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A PROPERTY OF CONFORMAL MARTINGALES

by

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Let  $Z$  be a complex-valued process and let  $X$  and  $Y$  be its real and imaginary parts respectively, so that  $Z = X + iY$ . Let  $\{\mathcal{F}_t, t \geq 0\}$  be an increasing family of complete  $\sigma$ -fields. We say that  $\{Z_t, \mathcal{F}_t, t \geq 0\}$  is a conformal martingale if both  $X$  and  $Y$  are continuous local martingales relative to  $\{\mathcal{F}_t\}$ , such that  $\langle X, Y \rangle_t \equiv 0$  and  $\langle X, X \rangle_t \equiv \langle Y, Y \rangle_t$ . If the point  $t=0$  is not included in the parameter set, we will say that  $\{Z_t, \mathcal{F}_t, t > 0\}$  is a conformal martingale if for all  $\delta > 0$ ,  $\{Z_t, \mathcal{F}_t, t \geq \delta\}$  is a conformal martingale. We refer the reader to (1) for the properties of conformal martingales. We want to call attention to the following property which, though elementary, is still curious.

Proposition : Let  $\{Z_t, \mathcal{F}_t, t > 0\}$  be a conformal martingale. Then, for a.e.  $\omega$ , one of the following happens.

- Either (i)  $\lim_{t \rightarrow 0} X_t(\omega)$  exists in the Riemann sphere,
- or (ii) for each  $\delta > 0$ ,  $\{\dot{X}_t(\omega), 0 < t < \delta\}$  is dense in  $\mathbb{C}$ .

Remark : Both possibilities can occur. Indeed, if  $B_t$  is a complex Brownian motion from 0 and if  $f$  is holomorphic in  $\mathbb{C} - \{0\}$ , then  $\{f(B_t), t > 0\}$  is a conformal martingale. If 0 is a removable singularity,  $\lim_{t \rightarrow 0} f(B_t)$  exists. If it is a pole,  $\lim_{t \rightarrow 0} f(B_t) = \infty$ , and if it is an essential singularity,  $\{f(B_t), 0 < t < \delta\}$  is dense in  $\mathbb{C}$  for each  $\delta > 0$ . Thus the above proposition is the analogue for conformal martingales of Weierstrass' theorem.

Proof : All we must show is that if (i) doesn't happen, (ii) does. The only fact about conformal martingales we will need is that if  $\{Z_t, t \geq t_0\}$  is a conformal martingale, it can be time-changed into a complex Brownian motion with a possibly finite lifetime ([1] or [2], p. 384). Thus all hitting probabilities for  $Z$  are dominated by those of Brownian motion.

Suppose (i) doesn't happen. Then there exist concentric circles  $C_1, C_2$  with a rational center  $z_0$  and rational radii  $r_1 < r_2$  respectively, such that the number of incrossings of  $(C_1, C_2)$  by  $Z_t(\omega)$  is infinite. Here, the number of incrossings  $v_{a,b}^{(\omega)}$  of  $(C_1, C_2)$  in  $(a,b)$  is defined to be the number of downcrossings (in the usual sense) of the interval  $(r_1, r_2)$  by the process  $\{|Z_t(\omega) - z_0|, a < t < b\}$ . Let  $D$  be a disc.

Suppose that  $D$  is not entirely contained in the interior of  $C_1$ . (If it is, we merely talk about outcrossings rather than incrossings in what follows.) The proposition will be proved if we can show that for any  $\delta > 0$ ,  $T_D < \delta$  a.s. on the set  $\{v_{0\delta} = \infty\}$ , where  $T_D = \inf \{t > 0 : Z_t \in D\}$ .

Let  $N$  be an integer.

$$\begin{aligned}
 (1) \quad P\{v_{0\delta} = \infty, T_D > \delta\} &\leq \lim_{n \rightarrow \infty} \{v_{\frac{1}{n}\delta} > N, T_D > \delta\} \\
 &= \lim_{n \rightarrow \infty} P\{T_D > \delta | v_{\frac{1}{n}\delta} > N\} P\{v_{\frac{1}{n}\delta} > N\} \\
 &\leq \lim_{n \rightarrow \infty} P\{T_D > \delta | v_{\frac{1}{n}\delta} > N\} .
 \end{aligned}$$

But this last probability involves only hitting probabilities, and hence can be dominated by the corresponding probability for Brownian motion. If  $P_B^z$  is the probability measure of Brownian motion starting from  $z$ , let :

$$\rho = \sup_{z \in C_2} P_B^z \{T_D < T_{C_1}\} < 1 .$$

It is easy to see, using the strong Markov property, that

$$P_B^z\{B_t \notin D, \forall t \in (\frac{1}{n}, \delta) | v_{\frac{1}{n}\delta} \geq N\} \leq \rho^{N-1}$$

Thus, from (1) we have :

$$(2) \quad P\{v_{0\delta} = \infty, T_D < \delta\} \leq \rho^{N-1} \rightarrow 0 \text{ as } N \rightarrow \infty ,$$

and we are done.

#### References :

- (1) R.K. GETTOOR and M.J. SHARPE : Conformal martingales , Invent. Math. 16 , pp. 271-308 (1972).
- (2) J.L. DOOB : Stochastic Processes , John Wiley and Sons , New York , 1953.