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by

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Introduction. Let M be a connected non-compact C^3-manifold and let L be a strictly elliptic second order differential operator on M with locally Lipschitz coefficients. If $K \subset M$ is compact with non-void interior let e_K denote its equilibrium potential. If h is a non-degenerate C^3-function on $M\setminus K$ valued in $]1, +\infty[$ the comparison lemma in question gives upper and lower estimates for $e_K(y)$ of the form $c\, a(h(y))$, where a is the equilibrium potential for $]1, \to \infty]$ corresponding to appropriate diffusions on $\mathbb{R}$ that are explicitly described in terms of h and the operator L.

A purely analytic (and short) proof of these estimates (for locally Hölder continuous coefficients) due to Azencott (and inspired by [6] and [11]) appears as part of the proof of proposition 5.2 in [2]. However, Malliavin [7] obtains these estimates by comparing the trajectories of various diffusions and this article, which presents Malliavin's ideas (and additional remarks), can be viewed as an illustration of the use of various probabilistic techniques.

In particular it is remarked that the functoriality of diffusions follows immediately once they are defined as solutions to the martingale problem.

1°. The diffusion associated with L. For simplicity (and also because [1] is not generally available) it will be assumed $L1=0$. An exposition of Azencott's arguments for this case is given in [12] (the general situation being considered in [1]).

Let M be a connected manifold of class C^2.

If (U,ϕ) is a chart of M and $u \in C^2(M)$ then

$$(Lu)\circ\phi^{-1}(x) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial v}{\partial x_i}$$

where $v\circ\phi = u$. The coefficients are assumed to be locally Hölder continuous and the matrix $(a_{ij}(x))$ to be positive definite.

Denote by $\Omega(M)$ the set of continuous functions $\omega : \mathbb{R}^+ \to M \cup \{\partial\}$ (the one-point compactification of M) that are absorbed by ∂ i.e. $\omega(t) = \partial$ implies $\omega(t+s) = \partial$ for all $s > 0$. Let $X_t : \Omega(M) \to M \cup \{\partial\}$ be the canonical coordinate maps $X_t(\omega) = \omega(t)$ and let $\underline{\underline{F}}_t = \sigma\{X_s | 0 \le s \le t\}$. Define $\delta(\omega) = \inf\{t | X_t(\omega) = \delta\}$.

Let $X(M) = (\Omega(M)\ ,\ \underline{\underline{F}}_t\ ,\ X_t\ ,\ \zeta)$.

Definition 1. A probability P on $(\Omega(M)\ ,\ \underline{\underline{F}}_\infty)$ is said to be a solution of the (x,L) -martingale problem if

(1) for all $f \in C_c^2(M)$, i.e. twice continuously differentiable and with compact support,

$$C_t^f = f o X_t - f o X_0 - \int_0^t Lf o X_s ds$$

is a martingale with respect to $(\Omega(M)\ ,\ \underline{\underline{F}}_t\ ,\ P)$; and

(2) $P\{X_0 = x\} = 1$.

Theorem 2. (Azencott; cf [12]).

For all $x \in M$ there is one and only one solution P^x to the (x,L) -martingale problem.

Definition 3. $(\Omega(M)\ ,\ \underline{\underline{F}}_t\ ,\ X_t\ ,\ P^x)_{x\in M} = (X(M)\ ,\ P^x)_{x\in M}$ is called the diffusion associated with L .

It turns out (see [12]) that because of the above uniqueness the process $(X(M)\ ,\ P^x)_{x\in M}$ satisfies the strong Markov property. In [2] Azencott characterizes its transition semigroup as the minimal sub-Markovian semigroup (P_t) such that for all $f \in C_c^2(M)$

(a) $(\frac{\partial}{\partial t} - L)P_t f = 0$, and

(b) $\lim_{t\to 0} P_t f(x) = f(x)$ for all $x \in M$.

2°. The functoriality of the diffusion.

Consider M_1, M_2 two connected C^2-manifolds and let $\psi : M_1 \to M_2$ be a proper map of class C^2 . Let $L_1\ ,\ L_2$ be two strictly elliptic operators on the corresponding manifolds for which $L_1 1$ and $L_2 1$ are both zero and such that, for all $f \in C_c^2(M_2)$

$$L_1(f o\psi) = (L_2 f) o\psi \ .$$

Let $(P^x)_{x\in M_1}$ and $(Q^y)_{y\in M_2}$ be the families of probabilities that define the corresponding diffusions.

Proposition 4. The canonical map $\Omega(\psi) : (M_1) \to \Omega(M_2)$ induced by ψ sends P^x to $Q^{\psi(x)}$ for all $x \in M_1$. In other words, the diffusion associated to L is a covariant functor on the obvious category.

Proof: The formula $X_t^2 o\Omega(\psi) = \psi o X_t^1$ for all $t > 0$ determines $\Omega(\psi)$. Let $x\in M_1$ and let $Q = \Omega(\psi)_* P^x$ be the image of P^x under $\Omega(\psi)$. It is then easy to see that Q is a solution of the $(\psi(x)\ ,\ L_2)$ -martingale problem.

Remarks. 1. The result is not new. It is merely another way of showing the functoriality of the semigroup with infinitesimal generator L which can be realised as the transition semigroup of a strong Markov process.

2. Use of [1] rather than [12] allows one to drop the condition that $L_i 1 = 0$.

3. A particular case of this situation is known as the theorem of Eells-Malliavin (see [8] p.168).

4. Let $M_1 = \mathbb{R}^n\backslash\{0\}$, $M_2 = \mathbb{R}^+$ and $\psi(x) = \|x\|$. If $L_1 = \Delta$ and $L_2 g = g'' + \frac{(n-1)}{\|x\|} g'$ then the image of Brownian motion on M_1 under ψ is the appropriate Bessel process on $\mathbb{R}^+$ (pointed out by J. Faraut).

5. The properness of ψ is used twice. First, to define $\Omega(\psi)$ and secondly to ensure that for all $f \in C_c^2(M_2)$ $(foX_t^2 - foX_0^2 - \int_0^t L_2\, foX_s^2 ds) o\Omega(\psi)$ is integrable (because it is exactly $(fo\psi)oX_t^1 - (fo\psi)oX_0^1 - \int_0^t L_1(fo\psi)oX_s^1 ds$!) . The arguments given on p.109 of [12] suggest that the first use is not essential and consequently the result is probably true as long as the integrability is preserved. The case of a fibration with non-compact fibre is perhaps worth considering

3^o. The associated increasing process.

Let M and L be as in 1^o and for $f \in C^2(M)$ let $C_t = C_t^f = foX_t - foX_0 - \int_0^t LfoX_s ds$. Then for all $x \in M$, (C_t) is a local martingale on $(\Omega(M)\ ,\ \underline{\underline{F}}_t\ ,\ \zeta\ ,\ P^x)$ in the sense that there is an increasing sequence (T_n) of stopping times $T_n < \zeta$ such that $(C_t^{T_n})$ is a uniformly integrable martingale for all n . Consequently it is natural to determine the increasing process associated with (C_t^2) up to ζ (i.e. to compute $< C,C >_t$) .

Theorem 5. Let $f \in C^2(M)$. Then, for each x , $< C,C >_t = \int_0^t \|\nabla f\|^2\ oX_s ds$ on $[0,\zeta)$ where $\|\nabla f\|^2 = Lf^2 - 2fLf$, which in local coordinates (U,ϕ) is $\Sigma a_{ij}(x)\ \frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_i}$ with $go\phi = f$ (the square of the length of the intrinsic gradient of f associated with L) .

Proof: Before giving a complete proof of this result the special case of $M = \mathbb{R}^n$ will be discussed and then an easy "proof" will be given which unfortunately contains a flaw .

1) $M = \mathbb{R}^n$. If σ is a positive square root of a then the solution P^y of the (y,L) -martingale problem can be constructed via the unique solution of the stochastic integral equation

$$Y_t = y + \int_0^t \sigma(Y_s)dB_s + \int_0^t b(Y_s)d_s$$

where (B_s) is Brownian motion on $\mathbb{R}^n$ and $b(x) = (b_1(x),\dots,b_n(x_1))$ (see Girsanov [5] theorem 3).

For $f \in C^2(M)$ Ito's lemma states that

$$foY_t = f(y) + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial y_i}(Y_s)dY_s^i + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial y_i \partial y_i}(Y_s)d < Y^{ic}, Y^{jc} >_s =$$

$$= f(y) + \sum_{i=1}^n \int_0^t \sum_{j=1}^n \frac{\partial f}{\partial y_i}(Y_s)\sigma_{ij}(Y_s)dB_s^j + \sum_{i=1}^n \int_0^t b_i(Y_s)\frac{\partial f}{\partial y_i}(Y_s)ds$$

$$+ \frac{1}{2}\sum_{i,j=1}^n \int_0^t a_{ij}(Y_s)\frac{\partial^2 f}{\partial y_i \partial y_j}(Y_s)ds$$

where Y_t^i and B_t^i denote the i -coordinate of the vector Y_t (resp. B_t) and $Y_t^{ic} = \int_0^t \sum_{j=1}^n \sigma_{ij}(Y_s)dB_s^j$

Consequently

$C_t^f = foY_t - foY_0 - \int_0^t LfoY_s ds = \sum_{j=1}^n \int_0^t (\text{grad } f.\sigma)_j oY_s dB_s^j$ and hence $< C^f, C^f >_t =$
$= \int_0^t \|\text{grad } f.\sigma\|^2 oY_s ds = \int_0^t \|\nabla f\|^2 oY_s ds$.

Remark. This shows that the result holds in general up to the exit time from a coordinate neighbourhood of x and so we could hope to prove it in general by patching things together with stopping times.

2) An incomplete proof for general M .

In the second article in [9] on the Littlewood-Paley inequalities MEYER defines a weak-type of infinitesimal generator A for the transition semigroup (P_t) of a "right" process. Specifically, $f \in D(A)$ and $Af = g$ if

1) f is bounded and universally measurable on the state space E ;

2) g is universally measurable on E and $U_p|g|$ is bounded for all $p > 0$ (where (U_p) is the associated resolvent) ; and

3) for all $p > 0$, $f = U_p(pf - g)$.

In the present context it is clear that, for each $f \in C_c^2(M)$, $P_t f - P_s f =$
$\int_s^t P_u Lf du$ and so $\frac{\partial}{\partial t} P_t f(x) = P_t Lf(x)$. Hence, $U_p Lf = \int_0^\infty e^{-pt}(\frac{\partial}{\partial t} P_t f)dt = -f+pU_pf$

which implies $f \in D(A)$ and $Af = Lf$. Now providing $D(A)$ is an algebra the computation of $< C^f, C^f > t$ as $Af^2 - fAf$ given in [9] p. 145 is applicable. However, it is not known whether $D(A)$ is an algebra and consequently the proof is incomplete.

The fact that $C_c^2(M)$ is an algebra does however play a key role in the following proof.

3) A martingale proof (suggested by both MEYER and YOR , details due to YOR, c.f. article by YOR in this volume).

First of all note that it suffices to prove the result for $f \in C_c^2(M)$ since the general result is obtained by using a sequence (T_n) of stopping times $T_n < \zeta$ that increase to ζ .

If $f \in C_c^2(M)$ then as noted above $f \in D(A)$ and so (C_t^f) is a martingale that is locally square integrable (MEYER [10] p. 143).

If $J_t = (C_t^f)^2$ and $\Gamma_t = foX_0 + \int_0^t LfoX_s ds$ then

$J_t = f^2 oX_t + \Gamma_t^2 - 2(foX_t)\Gamma_t = f^2 oX_t + \Gamma_t^2 - 2\{C_t^f + \Gamma_t\}\Gamma_t = f^2 oX_t - 2C_t^f \Gamma_t - \Gamma_t^2$.

Set $Q_t \equiv R_t$ if $(Q_t - R_t)$ is a local martingale. Then, since $f^2 \in C^2(M)$, $f^2 oX_t \equiv \int_0^t Lf^2 oX_s ds$, $C_t^f \Gamma_t \equiv \int_0^t C_s^f d\Gamma_s$ (see MEYER [10] theorem 38 p. 315) and $\Gamma_t^2 - \Gamma_0^2 = 2\int_0^t \Gamma_s d\Gamma_s$. It then follows that

$$J_t \equiv \int_0^t Lf^2 o X_s ds - 2\int_0^t \{C_s^f + \Gamma_s\} d\Gamma_s$$
$$= \int_0^t \{Lf^2 - 2fLf\} o X_s ds .$$

Consequently, $< C^f, C^f >_t - \int_0^t \{Lf^2 - 2fLf\} o X_s ds$ is a local martingale that is previsible and of bounded variation. Hence, it is constant. Since it vanishes at zero this completes the proof.

4. The corresponding time change

Let $f \in C^2(M)$ and consider the additive functional $A_t = \int_0^t \|\nabla f\|^2 o X_s ds$ ($= < C^f, C^f >_t$ where $C_t^f = f o X_t - f o X_0 - \int_0^t Lf o X_s ds$) defined for all $t > 0$ (note that $\|\nabla f\|^2(\delta) = 0$ by convention). Set $\sigma = \sup A_t$ and denote by T_s the stopping time equal to $\inf\{t < \zeta | A_t > s\}$ with the convention that $\inf \phi = \zeta$. Then $A_{T_s} = s \wedge \sigma$, for all $s \geqslant 0$.

As is shown in the appendix, the random variable $M_s = C^f_{T_s}$ is defined for all $s > 0$ (not only on $\{s<\sigma\}$) by a limit argument and satisfies

$$f o X_{T_s} = f o X_{T_0} + M_s + \int_0^{T_s} Lf o X_u du ,$$

where $f o X_{T_s}$ is defined appropriately if $\sigma \leq s$ (note that $T_s = \zeta$ if $\sigma \leq s$) .

If $\|\nabla f\|^2(u) > 0$ for all x , i.e. if f is non-degenerate, then $\int_0^{T_s} Lf o X_u du = \int_0^s (a o X_{T_v}) dv$ where $a = (Lf)/\|\nabla f\|^2$, providing $s \leq \sigma$. Consequently, for $s \leq \sigma$, if $Y_s = f o X_{T_s}$, $Y_s = Y_0 + M_s + \int_0^s (a o X_{T_v}) dv$.

MALLIAVIN [7] remarks that (M_s) is a Brownian motion. As shown in the appendix, this is so up to the $(\underline{\underline{F}}_{T_s})_s$ -stopping time σ . In other words, (M_s) is a stopped Brownian motion.

5°. The comparison lemma.

Let $K \subset M$ be compact with non-void interior and consider the equilibrium potential e_K of K on $M\backslash K$ i.e. $e_K(x) = E^x[T<\zeta]$ where T is the hitting time of K . It suffices to study e_K on a connected component of $M\backslash K$ and so to simplify notation it will be assumed that $M\backslash K$ is connected.

Denote by f a non-degenerate proper C^3 -function defined on a neighbourhood of $(\overset{\circ}{K})^c$ with values in $\mathbb{R}^+$. Then $f(M\backslash K)$ is an interval I with end points $a<b$ and it will be assumed that as $x \to \partial$ in M $f(x)$ converges to b . Replacing f by $\frac{1}{b-f}$ if $b<\infty$ it is clear that one can assume $b = +\infty$. To simplify matters it will be assumed that $\partial K = \{f = 1\}$.

Let u be a C^2 -function on I . Then $L(u o f) = \frac{1}{2}\|\nabla f\|^2 (u'' o f) + Lf(u' o f) = \frac{1}{2}\|\nabla f\|^2 \{u'' o f + a(u o f)\}$ where $a = \frac{Lf}{\|\nabla f\|^2}$ as can be seen by a computation in local coordinates. Consequently, the differential operator L modulo $\|\nabla f\|^2$ factors through f if and only if a is constant on the level hypersurfaces of f . In this case the "radial" behaviour of the diffusion associated with L on $M\backslash K$ can be reduced to that of a 1 -dimensional diffusion on $\mathbb{R}$. However, when this is not so estimates are obtained for the behaviour of e_K on $M\backslash K$ by studying two diffusions on $\mathbb{R}$.

Let $a^+(r) = \max\{Lf(x)/\|\nabla f\|^2(x) \mid f(x) = r\}$ and $a^-(r) = \min\{Lf(x)/\|\nabla f\|^2(x) \mid f(x) = r\}$. These two functions can be extended from $[1,\infty)$ to all of $\mathbb{R}$ so as to be Hölder-continuous functions on $\mathbb{R}$ since $f \in C^3$. The two diffusions in question are determined by the differential operators $D^+ = \frac{1}{2}\frac{d^2}{dr^2} + a^+\frac{d}{dr}$ and $D^- = \frac{1}{2}\frac{d^2}{dr^2} + a^-\frac{d}{dr}$.

Fix $x \in M\setminus K$ and hence $f(x) = r \in \mathbb{R}$. Let $U = \{1<f\}$ and let T be the hitting time of U^c for the diffusion on M. The strictly increasing process $(A_{t\wedge T})$, $A_t = \int_0^t \|\nabla f\|^2 o X_s ds$ determines a time change (T_s) such that if $Y_t = foX_{T_t}$ and $M_t = C^f_{t\wedge T}$ then:

(1) $Y_t = Y_0 + M_t + \int_0^t aoX_{T_v} dv$; and

(2) (M_t) is a Brownian motion stopped at $\sigma^T = \sup_t A_{t\wedge T}$.

Let α denote either a^+ or a^- and let (U_t) be the solution of the stochastic integral equation

$$U_t = r + \int_0^t (1oU_s)dM_s + \int_0^t \alpha oU_s ds \ ,$$

for the existence and uniqueness, see [5]) .

When $\alpha = a^{\pm}$ let $U_t^{\pm}$ denote the corresponding solution.

<u>Proposition 6</u>. (Comparison Lemma).

The following results hold:

(1) $U_t^- \le Y_t \le U_t^+$ for all $t < \sigma^T$

(2) $foX_T = \infty \Rightarrow U^+_{S^+} = \infty$

(3) $foX_T = 1 \Rightarrow U^-_{S^-} = 1$, where $S^{\pm}$ are the exit times from $(1,\infty)$ of the processes $(U_t^{\pm})$.

Proof: (MALLIAVIN [7]). To prove (1) first replace a^+ by $a^+ + \varepsilon$ and let $(U_t^{+\varepsilon})$ be the corresponding solution. Then the following modification of a lemma due to SKOROKHOD ([13] p.125) shows that $Y_t \le U_t^{+\varepsilon}$ for all $t > 0$. Let $G_t = U_t^{+\varepsilon} - Y_t$ and $k_t = a^+ oU_t^{+\varepsilon} - aoX_{T_t}$. Then $G_t = \varepsilon t + \int_0^t k_s ds$ P^x-a.s. since $P^x\{X_0 = x\} = 1$ implies $Y_0 = r$ P^x-a.s. Let $S = \inf\{t \mid 0<t<\sigma^T, G_t \le 0\}$. Then S is an $(\underline{\underline{F}}_{T_s})$ stopping time and $S > 0$ P^x-a.s. . This follows since k_t is continuous in t, $k_0 = a^+(r) - a(x) \geqslant 0$ and so $k_t(\omega) > -\varepsilon$ for $0 \le t < \delta(\omega)$. If $t' = S(\omega) < \infty$ then $G_S(\omega) = 0$, $k_S(\omega) = a^+(f(X_{T_{t'}}(\omega))) - a(X_{T_{t'}}(\omega)) \geqslant 0$. Since $k_t(\omega) > -\varepsilon$ if $S(\omega) - \delta_1(\omega) < t \leqslant S(\omega)$ it follows that for such t , $0 < G_t(\omega) - G_S(\omega) \leqslant \varepsilon(t - S(\omega)) - \varepsilon(t - S(\omega)) = 0$. Consequently, $S(\omega) = +\infty$ P^x-a.s. i.e. $G_t > 0$ on $\{t<\sigma^T\}$.

Similarly, replacing a^- by $a^- - \varepsilon$ it follows that $U_t^{-\varepsilon} \leqslant Y_t$ for all $t \geqslant 0$.

To complete the proof of (1) it therefore suffices to show that $\lim_{\varepsilon\downarrow 0} U_t^{\pm\varepsilon} = U_t^{\pm}$. In view of Girsanov's result on uniqueness [5], this follows from YAMADA [14] (Theorem 1.2).

The remaining statements (2) and (3) are immediate consequences of (1) since δ (the lifetime of the diffusion on $M\setminus K$) can be viewed as the entrance time of $K\cup\{\delta\}$ when starting from $x\in M\setminus K$.

6^0. Extension of the stopped Brownian motion (M_t) .

If the martingale (M_t) on $(\Omega(M), \underline{\underline{F}}_{T_t}, P^x)$ was in fact a Brownian motion then the solution (U_t) of the stochastic integral equation

$$U_t = r + M_t + \int_0^t \alpha o U_s ds$$

would describe the diffusion on $\mathbb{R}$ starting from r that corresponds to the infinitesimal generator $\frac{1}{2}\frac{d^2}{dr^2} + \alpha\frac{d}{dr}$.

Then the comparison lemma could be directly applied to show that $P^x\{U_{S^-}^- = 1\} \leqslant e_K(x)$ and $1 - e_K(x) \leqslant P^x\{U_{S^+}^+ = +\infty\} = 1 - P^x\{U_{S^+}^+ = 1\}$ where $e_K(x) = P^x\{f o X_T = 1\}$ and $T = T_{U^c}$. Hence, $P^x\{U_{S^+}^+ = 1\} \leqslant e_K(x) \leqslant P^x\{U_{S^-}^- = 1\}$, where the times $S^{\pm}$ now refer to exit from $(1, +\infty)$.

Therefore, in some way (U_t) has to be extended so as to describe the diffusion on $\mathbb{R}$. Malliavin does this by tacking onto the trajectories $t \rightsquigarrow U_t(\omega)$ the trajectories of the diffusion on $\mathbb{R}$ that start from $U_\sigma(\omega)$. As he points out "this identification is not completely straightforward and a little additional construction seems to be needed [7] ".

Rather than follow this route I propose to outline results of DAMBIS [3] which immediately permit (M_t) to be "extended" so as to give a Brownian motion (a trick used in [3]) on a larger probability space. This will then quickly give the desired extension of the process (U_t) .

Let $(\Omega, \underline{\underline{F}}_t, \underline{\underline{F}}, P)$ satisfy "les hypothèses droites". Denote by (X_t) and (Y_t) two right continuous martingales on this space and let T be a stopping time. Set $Z_t = X_{t\wedge T} + Y_t - Y_{t\wedge T}$.

Theorem 7. (DAMBIS [3]) (Z_t) is a right continuous martingale. Furthermore, if (X_t) and (Y_t) are square integrable with (A_t) and (B_t) the corresponding associated increasing processes , (Z_t) is square integrable and the increasing process (C_t) associated with (Z_t) is given by the formula

$$C_t = A_{t\wedge T} + B_t - B_{t\wedge T} \quad .$$

Corollary 7. Let (M_t) be a Brownian motion on $(\Omega, \underline{\underline{G}}_t, \underline{\underline{G}}, P)$ stopped at the stopping time T . Then there exists (i) a Brownian motion $(\overline{\Omega}, \overline{\underline{\underline{F}}}_t, \overline{\underline{\underline{F}}}, \overline{B}_t, \overline{P})$, (ii) an $(\overline{\underline{\underline{F}}}_t)$ -stopping time $\overline{T}$, and (iii) a map $\pi : \overline{\Omega} \to \Omega$ such that:

(1) $T o \pi = \overline{T}$;

(2) $M_t o \pi = \overline{B}_{t\wedge T}$; and

(3) $\pi_* \overline{P} = P$ on $\underline{\underline{G}}$.

Proof: (DAMBIS) Let (B_t) be a Brownian motion on $(\Lambda,\underline{\underline{H}}_t,\underline{\underline{H}},Q)$.
Let $\overline{\Omega} = \Omega \times \Lambda$, $\overline{\underline{\underline{F}}}_t = \underline{\underline{G}}_t \otimes \underline{\underline{H}}_t$, $\overline{\underline{\underline{F}}} = \underline{\underline{G}} \otimes \underline{\underline{H}}$ and $\overline{P} = P \otimes Q$. Define $\tilde{M}_t = M_t o\pi$ where $\pi(\omega,\lambda) = \omega$ and $\tilde{B}_t = B_t o\rho$ where $\rho(\omega,\lambda) = \lambda$. Then $(\tilde{M}_t)$ and $(\tilde{B}_t)$ are continuous martingales on $(\overline{\Omega},\overline{\underline{\underline{F}}}_t,\overline{\underline{\underline{F}}},\overline{P})$. Furthermore $(\tilde{M}_t)$ stops at $\overline{T} = To\pi$, which is an $(\overline{\underline{\underline{F}}}_t)$ -stopping time.

Define $\overline{B}_t = \tilde{M}_t+\tilde{B}_t -\tilde{B}_{t\wedge\overline{T}}$. Then $(\overline{B}_t)$ is a square integrable continuous martingale whose associated increasing process is $\overline{T}\wedge t+t -\overline{T}\wedge t = t$.

Proposition 8. Let (V_t) be the solution of the stochastic integral equation on $\overline{\Omega}$

$$(*) \quad V_t = r+\overline{B}_t + \int_0^t \alpha o V_s ds \ .$$

Let (U_t) be the solution of the stochastic integral equation on Ω

$$U_t = r+M_t + \int_0^t \alpha o U_s ds \ .$$

Then $U_t o\pi = V_{t\wedge\overline{T}}$.

Proof: $(V_{t\wedge\overline{T}})$ is the solution of the stochastic integral equation

$V_t = r+\overline{B}_{t\wedge\overline{T}} + \int_0^{t\wedge\overline{T}} \alpha o V_s ds$. Since $(U_t o\pi)$ solves the same equation the result follows.

Let $S^{\pm}$ be as in the statement of proposition 6. Let $R^{\pm}$ be the exit times from $(1, +\infty)$ for the diffusion on $\mathbb{R}$ with differential generator $D^{\pm}$. Since these diffusions when started from r can be realized by $V^{\pm}$ (solutions of equation $(*)$ with $\alpha = a^{\pm}$) $\pi^{-1}\{U^+_{S^+}= \infty\} \subset \{V^+_{R^+}= \infty\}$ and $\pi^{-1}\{U^-_{S^-}= 1\} \subset \{V^-_{R^-} = 1\}$. Consequently, the previous corollary (3) and the comparison lemma imply the following result.

Corollary 9.

$$P^x\{f(X_T) = \infty\} \leq \overline{P}^x\{V^+_{R^+} = \infty\} \quad \text{and}$$

$$P^x\{f(X_T) = 1\} \leq \overline{P}^x\{V^-_{Q^-}= 1\} \ .$$

Finally, in view of the equations $(*)$ with $\alpha = a^{\pm}$ it follows that $\overline{P}^x\{V^+_{R^+}= \infty\} = 1 -h^+(r)$ and $\overline{P}^x\{V^-_{R^-}= 1\} = h^-(r)$ where $h^{\pm}$ are the solutions of the Dirichlet problem for $D^{\pm}$ on $(1,\infty)$ with boundary value $1_{\{1\}}$.
Hence, this yields.

Corollary 8. $h^+(r) \leq e_K(x) \leq h^-(r)$.

Appendix. The definition and properties of (M_s)

1. Let $C^f_t = C_t = foX_t -foX_0 -\int_0^t LfoX_u du$. Then, for all x , (C_t) is a local martingale on $(\Omega,\underline{\underline{F}},\underline{\underline{F}}_t,P^x)$. Let (T_n) be a sequence of stopping times $T_n < \zeta$ that reduces (C_t) ([10] p.292) and the local martingale $(C^2_t -A_t)_t$, where $A_t = \int_0^f \|\nabla f\|^2 oX_s ds$.

2. Fix $s = a$ and consider $(C_{t\wedge T_a})_t$ with associated increasing process $(A_{t\wedge T_a})_t$. Then (T_n) reduces the local martingale $(C^2_{t\wedge T_a}-A_{t\wedge T_a})_t$.

Lemma . For all $a,b \geq 0$ $(C_{T_a \wedge T_n \wedge b})_n$ is a uniformly integrable martingale on $(\Omega,\underline{\underline{F}},\underline{\underline{F}}_{T_n},P^x)$.

Proof: $E[C^2_{T_a \wedge T_n \wedge b}] = E[A_{T_a \wedge T_n \wedge b}] \leq a$ and so, if $X_t = C_{T_a \wedge T_{(n+1)} \wedge t \wedge b}$, (X_t) is a uniformly integrable martingale relative to $(\underline{\underline{F}}_t)$. Hence, $E[X_{T_{(n+1)}} | \underline{\underline{F}}_{T_n}] = X_{T_n}$. The uniform integrability follows from the first inequality.

Define $C_{T_a \wedge t}$ to be $\lim_{n\to\infty} C_{T_a \wedge T_n \wedge t}$. Then $C_{T_a \wedge t}$ agrees with its usual value for $t < \zeta$ and if $T_a \wedge t \geq \zeta$ its value is given by this limit rather than by $C_\zeta = -foX_0 - \int_0^\zeta LfoX_u du$.

Lemma . $(C_{T_a \wedge t})_t$ is a uniformly integrable martingale on $(\Omega,\underline{\underline{F}},\underline{\underline{F}}_t,P^x)$.

Proof: Let $t_1 < t_2$ and $\wedge \in \underline{\underline{F}}_{t_1}$. Then $E[1_\wedge C_{T_a \wedge T_n \wedge t_1}] = E[1_\wedge C_{T_a \wedge T_n \wedge t_2}]$ and uniform integrability implies $E[1_\wedge C_{T_a \wedge t_1}] = E[1_\wedge C_{T_a \wedge t_2}]$.

Furthermore, $E[C^2_{T_a \wedge T_n \wedge t}] \leq a \; \forall n,t$ implies $E[C^2_{T_a \wedge t}] \leq a$ and so $(C_{T_a \wedge t})_t$ is uniformly integrable.

Corollary (c.f. DAMBIS [3] lemma 6). $(C_{T_a})_a$ is a martingale on $(\Omega,\underline{\underline{F}},\underline{\underline{F}}_{T_a},P^x)$.

Proof: If $a < b$ then $E[C_{T_a \wedge T_b} | \underline{\underline{F}}_{T_a}] = C_{T_a}$.

Lemma . $(C^2_{T_a} - \sigma \wedge a)_a$ is a martingale on $(\Omega,\underline{\underline{F}},\underline{\underline{F}}_{T_a},P^x)$.

Proof: For each n , $(C^2_{T_a \wedge T_n \wedge t} - A_{T_a \wedge T_n \wedge t})_t$ is a martingale. Since $C^2_{T_a \wedge T_n \wedge t} \to$ $\to C^2_{T_a \wedge t}$ in L^1 (by theorem 4.15 (iii) in [4]) and $A_{T_a \wedge T_n \wedge t} \to A_{T_a \wedge t}$ monotonically as $n \to \infty$ it follows that $(C^2_{T_a \wedge t}) - A_{T_a \wedge t})_t$ is also a martingale. By repeating the argument it follows that $C^2_{T_a} - A_{T_a})_a$ is a martingale where $C_{T_a} = \lim_{t\to\infty} C_{T_a \wedge t}$. Note that $A_{T_a} = \sigma \wedge a$.

Finally, the following result concludes the proof that (C_{T_a}) is a Brownian motion stopped at σ .

Proposition $a \rightsquigarrow C_{T_a}$ is continuous a.s.

Proof: It is obvious on $[0,\sigma)$. If $a = \sigma(\omega)$ then $T_a(\omega) = \zeta(\omega)$ and $C_{T_a}(\omega)$ is defined by a limit from the left. For $a > \sigma(\omega)$, $T_a(\omega) = \zeta(\omega)$ and so the result follows.

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