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by R. Cairoli and J.B. Walsh

Meyer's section theorem, Skorohod's embedding theorem, and a number of time-change theorems are all aspects of a fundamental principle underlying general theory of processes, to wit : there is a stopping time which will do almost anything one wants it to do*).

The corresponding principle for multiparameter processes operates only at a much-reduced level. It is not that there is any lack of stopping times. To the contrary, there is a great, even confusing, number of analogous objects. It is just that, by and large, they are of limited usefulness. We propose to illustrate one of these limits in this note.

In two-dimensional time, one analogue (there are others) of Brownian motion is the Brownian sheet $\{W_{s,t}, (s,t) \in \mathbb{R}_+^2\}$, which is characterized by the fact that it is a zero-mean Gaussian process with covariance function $\gamma(s,t;u,v) = (s \wedge u)(t \wedge v)$.

Question : can a given two-parameter martingale be time-changed into a Brownian sheet ?

The answer to this in the one-parameter case, given by the Dubins-Schwarz theorem, is "yes", and the time-change can

*) Thus, while one can't find a stopping time which will boil an egg, he can find one which will keep the egg from being hard boiled.

be constructed as follows. Let $\{M_t, t \in \mathbb{R}_+\}$ be a continuous martingale with unbounded paths and let $\langle M\rangle_t$ be the continuous increasing process, with $\langle M\rangle_o = o$, for which $M_t^2 - \langle M\rangle_t$ is a martingale. If $T_t = \inf\{s : \langle M\rangle_s > t\}$ is the inverse of $\langle M\rangle_t$, then $\{M_{T_t}, t \in \mathbb{R}_+\}$ is a Brownian motion.

Notice that the time-change depends only on the increasing process. Thus, to make our question more specific, we ask if a given two-parameter martingale can be transformed into a Brownian sheet via a time-change which depends only on the increasing process $\langle M\rangle_{s,t}$. (See [1] for a discussion of the increasing process associated with a two-parameter martingale.)

We will see that the answer to this question is "no", even if we restrict ourselves to strong martingales [1].

Let $\{W_{s,t}, (s,t) \in \mathbb{R}_+^2\}$ be a Brownian sheet, let

$$\phi(s,t) = \begin{cases} 1 & \text{if } st \leq 1, \\ 2 & \text{if } st > 1, \end{cases}$$

and define

(1) $$M_{s,t} = \iint_{oo}^{st} \phi(u,v)\,dW_{u,v}.$$

M is a strong martingale with increasing process

(2) $$\langle M\rangle_{s,t} = \iint_{oo}^{st} \phi^2(u,v)\,du\,dv.$$

This process is deterministic, so that any time-change depending only on it must be deterministic, i.e. of the form $(s,t)\to\Gamma(s,t)$, where Γ is a fixed mapping of $\mathbb{R}_+^2$ onto itself. Thus, the problem reduces to the simpler, but still not quite trivial, one of finding a mapping Γ of the positive quadrant onto itself such that $\{M_{\Gamma(s,t)}, (s,t)\in\mathbb{R}_+^2\}$ is a Brownian sheet.

Some notation : S will denote the open quadrant $\{(s,t):s>o,t>o\}$, H_c and S_c the sets $\{(s,t)\in S:\ st = c\}$ and $\{(s,t)\in S:\ st \leq c\}$ respectively $(c > o)$. We say $(s,t) \prec (u,v)$ if $s \leq u$ and $t \leq v$. When we write $z\wedge z'$ and $z\vee z'$ for elements of $\mathbb{R}_+^2$, we mean the inf and sup respectively relative to the partial order "$\prec$". Since all the processes we consider vanish on the axes, we need only consider mappings on the open set S. A mapping Γ of S onto itself is <u>order-preserving</u> if, for $z,z'\in S$, $z \prec z'$ if and only if $\Gamma(z) \prec \Gamma(z')$. An order-preserving map is necessarily one-to-one.

If we speak about a martingale without indicating the σ-fields, it is understood that the natural σ-fields are intended. $\mathcal{F}_z$ will always refer to the σ-fields generated by W, suitably completed. These fields satisfy the conditional independence hypothesis (F4) of [1]:

(F4) For each $(s,t)\in\mathbb{R}_+^2$, the fields $\mathcal{F}_{s,\infty}$ and $\mathcal{F}_{\infty,t}$ are conditionally independent given $\mathcal{F}_{s,t}$.

<u>Lemma 1</u>. Let Γ and Γ' be order-preserving mappings of S onto itself. If $\Gamma(z) = \Gamma'(z)$ for each z in some H_c, then $\Gamma \equiv \Gamma'$.

Proof. If $z \in S$, there are unique $z_1, z_2 \in H_c$ such that either $z = z_1 \vee z_2$ or $z = z_1 \wedge z_2$. If, for instance, $z = z_1 \vee z_2$, then $\Gamma(z) = \Gamma(z_1 \vee z_2) = \Gamma(z_1) \vee \Gamma(z_2)$, since Γ preserves order. But this equals $\Gamma'(z_1) \vee \Gamma'(z_2) = \Gamma'(z_1 \vee z_2) = \Gamma'(z)$.

Lemma 2. Let $\{X_z, \mathcal{G}_z, z \in S\}$ be a martingale whose σ-fields $\mathcal{G}_z$ satisfy (F4), and such that $P\{X_z = X_{z'}\} < 1$ if $z \neq z'$. Then

$$E\{X_{z'} | X_z\} = X_z \text{ if and only if } z \prec z'. \tag{3}$$

Proof. Note that

$$E\{X_{z'} | X_z\} = E\{E\{X_{z'} | \mathcal{G}_z\} | X_z\} = E\{X_{z \wedge z'} | X_z\}, \tag{4}$$

where we have used (F4) to get the second equality. Suppose $E\{X_{z'} | X_z\} = X_z$. Then

$$X_z = E\{X_{z \wedge z'} | X_z\}. \tag{5}$$

On the other hand, $z \wedge z' \prec z$ and X is a martingale, so

$$X_{z \wedge z'} = E\{X_z | X_{z \wedge z'}\}. \tag{6}$$

By p.314 of [2], (5) and (6) together imply that $X_z = X_{z \wedge z'}$. If follows that $z = z \wedge z'$, so that $z \prec z'$. This establishes (3) in one direction. The other direction is clear, so we are done.

Lemma 3. Let $\{X_z, \mathcal{F}_z, z \in S\}$ be a martingale with the property that $P\{X_z = X_{z'}\} < 1$ if $z \neq z'$. Let Γ be a mapping of S onto itself and set $Y_z = X_{\Gamma(z)}$. If $\{Y_z, z \in S\}$ is a martingale with the same property and whose natural σ-fields satisfy (F4), then Γ is order-preserving.

Proof. We apply Lemma 2 to both X and Y :

$$E\{Y_{z'}|Y_z\} = Y_z \text{ if and only if } z \prec z',$$

and

$$E\{X_{\Gamma(z')}|X_{\Gamma(z)}\} = X_{\Gamma(z)} \text{ if and only if } \Gamma(z) \prec \Gamma(z').$$

It follows that

$$z \prec z' \text{ if and only if } \Gamma(z) \prec \Gamma(z').$$

Remarks. 1) The mapping $\Gamma(s,t) = (st,t)$ is not order-preserving, even though $\{W_{\Gamma(z)}, z\in S\}$ is a martingale. Its natural σ-fields do not satisfy (F4), however.

2) Lemmas 1-3 have been stated for the parameter set S for simplicity. They hold, with no change in proof, if the parameter set is some S_c.

Lemma 4. Let $S_\infty = S$ and let $\mathbb{G}_c (o < c \leq \infty)$ be the group of all mappings Γ of S_c onto itself which have the property that $\{W_{\Gamma(z)}, z\in S_c\}$ is a Brownian sheet. Then $\mathbb{G}_c$ is generated by the mappings

$$\Gamma_\lambda : \Gamma_\lambda(s,t) = (\lambda s, \tfrac{t}{\lambda}) \ (\lambda > o) \text{ and } \Gamma_+ : \Gamma_+(s,t) = (t,s)$$

on S_c, and, consequently, each of its elements can be uniquely extended to an element of $\mathbb{G}_\infty$.

Proof. Let $\mathbb{G}$ be the group generated by the mappings

Γ_λ and Γ_+ on S_c. W is a Gaussian process with covariance function $\gamma(s,t;u,v) = (s\wedge u)(t\wedge v)$. A mapping Γ is in $\mathbb{G}_c$ if and only if it leaves γ invariant on S_c. Γ_λ and Γ_+ do this, so $\mathbb{G} \subset \mathbb{G}_c$. We must show that $\mathbb{G}_c \subset \mathbb{G}$. If $\Gamma \in \mathbb{G}_c$, Γ preserves order (Lemma 3, remark 2) and is determined by its action on any one of the $H_{c'}(c' \leq c, c' < \infty)$ (Lemma 1). Furthermore, $\Gamma(H_{c'}) = H_{c'}$, since, if $\Gamma(s,t) = (s',t')$,

$$st = \gamma(s,t;s,t) = \gamma(s',t';s',t') = s't'.$$

Suppose for simplicity that $H_1 \subset S_c$. Let $o < a < b$, so that $(a,\frac{1}{a})$ and $(b,\frac{1}{b})$ are distinct points of H_1. Let their images be $(a',\frac{1}{a'})$ and $(b',\frac{1}{b'})$ respectively. Since γ is invariant under Γ,

$$(7) \qquad (a\wedge b)(\tfrac{1}{a}\wedge \tfrac{1}{b}) = (a'\wedge b')(\tfrac{1}{a'}\wedge \tfrac{1}{b'}).$$

There are two cases, according to whether $a' < b'$ or $b' < a'$.

Case 1 : $a' < b'$. In this case, (7) says that $\frac{a}{b} = \frac{a'}{b'}$, so that, if $\lambda = \frac{a'}{a}$, $\Gamma(a,\frac{1}{a}) = \Gamma_\lambda(a,\frac{1}{a})$ and $\Gamma(b,\frac{1}{b}) = \Gamma_\lambda(b,\frac{1}{b})$. It is not hard to verify that if z is a third point of H_1, $\Gamma(z) = \Gamma_\lambda(z)$, so that $\Gamma = \Gamma_\lambda$ on H_1, and hence, since Γ is determined by its action on H_1, on all of S_c.

Case 2 : $b' < a'$. Then (7) implies that $\frac{a}{b} = \frac{b'}{a'}$, so that, if $\lambda = aa'$, $b' = \lambda\frac{1}{b}$. Thus $\Gamma(a,\frac{1}{a}) = \Gamma_\lambda\Gamma_+(a,\frac{1}{a})$ and $\Gamma(b,\frac{1}{b}) = \Gamma_\lambda\Gamma_+(b,\frac{1}{b})$. It then follows as in case 1 that $\Gamma = \Gamma_\lambda\Gamma_+$ on S_c, and hence that $\Gamma \in \mathbb{G}$.

We can now come to the point. Lemmas 1, 3 and 4 show us that we have very few deterministic time-changes at our disposal, so the following proposition comes as no surprise.

Proposition. The martingale M defined in (1) can not be transformed into a Brownian sheet by any time-change depending only on $\langle M \rangle$.

Proof. As remarked before, $\langle M \rangle$ is deterministic, so that we need only consider deterministic time-changes. Thus, suppose there exists a mapping Γ of S onto itself which transforms M into a Brownian sheet. Now M is already a Brownian sheet on S_1, for $\phi \equiv 1$ there. Thus, by Lemma 4, there is a $\Lambda \in \mathfrak{G}_\infty$ for which $\Lambda = \Gamma$ on S_1. Notice that $\Lambda^{-1}\Gamma$ must also transform M into a Brownian sheet. Clearly $P\{M_z = M_{z'}\} < 1$ if $z \neq z'$, so that, by Lemma 3, $\Lambda^{-1}\Gamma$ is order-preserving. But $\Lambda^{-1}\Gamma \equiv I$, the identity, on S_1. By Lemma 1, $\Lambda^{-1}\Gamma \equiv I$, and we are forced to conclude that M itself is already a Brownian sheet. This is a contradiction, and we are done.

References

[1] R. Cairoli and J.B. Walsh. Stochastic integrals in the plane. Acta mathematica, Vol. 134, 1975, p. 111-183.

[2] J.L. Doob. Stochastic processes, John Wiley & Sons, New York, 1953.