha r^{\alpha}$,

(3) X satisfies

(3i) $\pi(X)\in DNA(cPois(\mu\circ\pi^{-1}))$ for every continuous linear π with finite dimensional range.

(3ii) there exists (for all) a family $\{F_m\}$ of finite dimensional subspaces of B such that $F_1=\{0\}$, $F_m\uparrow$, $\overline{\cup F_m}=B$ and $\lim_m \limsup_n nP\{d(X,F_m)>n^{1/\alpha}\}=0$ (where it is understood that all these $\limsup_n$ are finite).

If moreover X and σ are symmetric, then (1)-(3) are equivalent to:

(4) X satisfies (3i) for $\pi=f\in B'$ and (3ii).

Remark. It is easy to check that $\mu\circ\pi^{-1}$ is the Lévy measure of a stable p.m. of order α in $\pi(B)$, and that if μ itself is the Lévy measure of a stable p.m. ρ on B, then $c\text{Pois}(\mu\circ\pi^{-1})$ is a shift of $\rho\circ\pi^{-1}$. It is also easy to check that there is no loss of generality in assuming $\lim_m \sup_{t>0} t^\alpha P\{d(X,F_m)>t\}=0$ instead of the limit in (3ii).

Proof. (1)<=>(2) because the sets of the form $\{x/\|x\|\in W,\ r<\|x\|\leq s\}$ are a convergence determining class in $S_B\times[\delta,\infty)$ for every $\delta>0$ (see e.g. [8], Theorem 1.3.1).

(1) =>(3). The particular form of μ implies that
(a) $\mu(tC)=t^{-\alpha}\mu(C)$ for every $C\in\mathcal{B}$ (change of variables), and
(b) $\mu\{x:\pi(x)=t\}=0$ and $\mu\{x:d(x,F)=t\}=0$ for every $t\neq 0$, linear and continuous π and closed subspace F (by Fubini). Therefore, (2.2') implies:

$$(2.5)\qquad \begin{cases} nL(X/n^{1/\alpha})\circ\pi^{-1}|\{y\in\pi(B):\|y\|>\delta\}\to_{w*}\mu\circ\pi^{-1}|\{\|y\|>\delta\} \\ nP\{d(X,F)>tn^{1/\alpha}\}\to t^{-\alpha}\mu\{d(X,F)>1\}. \end{cases}$$

This already gives (3i) by the finite dimensional theorem on domains of normal attraction. To prove (3ii) just note that $\mu\{x:d(x,F_m)>1\}\downarrow 0$ (as $\mu(B_1^c)<\infty$). This and (2.5) imply (3).

(3) =>(1). Condition (3ii) ensures that the family of finite measures $\{nL(X/n^{1/\alpha})|B_\delta^c\}_{n=1}^\infty$ is flatly concentrated for every $\delta\geq 0$ (see [1] for the definition) and condition (3i) that the one dimensional marginals are tight (just apply (2.1) to $X\circ f^{-1}$, $f\in B'$). Hence, by [1], Theorem 2.3 $\{nL(X/n^{1/\alpha})|B_\delta^c\}$ is tight for each $\delta>0$. Let ν^δ be a limit of this sequence for some $\delta>0$; by a diagonal argument we can construct ν such that $\nu\{0\}=0$, $\nu|B_\delta^c=\nu^\delta$, and a subsequence $\{n_k\}$ such that $n_k L(X/n_k^{1/\alpha})|B_\tau^c\to_{w*}\nu|B_\tau^c$ for every $\tau<\delta$. By (3i) and the finite dimensional CLT, μ and ν coincide on all cylinder sets at a positive distance from zero; since these sets form a semi-ring which generates the σ-ring of all Borel sets not containing zero, we conclude that $\mu=\nu$ (note $\mu\{0\}=\nu\{0\}=0$). This proves (2.2)'.

(3)<=>(4) in case of symmetry. If (3i) holds for $\pi=f\in B'$, then by symmetry and the Cramér-Wold theorem, it also holds for every continuous linear π with finite dimensional range. □

Remark. In general, conditions (2), (3) or (4) are hard to verify. However (3), and particularly (4), are adequate in some particular situations, notably if $X=\Sigma\theta_i x_i$, θ_i real rv's and $x_i\in B$, as in this case there is a natural choice for F_m. This is always the case in spaces with a Schauder basis.

Part of the proof of the main theorem is based on some interesting properties of series of the form $\Sigma\theta_i x_i$ where the θ_i are truncations of stable variables. These results, collected in the next lemma, are due to Marcus and Woyczynski [16]. The proof of 2.2(i) departs slightly from [16].

2.2. Proposition. Let $\{x_i\}\subset B$ be such that $\Sigma\|x_i\|^\alpha<\infty$, $\{\phi_i\}$ independent symmetric stable rv's of order α with Lévy measure $d\mu(x)=dx/|x|^{1+\alpha}$, $x\neq 0$, $\mu\{0\}=0$ (i.e. with ch.f.'s $e^{-c|t|^\alpha}$ for some constant c), let $\phi_i''=\phi_i I_{\{|\phi_i|>c_i\}}$ with $c_i>0$ such that $\Sigma P\{|\phi_i|>c_i\}<\infty$, and $\rho_i=\phi_i'' I_{\{|\phi_i''|\leq d_i\}}$ for some sequence $\{d_i\}$.

Then,

(i) $\lim_{t\to\infty} t^\alpha P\{\|\Sigma_{i=1}^\infty \phi_i'' x_i\|>t\}=2\Sigma_{i=1}^\infty\|x_i\|^\alpha/\alpha$,

and

(ii) $\lim_{t\to\infty} t^\alpha P\{\|\Sigma_{i=1}^\infty \rho_i x_i\|>t\}=0$.

Proof. We first prove (i). By the Borel-Cantelli lemma, the series $\Sigma\phi_i'' x_i$ converges. Define

$$F(t) = P\{\|\Sigma_{i=1}^\infty \phi_i'' x_i\|>t\}.$$

We will show first that there exists $c>0$ and $t_o>0$ such that for $t>t_o$,

$$F(t)\leq c\Sigma_{i=1}^\infty\|x_i\|^\alpha/t^\alpha. \tag{2.6}$$

If $G(t)=P\{\sup_i\|\phi_i'' x_i\|>t\}$, then a result of Hoffmann-Jorgensen [11] asserts that

$$F(3t)\leq G(t) + 4F^2(t).$$

The properties of the tails of ϕ_i (e.g.(2.1)) imply that given $\varepsilon>0$ there exists t_1 such that for $t>t_1$,

$$G(t)\leq\Sigma_{i=1}^\infty P\{|\phi_i''|>t/\|x_i\|\}\leq 2(1+\varepsilon)\Sigma_{i=1}^\infty\|x_i\|^\alpha/\alpha t^\alpha.$$

If $4F^2(t)>\frac{1}{2}F(3t)$ from some t on, then there exist $\beta,\gamma>0$ such that from some other t on, $F(t)<e^{-\beta t^\gamma}$ and (2.6) is satisfied. So we may assume that there exists a sequence $t_k\uparrow\infty$ such that $4F^2(t_k)\leq\frac{1}{2}F(3t_k)$.

This, together with the last two inequalities yields:

$$F(3t_k)\leq 2G(t_k)\leq 4(1+\varepsilon)\Sigma_{i=1}^\infty\|x_i\|^\alpha/\alpha t_k^\alpha.$$

$$F(9t_k)\leq G(3t_k)+4F^2(3t_k)$$

$$\leq 2(1+\varepsilon)\Sigma_{i=1}^\infty\|x_i\|^\alpha/\alpha(3t_k)^\alpha+64(1+\varepsilon)^2(\Sigma_{k=1}^\infty\|x_i\|^\alpha/\alpha t_k^\alpha)^2.$$

Hence, from some t_{k_o} on we will have

$$F(9t_k)\leq 4(1+\varepsilon)\Sigma_{i=1}^\infty\|x_i\|^\alpha/3^\alpha\alpha t_k^\alpha,$$

and by recurrence

$$F(3^j t_k)\leq 4(1+\varepsilon)\Sigma_{i=1}^\infty\|x_i\|^\alpha/(3^{j-1}t_k)^\alpha\alpha.$$

Now (2.6) follows by interpolation.

We will obtain (i) from (2.6) and the following obvious inequalities (used by Feller [10],p.278, and also in [2],[6] and [16]): if X_1, X_2 are independent, and $|||\cdot|||$ is a seminorm, then

$$(2.7)\quad \begin{cases} P\{|||X_1+X_2|||>t\}\geq P\{|||X_1|||>t(1+\varepsilon)\}P\{|||X_2|||<t\varepsilon\} \\ \qquad +P\{|||X_2|||>t(1+\varepsilon)\}P\{|||X_1|||<t\varepsilon\}, \\ P\{|||X_1+X_2|||>t\}\leq P\{|||X_1|||>t(1-\varepsilon)\}+P\{|||X_2|||>t(1-\varepsilon)\} \\ \qquad +P\{|||X_1|||>t\varepsilon\}P\{|||X_2|||>t\varepsilon\} \end{cases}$$

for every $t>0$ and $0<\varepsilon<1$. Repeated application of the first inequality together with (2.1) gives that for each $n\in N$ and $\varepsilon\in(0,1)$.

$$\liminf_{t\to\infty} t^{\alpha}P\{\|\Sigma_{i=1}^{\infty}\phi_i''x_i\|>t\} \geq$$
$$\geq 2(1+\varepsilon)^{-\alpha}\|x_1\|^{\alpha}/\alpha+2(1+\varepsilon)^{-2\alpha}\|x_2\|^{\alpha}/\alpha +\ldots$$
$$+2(1+\varepsilon)^{-(n-1)\alpha}\|x_{n-1}\|^{\alpha}/\alpha+(1+\varepsilon)^{-n\alpha}\liminf_{t\to\infty} t^{\alpha}P\{\|\Sigma_{i=n}^{\infty}\phi_i''x_i\|>t\}.$$

Letting $\varepsilon\downarrow 0$ we get that for each n,

$$(2.8)\qquad \liminf_{t\to\infty} t^{\alpha}P\{\|\Sigma_{i=1}^{\infty}\phi_i''x_i\|>t\}\geq 2\Sigma_{i=1}^{n-1}\|x_i\|^{\alpha}/\alpha.$$

The second inequality gives that for every $n\in N$ and $\varepsilon\in(0,1)$

$$\limsup_{t\to\infty} t^{\alpha}P\{\|\Sigma_{i=1}^{\infty}\phi_i''x_i\|>t\}\leq 2(1-\varepsilon)^{-\alpha}\|x_1\|^{\alpha}/\alpha+\ldots+2(1-\varepsilon)^{-(n-1)\alpha}\|x_{n-1}\|^{\alpha}/\alpha$$
$$+(1-\varepsilon)^{-n\alpha}\limsup_{t\to\infty} t^{\alpha}P\{\|\Sigma_{i=n}^{\infty}\phi_i''x_i\|>t\}.$$

Since ε is arbitrary application of (2.6) gives

$$(2.9)\qquad \limsup_{t\to\infty} t^{\alpha}P\{\|\Sigma_{i=1}^{\infty}\phi_i''x_i\|>t\}\leq 2\Sigma_{i=1}^{n-1}\|x_i\|^{\alpha}/\alpha+C\Sigma_{i=n}^{\infty}\|x_i\|^{\alpha}.$$

But since n is arbitrary in (2.8) and (2.9), the limit (i) follows at once.

Finally we prove (ii). Note that $\Sigma_{i=1}^{\infty}\rho_i x_i$ exists also by Borel-Cantelli. Each ρ_i being bounded, for each $n\in N$ there exists M_n such that $\|\Sigma_{i=1}^{n-1}\rho_i x_i\|\leq M_n$.

Hence

$$P\{\|\Sigma_{i=1}^{\infty}\rho_i x_i\|>t\}\leq P\{\|\Sigma_{i=1}^{n-1}\rho_i x_i\|>M_n\}+P\{\|\Sigma_{i=n}^{\infty}\rho_i x_i\|>t-M_n\}$$
$$=P\{\|\Sigma_{i=n}^{\infty}\rho_i x_i\|>t-M_n\}.$$

By symmetry, the variables $\Sigma\phi_i''x_i$ and $\Sigma\rho_i x_i-\Sigma(\phi_i''-\rho_i)x_i$ are identically distributed and therefore, using (2.6) we get

$$(2.10)\qquad P\{\|\Sigma_{i=n}^{\infty}\rho_i x_i\|>t-M_n\}\leq 2P\{\|\Sigma_{i=n}^{\infty}\phi_i''x_i\|>t-M_n\}$$
$$\leq C\Sigma_{i=n}^{\infty}\|x_i\|^{\alpha}/\alpha(t-M_n)^{\alpha}$$

for t large enough. The last two sets of inequalities yield

$$\lim_{t\to\infty} t^{\alpha} P\{\| \Sigma_{i=1}^{\infty} \rho_i x_i \| > t\} \leq K \Sigma_{i=n}^{\infty} \| x_i \|^{\alpha}$$

which gives the limit (ii) as $n\to\infty$. □

The next theorem, which is the main result in this section, determines exactly the Banach spaces where the "classical condition" (2.2)' is necessary and/or sufficient for $X \in DNA(\rho)$. The following observation is pertinent: $\rho=\delta_x, x \in B$, is a stable law of order α for every α; we will write $X \in DNA_\alpha(\delta_x)=DNA_\alpha(\delta_0)$ if there exists $\{b_n\}\subset B$ such that $\Sigma_{i=1}^n X_i/n^{1/\alpha}-b_n \to 0$ in probability.

2.3. Theorem. Let B be a separable Banach space, ρ a stable p.m. of order α on B, with associated Lévy measure μ, and X a B-valued rv. Then:

(1) If $X \in DNA(\rho)$ then

(2.2)' $nL(X/n^{1/\alpha})|B_\delta^c \to_{w*} \mu|B_\delta^c$ for all $\delta>0$ (with no restrictions on B).

(2) If B is of type p Rademacher for some $p>\alpha$ then condition (2.2)' is also sufficient for $X \in DNA(\rho)$.

(3) If condition (2.2)' is sufficient for $X \in DNA_\alpha(\delta_0)$ then B is of stable type α.

(4) If condition (2.2)' is sufficient for $X \in DNA(\rho)$ for all B-valued X and stable non-degenerate p.m.'s ρ of order α, then B is of stable type α.

Remarks. 1. By Proposition 2.2 in Maurey and Pisier [19], B is of type p-Rademacher for some $p>\alpha$ if and only if it is of type α stable, $\alpha \in (0,2)$. Hence 2.3(2,3) characterize Banach spaces of stable type α.

2. In (3), it is enough to consider random variables X defined through series, as it will become apparent in the proof.

3. 2.1(1) was proved by Araujo and Giné [7] in general, and by Marcus and Woyczynski [17,18] in some particular cases; (2), by Araujo and Giné [7] in general and simultaneously and with different methods, by Woyczynski [23] in the symmetric case (although both rely on work of Le Cam [15] for their proofs; the relevance of Le Cam's work to this subject seems to have been first noticed by A. Araujo); [23] gives (2.2') in the form 2.1(2) and [7], in the forms 2.1(3 and 4). (3) and (4) are due to Marcus and Woyczynski [17]; Mandrekar and Zinn [16] have another proof. The proof of (3)-(4) here is borrowed from [16] and [17].

Proof. (1) is an immediate corollary of Theorem 1.2.

(2). We will prove that under the stated conditions, (1)-(3) in Theorem 1.2 are satisfied, and therefore that the conclusion in (2) holds. (2.2)' implies (2.5) (proof of Proposition 2.1). From the first limit in (2.5) we conclude that

$$\sup_{t>0} t^{\alpha} P\{|f(X)|>t\}=c_f<\infty$$

for every $f \in B'$. Then, as in (2.4) we get $\lim_{\delta\downarrow 0}\sup_n n^{1-2/\alpha}\int_{\|X\|\le\delta n^{1/\alpha}} f^2(X)dP$ $\le\lim_{\delta\downarrow 0}\sup_n n^{1-2/\alpha}\int_{|f(X)|\le\delta\|f\| n^{1/\alpha}} f^2(X)\,dP \le \lim_{\delta\downarrow 0} 2c_f(2-\alpha)^{-1}\delta^{2-\alpha}\|f\|^{2-\alpha}=0$,

and this gives condition (2) in Theorem 1.2 with $\psi(f,f)=0$. Let now $F_n\uparrow$, $\overline{\cup F_m}=B$, F_m finite dimensional subspaces of B. Since B/F_m is of type p Rademacher with the same difining constant as B, C_p, if X_i are i.i.d. with $L(X_i)=L(X)$, we get by (3ii) in Proposition 2.1 (and the remark after it) that

$$\begin{aligned}\sup_n Ed^p[(S_{n,\delta}-ES_{n,\delta})/n^{1/\alpha},F_m] &\le C_p \sup_n \Sigma_{j=1}^n En^{-p/\alpha}Ed^p(X_{j,\delta n^{1/\alpha}}-EX_{j,\delta n^{1/\alpha}},F_m)\\ &\le 2^p C_p \sup_n n^{1-p/\alpha} Ed^p(X_{\delta n^{1/\alpha}},F_m)\\ &\le 2^p C_p p(p-\alpha)^{-1}\delta^{p-\alpha}\sup_{t>0} tP\{d(X,F_m)>t^{1/\alpha}\}\to 0 \text{ as } m\to\infty\end{aligned}$$

where the last inequality is proved with a computation similar to (2.4) (with p instead of 2). So (3) in Theorem 1.2 is also proved. This ends up this part of the theorem as (1) in 1.2 is precisely (2.2)'.

(3) Assume B is not of stable type α but satisfies the hypothesis in statement (3). Then there exists some sequence $\{x_i\}\subset B$ such that $\Sigma\|x_i\|^{\alpha}<\infty$ but $\Sigma\phi_i x_i$ does not converge a.s., and therefore, does not converge in probability either. (As in Proposition 2.2, ϕ_i are real i.i.d. rv's with ch.f. $e^{-c|t|^{\alpha}}$). So, for some $\varepsilon>0$ and sequence $n_k\uparrow\infty$ we have

$$P\{\|\Sigma_{i=n_k}^{n_{k+1}-1}\phi_i x_i\|\ge\varepsilon\}\ge\varepsilon.$$

If ϕ_i'' are as in Prop. 2.2, then the variables $X^k=\Sigma_{i=n_k}^{n_{k+1}-1}\phi_i'' x_i$, symmetric, belong to the DNA of $\Sigma_{i=n_k}^{n_{k+1}-1}\phi_i x_i$ (as $\phi_i''\in DNA(\phi_i)$ and the sums are finite). Therefore, if $\phi_{i,r}$ are independent copies of ϕ_i, there exist $m_k\to\infty$ such that

(2.11) $\quad P\{\|\Sigma_{i=n_k}^{n_{k+1}-1}(\phi''_{i,1}+\ldots+\phi''_{i,m_k})x_i/m_k^{1/\alpha}\|>\varepsilon/2\}\ge\varepsilon/2.$

Take now $\rho_{i,r}=\phi''_{i,r}I_{\{|\phi''_{i,r}|\le d_k\}}$ for $n_k\le i<n_{k+1}$ $(\rho_i=\phi_i''I_{\{|\phi_i''|\le d_k\}})$ where $\{d_k\}$ is chosen so that

(2.12) $\quad P\{\rho_{i,r}=\phi''_{i,r}: n_k\le i<n_{k+1}, r\le m_k\}>1-\varepsilon/4.$

Then, (2.11) and (2.12) give

$$P\{\|\Sigma_{i=n_k}^{n_{k+1}-1}(\rho_{i,1}+\ldots+\rho_{i,m_k})x_i/m_k^{1/\alpha}\|>\varepsilon/2\}\ge\varepsilon/4.$$

By a symmetry argument just as in (2.10) we obtain

$$P\{\|\Sigma_{i=1}^{\infty}(\rho_{i,1}+\ldots+\rho_{i,m_k})x_i/m_k^{1/\alpha}\|\geq\varepsilon/2\}\geq\varepsilon/8$$

thus proving that the series $X=\Sigma_{i=1}^{\infty}\rho_i x_i$ does not satisfy $\Sigma_{i=1}^{n}X_i/n^{1/\alpha}\to 0$ in probability. However, by Proposition 2.2, $nP\{\|X\|>n^{1/\alpha}\}\to 0$. Hence (2.2)' is not sufficient for $X\in DNA_\alpha(\delta_0)$ in B.

(4) We will prove that if B satisfies the hypothesis in (4), then it satisfies that in (3).

Let X be a B-valued symmetric rv verifying (2.2)' for $\mu=0$, let ϕ be a standard real symmetric stable rv of order α ($\phi(t)=e^{-c|t|^\alpha}$ as in 2.2) independent of X, and let $x\in B, \|x\|=1$. Define $Y=X+\phi x$. Apply inequalities (2.7) for a continuous seminorm $|||\cdot|||$ with $X_1=X$, $X_2=\phi x$, multiply by t^α and take limits as $t\to\infty$, to get

$$\lim_{t\to\infty} t^\alpha P\{|||Y|||>t\}=2|||x|||^\alpha/\alpha$$

(as $nP\{|||X|||>n^{1/\alpha}\}\to 0$, $nP\{|\phi|>n^{1/\alpha}\}\to 2/\alpha$). With $|||\cdot|||=d(\cdot,F_m)$, F_m as in Proposition 2.1(3), the last limit gives condition (3ii) in 2.1 because $d(x,F_m)\to 0$. It is easy to see that the following analogues of (2.7) also hold true (Feller [10]p.278): if ξ_1 and ξ_2 are real independent rv's then

$$(2.13)\quad \begin{cases} P\{\xi_1+\xi_2>t\}\geq P\{\xi_1>t(1+\varepsilon)\}P\{\xi_2>-t\varepsilon\}+P\{\xi_2>t(1+\varepsilon)\}P\{\xi_1>-t\varepsilon\} \\ P\{\xi_1+\xi_2>t\}\leq P\{\xi_1>t(1-\varepsilon)\}+P\{\xi_2>t(1-\varepsilon)\}+P\{\xi_1>t\varepsilon\}P\{\xi_2>t\varepsilon\} \end{cases}$$

and analogously for $P\{\xi_1+\xi_2<-t\}$. Applying these inequalities to f(Y), $f\in B'$, and proceeding as before, we get $\lim_{t\to\infty}t^\alpha P\{f(Y)>t\}=|f(x)|^\alpha/\alpha =$

$$= \lim_{t\to\infty}P\{f(Y)<-t\}$$

which proves $f(Y)\in DNA(f(x)\phi)$. Hence, Y satisfies 2.1(4), and Proposition 2.1 implies that Y satisfies condition (2.2)' with μ defined as $d\mu(r,s)=d\sigma(s)dr/r^{1+\alpha}$ ($\mu\{0\}=0$) and $\sigma=\frac{1}{2}(\delta_x+\delta_{-x})\neq 0$, which is the Lévy measure of the stable variable ϕx. Hence the hypothesis on B implies $Y\in DNA(\phi x)$. Therefore, if $Y_i=X_i+\phi_i x, X_i, \phi_i$ independent and distributed as X and ϕ, we get $L(\Sigma_{i=1}^{n}X_i/n^{1/\alpha}+\Sigma_{i=1}^{n}\phi_i x/n^{1/\alpha})\to L(\phi x)$. But since $L(\Sigma_{i=1}^{n}\phi_i x/n^{1/\alpha})=L(\phi x)$, this implies $\Sigma_{i=1}^{n}X_i/n^{1/\alpha}\to 0$ in probability, i.e. the hypothesis in (3). □

Theorem 2.3 in the case of symmetric variables takes a pleasant form if one uses Proposition 2.1, (1)<=>(4). We will give an application:

<u>2.4. Proposition</u>. (1). Let B be of type p-Rademacher for some $p>\alpha$ ($0<\alpha<2$), let $\{\phi_i\}$ be real independent, symmetric stable rv's of order α as in Proposition 2.2 and let $\{\psi_i\}$ be real independent symmetric rv's

such that $\psi_i \in DNA(\phi_1)$. Then if $X=\Sigma_{i=1}^{\infty}\psi_i x_i$ ($x_i \in B$) exists and

(2.14) $\lim\sup_m \lim\sup_{t\to\infty} t^{\alpha}P\{\|\Sigma_{i=m}^{\infty}\psi_i x_i\|>t\}=0$,

we have that $\Sigma_{i=1}^{\infty}\phi_i x_i$ exists, $X\in DNA(\Sigma_{i=1}^{\infty}\phi_i x_i)$, and

$\lim_{t\to\infty} t^{\alpha}P\{\|\Sigma_{i=1}^{\infty}\psi_i x_i\|>t\}=2\Sigma\|x_i\|^{\alpha}/\alpha<\infty$.

(2) If (2.14) implies that $\Sigma\phi_i x_i$ exists, then B is of type α-stable.

Proof. By (2.14), if F_m = linear span of $\{x_1,\ldots,x_m\}$, then

$$\lim_m \lim\sup_n nP\{d(X,F_m)>n^{1/\alpha}\}\leq \lim\sup_m \lim\sup_n nP\{\|\Sigma_{i=m+1}^{\infty}\psi_i x_i\|>n^{1/\alpha}\}=0$$

i.e. $X=\Sigma\psi_i x_i$ satisfies 2.1(3ii). By (2.7) and (2.1), for every $n,m>0$,

$$2\Sigma_{i=1}^{n}\|x_i\|^{\alpha}/\alpha\leq\lim\inf_{t\to\infty} t^{\alpha}P\{\|\Sigma_{i=1}^{\infty}\psi_i x_i\|>t\}\leq$$
$$\leq\lim\sup_{t\to\infty} t^{\alpha}P\{\|\Sigma_{i=1}^{\infty}\psi_i x_i\|>t\}\leq 2\Sigma_{i=1}^{m}\|x_i\|^{\alpha}/\alpha+$$
$$+\lim\sup_{t\to\infty} t^{\alpha}P\{\|\Sigma_{i=m+1}^{\infty}\psi_i x_i\|>t\}$$

(as in the proof of Proposition 2.2, (2.8) and (2.9)). Hence letting first n and then m tend to ∞, we get that $\Sigma_i\|x_i\|^{\alpha}<\infty$ and that $\lim_{t\to\infty} t^{\alpha}P\{\|\Sigma_{i=1}^{\infty}\psi_i x_i\|>t\}=2\Sigma_{i=1}^{\infty}\|x_i\|^{\alpha}/\alpha$. In particular, by Theorem 1.1 $\Sigma\phi_i x_i$ exists.

The same argument using inequalities (2.13) for $f(\Sigma_i\psi_i x_i)$, $f\in B'$, gives us that

$$\lim_{t\to\infty} t^{\alpha}P\{f(\Sigma_i\psi_i x_i)>t\} = \Sigma|f(x_i)|^{\alpha}/\alpha = \lim_{t\to\infty} t^{\alpha}P\{f(\Sigma_i\psi_i x_i<-t\},$$

i.e. that $f(\Sigma_i\psi_i x_i)\in DNA(\Sigma_i\phi_i f(x_i))$, or 2.1(3i) for $\pi=f$.

Hence, Proposition 2.1 and Theorem 2.3 show that $\Sigma_i\psi_i x_i\in DNA(\Sigma_i\psi_i x_i)$.

(2). If B is not of type α stable, let $\{x_i\}\subset B$ be such that $\Sigma\|x_i\|^{\alpha}<\infty$ but such that $\Sigma\phi_i x_i$ is divergent, $\{\phi_i\}$ as in 2.2. Let ϕ_i'' be as in 2.2 too. Then $\phi_i''\in DNA(\phi_1)$ and by 2.2(i), $\lim_m \lim_{t\to\infty} t^{\alpha}P\{\|\Sigma_{i=m}^{\infty}\phi_i'' x_i\|>t\}=2\lim_m\Sigma_{i=m}^{\infty}\|x_i\|^{\alpha}/\alpha=0$, i.e. (2.14) holds. □

Remark. 2.4 (1) is essentially contained in [7], and [17] has a slightly weaker version. 2.4 (2) is in [17].

The next natural question about domains of attraction is to obtain results on DNA in those spaces where this theory does not hold, in particular, in C(S), S compact metric. A first result in this direction can be found in Araujo and Giné [7] (see Section 4 in this paper).

3. General Domains of attraction. In this section we consider the same questions as in the preceding one, but for the general case. The main difference here is that we must borrow some results from the theory of regularly varying functions.

We start by giving several equivalent formulations of the classical conditions (Feller [10],p.313).

3.1. Proposition. Let σ be a finite Borel measure on S_B, $\sigma\neq 0$, $\mu=\mu(\alpha,\sigma)$, and let X be a B-valued rv. Then the following are equivalent:

(1) there exists $\{a_n\}\subset R_+, a_n\uparrow\infty, a_n/a_{n+1}\to 1$ such that

(3.1) $nL(X/a_n)|B^c_\delta \to_{w*} \mu|B^c_\delta$ for every $\delta>0$.

(2) there exists $\{a_n\}\subset R_+, a_n\uparrow\infty, a_n/a_{n+1}\to 1$ such that

(2i) $\pi(X)\in DA_{\{a_n\}}(c\mathrm{Pois}(\mu\circ\pi^{-1}))$ for every continuous linear π on B with finite dimensional range (or $f(X)\in DA_{\{a_n\}}(c\mathrm{Pois}(\mu\circ f^{-1}))$ for every $f\in B'$, if μ and X are symmetric).

(2ii) there exists also a sequence $\{F_m\}$ of finite dimensional subspaces of B, $F_m\uparrow$, $\overline{\cup F_m}=B$ such that

(3.2) $\lim_m \sup_n nP\{d(X,F_m)>a_n\}=0$.

(3) the function $t^\alpha P\{\|X\|>t\}$ is slowly varying and

(3.3) $P\{X/\|X\|\in W, \|X\|>t\}/P\{\|X\|>t\}\to\sigma(X)/\sigma(S)$

for every Borel set $W\subset S$ such that $\sigma(\partial W)=0$.

Proof. (1) =>(2). First note that if $|||\cdot|||$ is a continuous seminorm such that $\mu\{|||x|||=r\}=0$ for all $r>0$ (as is the case for $|||x|||=f(x)$, $f\in B'$, or $|||x|||=d(x,F)$, F a closed subspace -see proof of 2.1, (1) =>(3)), then (1) implies that the function $t\to t^\alpha P\{|||X|||>t\}$ is slowly varying: if n_t is the largest n such that $a_n\leq t$, then

(3.4) $[n_t/(n_t+1)](n_t+1)P\{|||X|||>a_{n_t+1}\}/n_tP\{|||X|||>a_{n_t}u\}\leq P\{|||X|||>t\}/P\{|||X|||>tu\}$

$\leq[(n_t+1)/n_t]n_tP\{|||X|||>a_{n_t}\}/(n_t+1)P\{|||X|||>a_{n_t+1}u\}$;

Since $\mu\{|||x|||>t\}/\mu\{|||x|||>tu\}=u^\alpha$ (change of variables) we get from (1) and the previous inequalities that

$$\lim_{t\to\infty}P\{|||X|||>t\}/P\{|||X|||>tu\}=u^\alpha,$$

i.e. that $P\{|||X|||>t\}$ is regularly varying with exponent $-\alpha$.

Suppose now that (1) holds. For $f\in B'$, consider $|||x|||=|f(x)|$. Then by [10]p.281, $\lim_{t\to\infty}t^2P\{|f(X)|>t\}/\int_0^t uP\{|f(X)|>u\}du=2-\alpha$, and therefore we can perform a computation analogous to (2.4):

(3.5) $\lim_{\delta\downarrow 0}\lim\sup_n na_n^{-2}\int_{|f(X)|\leq\delta a_n}f^2(X)dP$

$\leq\lim_{\delta\downarrow 0}\lim\sup_n 2na_n^{-2}\int_0^{\delta a_n}uP\{|f(X)|>u\}du$

$\leq \lim_{\delta\downarrow 0} \limsup_n 2na_n^{-2}(2-\alpha)^{-1}\delta^2 a_n^2 P\{|f(X)|>\delta a_n\}$

$= \lim_{\delta\downarrow 0} 2(2-\alpha)^{-1}\delta^{2-\alpha}\mu\{|f(X)|>1\}=0.$

Since, as in (2.5), we also have from (3.1) that

$$nL(X/a_n)\circ\pi^{-1}|\{\|y\|>\delta\} \to_{w*} \mu\circ\pi^{-1}\{\|y\|>\delta\}$$

for every $\delta>0$, we conclude by the f.d. CLT that $\pi(X)\in DA_{\{a_n\}}(c\mathrm{Pois}(\mu\circ\pi^{-1}))$. Finally, (2ii) follows also as in the proof of 2.1 ((1) =>(3)) with a_n for $n^{1/\alpha}$.

(2) =>(1). By (2ii) and (2i) for $\pi=f\in B'$, $\{nL(X/a_n)|B_\delta^c\}$ is flatly concentrated and has uniformly tight one dimensional marginals, hence it is a uniformly tight sequence. The unicity of the limit follows from (2i) as in 2.1 ((3 =>(1)). The symmetric case can be treated via Crâmer-Wold as in 2.1.

(1) =>(3). We have already seen above that (1) implies that $t^\alpha P\{\|X\|>t\}$ is slowly varying. The rest is also easy: obviously for any integer k, $P\{X/\|X\|\in W, \|X\|>a_{n+k}\}/P\{\|X\|>a_n\}\to\sigma(W)/\sigma(S)$ by (1); therefore, if n_t is as in (3.4), we obtain (3.3) by taking limits in the obvious inequality

$$P\{X/\|X\|\in W, \|X\|>a_{n_t+1}\}/P\{\|X\|>a_{n_t}\} \leq P\{X/\|X\|\in W, \|X\|>t\}/P\{\|X\|>t\}$$

$$\leq P\{X/\|X\|\in W, \|X\|>a_{n_t}\}/P\{\|X\|\ a_{n_t+1}\}.$$

(3) =>(1). If $t^\alpha P\{\|X\|>t\}$ is slowly varying, and if

$$a_n=\sup\{t:\ nP\{\|X\|>t\}\geq\sigma(S)/\alpha\}$$

then the properties of slowly varying functions show that $a_n\uparrow\infty$, $a_n/a_{n+1}\to 1$ and $\lim_n nP\{\|X\|>a_n\}=\sigma(S)/\alpha$ (these properties of $\{a_n\}$ follow easily from the representation theorem for slowly varying functions, [10] page 282). Hence, by (3.3),

$$P\{X/\|X\|\in W, \|X\|>ta_n\}\to t^{-\alpha}\sigma(W)/\alpha=\mu\{x/\|x\|\in W, \|x\|>t\}$$

for all $t>0$ and σ-continuity set $W\subset S$. Now (1) follows as in 2.1((1)<=>(2)). □

Remark. (3) is interesting in that it does not presuppose knowledge of $\{a_n\}$, although in fact $\{a_n\}$ is implicit in the function $t^\alpha P\{\|X\|>t\}$. This condition appears first in Mandrekar and Zinn [16]. For a somewhat more complicated but equivalent condition of this kind, see Araujo and Giné [7],4.10 (i a and b) (Kuelbs and Mandrekar [14], (4.2), for the Hilbert space case).

3.2. Theorem. Let B be a separable Banach space, ρ a stable p.m. of order α on B with associated Lévy measure $\mu=\mu(\alpha,\sigma)$, $\sigma\neq 0$, and X a B-valued rv. Then:

(1) If $X \in DA(\rho)$ then condition 3.1(3) holds (and also 3.1(1) and 3.1(2) for the sequence $\{a_n\}$ such that $X \in DA_{\{a_n\}}(\rho)$).

(2) If B is of type p Rademacher for some $p>\alpha$, then condition 3.1(3) is also sufficient for $X \in DA(\rho)$. 3.1(1) or 3.1(2) for $\{a_n\}$ imply $X \in DA_{\{a_n\}}(\rho)$.

(3) If condition 3.1(3) (3.1(2) or 3.1(1) for some $\{a_n\}$)is sufficient for $X \in DA(\rho)$ ($X \in DA_{\{a_n\}}(\rho)$), then B is of type α stable.

Proof. (3) is contained in Theorem 2.3(3). (1) is just condition (1) in Theorem 1.2. So we need only see (2). We will use Theorem 1.2. Condition 3.1(3) implies 3.1(1) for some sequence $\{a_n\}$ such that $a_n \uparrow \infty, a_n/a_{n+1} \to 1$, by the last proposition. Therefore, the triangular array $X_{nk}=X_k/a_n$, $n \in \mathbb{N}$, $k \le n$, where the X_k are independent copies of X, is infinitesimal and satisfies condition 1.2 (1). Moreover, by (3.5), it also satisfies 1.2 (2) with $\psi(f,f)=0$. And if F_m is a sequence of f.d. subspaces of B, $F_m \uparrow$, $\overline{\cup F_m}=B$, and $K_m=\mu\{d(X,F_m)>1\}$, since $K_m \to 0$ and the function $t^\alpha P\{d(X,F_m)>t\}$ is slowly varying (take $|||x|||=d(x,F_m)$ in (3.4)), the theorem in [10] p.281 and (3.1) give, in analogy with (3.5), that

$$\lim_m \limsup_n Ed^p(\Sigma_{k=1}^n X_{nk\delta} - EX_{nk\delta}, F_m) \le$$

$$\le \lim_m \limsup_n C_p \Sigma_{k=1}^n Ed^p(X_{nk\delta} - EX_{nk\delta}, F_m)$$

$$\le 2^p C_p \lim_m \limsup_n n a_n^{-p} Ed^p(X_{\delta a_n}, F_m)$$

$$\le 2^p C_p \lim_m \limsup_n n a_n^{-p} p \int_0^{\delta a_n} u^{p-1} P\{d(X,F_m)>u\} du$$

$$= 2^p C_p \lim_m \limsup_n n a_n^{-p} p(p-\alpha)^{-1} (\delta a_n)^p P\{d(X,F_m)>\delta a_n\}$$

$$= 2^p C_p p(p-\alpha)^{-1} \delta^{p-\alpha} \lim_m \lim_n nP\{d(X,F_m)>a_n\}$$

$$= 2^p C_p p(p-\alpha)^{-1} \delta^{p-\alpha} \lim_m K_m = 0.$$

And this is condition 1.2(3). Hence $\{\Sigma_{k=1}^n X_n/a_n\}$ is shift convergent to ρ. □

Next we generalize the two propositions on series given in the previous section.

3.3. Corollary. Let B be of type α-stable, ψ_i symmetric independent, h a slowly varying function and $\{x_i\} \subset B$, such that $\Sigma_{i=1}^\infty \psi_i x_i$ exists and

(i) $\lim_{t\to\infty} t^\alpha P\{|\psi_i|>t\}/h(t)=2$ for every $i \in \mathbb{N}$,

(ii) $\lim\sup_m \lim\sup_{t\to\infty} t^\alpha P\{\|\Sigma_{i=m}^\infty \psi_i x_i\|>t\}/h(t)=0$.

Then $\Sigma_{i=1}^\infty \phi_i x_i$ exists (where $\{\phi_i\}$ is as in 2.2), $\lim_{t\to\infty} t^\alpha P\{\|\Sigma_{i=1}^\infty \psi_i x_i\|>t\}/h(t)$ $= 2\Sigma_{i=1}^\infty \|x_i\|^\alpha/\alpha$, and $\Sigma_{i=1}^\infty \psi_i x_i \in DA_{\{a_n\}}(\Sigma_{i=1}^\infty \phi_i x_i)$ where $a_n=\sup\{t: nt^{-\alpha}h(t)\geq 1\}$.

Proof. By the definition of slowly varying function we have that, exactly as in 2.4,

$2\Sigma_{i=1}^n \|x_i\|^\alpha/\alpha \leq \lim\inf_{t\to\infty} t^\alpha P\{\|\Sigma_{i=1}^\infty \psi_i x_i\|>t\}/h(t)$

$\leq \lim\sup_{t\to\infty} t^\alpha P\{\|\Sigma_{i=1}^\infty \psi_i x_i\|>t\}/h(t) \leq 2\Sigma_{i=1}^m \|x_i\|^\alpha/\alpha + \lim\sup_{t\to\infty} t^\alpha P\{\|\Sigma_{i=m+1}^\infty \psi_i x_i\|>t\}/h(t)$

for arbitrary n and m in $\mathbb{N}$. Therefore, the tail behavior of $\Sigma_{i=1}^\infty \psi_i x_i$ is as stated and $\Sigma_{i=1}^\infty \|x_i\|^\alpha<\infty$; in particular, $\Sigma_{i=1}^\infty \phi_i x_i$ exists. The same type or argument shows that

$\lim_{t\to\infty} t^\alpha P\{\Sigma_{i=1}^\infty \psi_i f(x_i)>t\}/h(t) = \lim_{t\to\infty} t^\alpha P\{\Sigma_{i=1}^\infty \psi_i f(x_i)<-t\}/h(t) = \Sigma_{i=1}^\infty |f(x_i)|^\alpha/\alpha$ for every $f\in B'$. This implies that $f(\Sigma_{i=1}^\infty \psi_i x_i) \in DA_{\{a_n\}}(\Sigma_{i=1}^\infty \psi_i f(x_i))$ as $t^\alpha P\{\Sigma_{i=1}^\infty \psi_i f(x_i)>t\}$ is slowly varying and $a_n \uparrow \infty, a_n/a_{n+1}\to 1$ and $na_n^{-\alpha} h(a_n)\to 1$ by standard facts on slowly varying functions, as mentioned above.

Using the computation at the start of this proof and the properties of $\{a_n\}$, and setting F_m=linear span of $\{x_1,\ldots,x_n\}$, we obtain

$0 = \lim_m 2\Sigma_{i=m}^\infty \|x_i\|^\alpha/\alpha = \lim_m \lim_{t\to\infty} t^\alpha P\{\|\Sigma_{i=m}^\infty \psi_i x_i\|>t\}/h(t)$

$= \lim_m \lim_n nP\{\|\Sigma_{i=m}^\infty \psi_i x_i\|>a_n\} \geq \lim_m \lim\sup_n nP\{d(X,F_m)>a_n\}$.

Now, Theorem 3.2(2) gives the result. ∎

With the next corollary we obtain concrete examples of application of 3.2 with $a_n \neq n^{1/\alpha}$.

3.4 Corollary. Let B be of type α-stable, ψ real symmetric such that $\psi\in DA(\phi)$, φ symmetric stable of order α as usual, and let $h(t)=t^\alpha P\{|\psi|>t\}/2$. Let $c_i>0$ be such that $\Sigma_{i=1}^\infty P\{|\psi|>c_i\}<\infty$ and $\{x_i\}\subset B$ such that for some $t_o>0$,

(3.6) $\Sigma_{i=1}^\infty \|x_i\|^\alpha \sup_{t>t_o} [h(t/\|x_i\|)/h(t)]<\infty$.

Then, if ψ_i, ϕ_i are independent copies of ψ, φ, and $\theta_i=\psi_i I_{\{|\psi_i|>c_i\}}$, we have that $\Sigma_{i=1}^\infty \theta_i x_i \in DA_{\{a_n\}}(\Sigma_{i=1}\phi_i x_i)$, where the a_n are as in 3.3.

Proof. Note that by (3.6), $\Sigma_{i=1}^\infty \|x_i\|^\alpha<\infty$ and $\Sigma_{i=1}^\infty \|x_i\|^\alpha h(t/\|x_i\|)<\infty$ for all t>0. By the previous corollary it is enough to show that

(3.7) $\lim_{t\to\infty} t^\alpha P\{\|\Sigma_{i=1}^\infty \theta_i x_i\|>t\}/h(t) = 2\Sigma_{i=1}^\infty \|x_i\|^\alpha/\alpha$

(then 3.3(ii) follows applying (3.7) to the tail sums). The proof is similar to that of 2.2(i). If $F(t)=P\{\|\Sigma_{i=1}^\infty \theta_i x_i\|>t\}$, then

(3.8) $F(t)\le C\Sigma_{i=1}^{\infty}\|x_i\|^{\alpha}h(t/\|x_i\|)/t^{\alpha}$:

just note that as in 2.2 we can obtain

$$F(3^j t_k)\le C\Sigma_{i=1}^{\infty}\|x_i\|^{\alpha}h(3^{j-1}t_k/\|x_i\|)/(3^{j-1}t_k)^{\alpha}$$

for some sequence $t_k\uparrow\infty$ and all natural j (the only additional fact needed to prove this, besides use of the definition of slow variation, is that $\Sigma_{i=1}^{\infty}\|x_i\|^{\alpha}h(t/\|x_i\|)/t^{\alpha}\to 0$ as $t\to\infty$; because of (3.6) and the fact that $h(t)/t^{\varepsilon}\to 0$ for all $\varepsilon>0$ at $t\to\infty$, this follows by dominated convergence). Then, again as in 2.2, by (3.8),

$$2\Sigma_{i=1}^{n}\|x_i\|^{\alpha}/\alpha\le\liminf_{t\to\infty}t^{\alpha}P\{\|\Sigma_{i=1}^{\infty}\theta_i x_i\|>t\}/h(t)$$

$$\le\limsup_{t\to\infty}t^{\alpha}P\{\|\Sigma_{i=1}^{\infty}\theta_i x_i\|>t\}/h(t)$$

$$\le 2\Sigma_{i=1}^{m}\|x_i\|^{\alpha}/\alpha+C\limsup_{t\to\infty}\Sigma_{i=m+1}^{\infty}\|x_i\|^{\alpha}h(t/\|x_i\|)/h(t)$$

$$\le 2\Sigma_{i=1}^{m}\|x_i\|^{\alpha}/\alpha+C\Sigma_{i=m+1}^{\infty}\|x_i\|^{\alpha}\lim_{t\to\infty}h(t/\|x_i\|)/h(t)$$

$$=2\Sigma_{i=1}^{m}\|x_i\|^{\alpha}/\alpha+C\Sigma_{i=m+1}^{\infty}\|x_i\|^{\alpha}$$

for every n, $m\in N$, and this proves (3.7). □

Remark. Here there are some examples for which condition (3.6) is satisfied.

1) If $h(t)\to c$ as $t\to\infty$, then (3.6) reduces to $\Sigma_{i=1}^{\infty}\|x_i\|^{\alpha}<\infty$ and Corollary 3.4, to 2.2 and 2.4.

2) If $\Sigma_{i=1}^{\infty}\|x_i\|^{\alpha-\varepsilon}<\infty$ for some $\varepsilon>0$, then (3.6) is satisfied for any slowly varying function h. In fact, by the representation theorem [10], p.282, there exists t_o such that for $t>t_o$, $h(t/\|x_i\|)/h(t)\le$ $\le 2\exp(\log(t/\|x_i\|)-\log t)=2\|x_i\|^{-\varepsilon}$.

3) If h(t) is eventually decreasing as $t\to\infty$ (for instance if $h(t)\approx c(\log t)^{\beta}$ $\beta<0$) then (3.6) reduces to $\Sigma_{i=1}^{\infty}\|x_i\|^{\alpha}<\infty$.

4) If $h(t)\approx c(\log t)^{\beta}$, $\beta>0$, then (3.6) is equivalent to $\Sigma_{i=1}^{\infty}\|x_i\|^{\alpha}(\log\|x_i\|^{-1})^{\beta}<\infty$.

4. Preliminary results in the case B=C(S). The results obtained so far for B=C(S) still seem to be far from definitive. In general they are consequences of the Dudley-Fernique theory for sample continuous Gaussian processes. A first question is how to generate stable p.m.'s on C(S). The following theorem, obtained independently by A. Araujo and M.B.Marcus (private communication) and by this author, gives a way for this. We recall first a definition from [4]: a continuous pseudo-distance e on S is L.P.I (implies Lipschitz paths) if there exists a continuous pseudo-dis-

tance ρ on S such that every Gaussian process X on S with the property that $E(X(s)-X(t))^2 \leq C(e(s,t))^2$, $C>0$, $s,t\in S$, has a version with almost all its sample paths in $\mathrm{Lip}(\rho)$. For instance if $\int_0 H^{1/2}(S,e,x)dx<\infty$, where H is metric entropy or, more generally, if X satisfies the condition of "mesure majorante" of Fernique, then e is L.P.I. (see the references in [4]). We choose a point $a\in S$ and set $\|x\|_e = \sup_{s\neq t} |x(s)-x(t)|/e(s,t)+|x(a)|$ for all $x\in C(S)$. In what follows, S is a compact metric space.

4.1. Theorem. Let $U=\{x\in C(S):\|x\|=1\}$. Let $\{\sigma_i\}_{i\in I}$ be a family of finite positive measures on U and $\alpha\in(0,2)$ such that

$$\sup_{i\in I}\int_U \|u\|_e^\alpha d\sigma_i(u)<\infty. \tag{4.1}$$

Then for each $i\in I$ the measure $\mu_i=\mu(\alpha,\sigma_i)$ is the Lévy measure associated to a stable p.m. $\rho_{i,\delta}=c_\delta \mathrm{Pois}\mu_i$ on C(S), and the family of p.m.'s $\{\rho_{i,\delta}\}_{i\in I}$ is relatively compact for every $\delta>0$.

Proof. By Theorem 4.10 in [4] it is enough to show that $\{\mu_i | \|x\|_e>1\}_{i\in I}$ is a relatively compact family of finite measures and that $\sup_{i\in I}\int\min(1,\|x\|_e^2)d\mu_i(x)<\infty$. But the hypothesis (4.1) easily implies these two conditions:

$$\sup_{i\in I}\mu_i\{x:\|x\|_e>m\}= \sup_{i\in I}\int_U\int_{m/\|u\|_e}^\infty r^{-1-\alpha}drd\sigma_i(u)$$

$$= \alpha^{-1}m^{-\alpha}\sup_{i\in I}\int_U \|u\|_e^\alpha d\sigma_i(u)\to 0 \quad \text{as } m\to\infty,$$

and this proves the first condition as the sets $\{x:\|x\|_e\leq m\}$ are compact in C(S); also,

$$\sup_{i\in I}\int\min(1,\|x\|_e^2)d\mu_i(x)\leq \sup_{i\in I}\int_U\int_0^{1/\|u\|_e} r^{1-\alpha}\|u\|_e^2 drd\sigma_i(u)$$

$$+\sup_{i\in I}\int_U\int_{1/\|u\|_e}^\infty r^{-1-\alpha}drd\sigma_i(u)$$

$$=\left[(2-\alpha)^{-1}+\alpha^{-1}\right]\sup_{i\in I}\int_U \|u\|_e^\alpha d\sigma_i(u)<\infty. \qquad \square$$

An immediate corollary:

4.2. Corollary. If $\Sigma_{j=1}^\infty \|x_j\|_e^\alpha<\infty$ and $\{\phi_i\}$ are i.i.d. real symmetric rv's stable of order α, then the series $\Sigma_{j=1}^\infty x_i\phi_i$ converges a.s. in C(S) and is a stable of order α, symmetric, C(S)-valued rv.

Proof. Apply Theorem 4.1 to $\sigma=\Sigma_{j=1}^\infty \|x_j\|^\alpha(\delta_{x_j/\|x_j\|}+\delta_{-x_j/\|x_j\|})$. $\square$

A. Araujo and M.B. Marcus have stronger results for stationary stable processes (as stated by Marcus in this conference) and examples on 4.2.

Next we give a sufficient condition for a C(S)-valued rv X to be in the domain of normal attraction of a stable law of order $\alpha\in(0,2)$. It is taken from [7] and is based on some of the work in [4]. More general cases can be covered with the same technique.

4.3. Theorem. Let X be a C(S)-valued rv, e a L.P.I. pseudo-distance in S and M a non-negative real rv such that:

(i) the finite dimensional distributions of X belong to the domain of normal attraction of a stable p.m. of order α in Euclidean space,

(ii) $\lim\sup_{t\to\infty} t^{\alpha}P\{M>t\}<\infty$,

(iii) $|X(\omega,s)-X(\omega,t)|\leq M(\omega)e(s,t)$ for every $s,t\in S$ and almost all $\omega\in\Omega$.

Then X is in the domain of normal attraction of a stable p.m. on C(S).

Proof. By (i) and considerations on centering as in [4], it is enough to prove that $\{L(S_n/n^{1/\alpha})\}$ is shift tight, where $S_n=\Sigma_{j=1}^{n}X_j$, X_j i.i.d. with $L(X_j)=L(X)$. Then, by Theorems 3.1 and 4.10 in [4] it suffices to check that the measures $\mu_n=\Sigma_{j=1}^{n}L(X_j/n^{1/\alpha})$ satisfy the requirements of μ_i in the proof of 4.1. But this follows as in previous proofs:

$$\lim\sup_n nP\{\|X\|_e/n^{1/\alpha}>m\}\leq\lim\sup_n nP\{M>2^{-1}mn^{1/\alpha}\}$$
$$+\lim\sup_n nP\{|X(a)|>2^{-1}mn^{1/\alpha}\}\to 0$$

as $M\to\infty$; also, the computations (2.4) show that $\sup_n n\int\min(1,M^2/n^{2/\alpha})dP<\infty$ and therefore, using (i) we get $\sup_n n\int\min(1,\|X\|_e^2/n^{2/\alpha})dP<\infty$.□

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