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Germ-Field Markov Property for Multiparameter Processes

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0. Introduction: In recent years, interest has grown in the study of Markov property for multiparameter stochastic processes ([12], [8], [9], [3]) motivated by work on the Markov property for the so-called Lévy Brownian motion ([5], [7], [1], [2]). Unfortunately the general theory is not unified in the sense that various definitions are proposed without showing their equivalence. In view of this situation, it seems natural to show equivalences of these various definitions. In [4], F. Knight showed that in one-dimension various other equivalent definitions of "germ-field" Markov Property are possible if such property is presumed to hold on each set of the class of intervals $\{(0,t);\ t \text{ real}\}$. In section 2 we give an extension of the work in [4]. We show that in Gaussian case all definitions of Markov property coincide with the one presented in [3].

We need the following definition and Lemma throughout the paper.

0.1 Definition ([6], p. 30). Let $(\Omega,\mathcal{F},P)$ be a probability space and $\mathcal{A},\mathcal{B},\mathcal{G}$ be sub-σ-fields of $\mathcal{F}$. Then $\mathcal{A}$ and $\mathcal{B}$ are said to be conditionally independent $\mathcal{G}$ if $P(A \cap B|\mathcal{G}) = P(A|\mathcal{G})P(B|\mathcal{G})$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

0.2 Lemma ([4]). Let $\mathcal{A}$ and $\mathcal{B}$ be sub-σ-fields of $\mathcal{F}$ conditionally independent given $\mathcal{G}$.

(a) If $\tilde{\mathcal{G}}$ is a sub-σ-field satisfying $\mathcal{G} \subseteq \tilde{\mathcal{G}} \subseteq \mathcal{G} \vee \mathcal{B}$ then $\mathcal{A}$ and $\mathcal{B}$ are conditionally independent given $\tilde{\mathcal{G}}$.

(b) $\mathcal{G}' \subseteq (\mathcal{G} \vee \mathcal{B})$ then $\mathcal{A}$ and $\mathcal{G}'$ are conditionally independent given $\mathcal{G}$.

1. Markov Property on an open set: Let $(\Omega,\mathcal{F},P)$ be a probability space and T be an open subset of R^n. Let $\{X_t,\ t \in T\}$ be a family of real (or complex)-valued random variables on $(\Omega,\mathcal{F},P)$. We associate the following σ-algebras with $\{X_t,\ t \in T\}$;

$\mathcal{F}(X:A) = \sigma\{X_t,\ t \in A\}$[1)] for all $A \subseteq T$;

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1) $\sigma\{\ \}$ denotes σ-algebra generated by $\{\ \}$.

$\Sigma_X(A) = F(X:A)$ if A is open subset of T;

$\Sigma_X(A) = \cap\Sigma_X(0)$ if A is a closed subset of A.

Here intersection is over all open subsets $0 \subseteq T$ containing A.

1.1 Definition (Markov Property). Let $\{X_t, t \in T\}$ be a stochastic process defined on (Ω, F, P). We say that it has Markov property on a subset A of T if $\Sigma_X(\bar{A})$ and $\Sigma_X(\overline{T\setminus A})$ are conditionally independent given $\Sigma_X(\partial A)$, where ∂A is (topological) boundary of A.

1.2 Theorem: Let $\{X_t, t \in T\}$ be a stochastic process and D is an open subset of T. Then the following are equivalent:

(a) $\{X_t, t \in T\}$ has Markov property on D;

(b) $F(X:\bar{D})$ and $F(X:D^c)$ are conditionally independent given $\Sigma_X(\partial D)$

(c) $F(X:D)$ and $\Sigma_X(D^c)$ are conditionally independent given $\Sigma_X(\partial D)$.

Proof: (a) implies (b) and (c). We observe that (b) is equivalent to

(1.3) $\begin{cases} F(X:\bar{D}) \text{ and } F(X:D^c) \text{ are conditionally independent given} \\ F(X:0) \text{ for all open sets } 0 \text{ containing } \partial D. \end{cases}$

By Martingale convergence theorem we get (1.3) implies (b). To see the converse implication we use Lemma 0.2(a) with $G = \Sigma_X(\partial D)$, $\tilde{G} = F(X:0 \cap \bar{D})$, $B = F(X:\bar{D})$ and $A = F(X:D^c)$ to get $F(X:\bar{D})$ and $F(X:D^c)$ are conditionally independent given $F(X:0 \cap \bar{D})$. Now choose $G = F(X: 0 \cap \bar{D})$, $\tilde{G} = F(X:0)$ and $B = F(X:D^c)$ in Lemma 0.2(a) to get (b). Similar arguments show that (a) is equivalent to

(1.4) $\begin{cases} \Sigma_X(\bar{D}) \text{ and } \Sigma_X(D^c) \text{ are conditionally independent given } F(X:0) \\ \text{for all open } 0 \text{ containing } \partial D \end{cases}$

and (c) is equivalent to

(1.5) $\begin{cases} F(X:D) \text{ and } \Sigma_X(D^c) \text{ are conditionally independent given } F(X:0) \\ \text{for all open } 0 \text{ containing } \partial D. \end{cases}$

Now (1.3), by Lemma 0.2(b) with $G = F(X:0)$, $G' = \Sigma_X(\overline{D})$, $B = F(X:\overline{D})$ and $A = F(X:D^c)$, we get $\Sigma_X(\overline{D})$ and $F(X:D^c)$ are conditionally independent given $F(X:0)$ for each 0 containing ∂D. Another use of Lemma 0.2(b) with $G' = \Sigma_X(D^c)$, $B = F(X:D^c)$, $G = F(X:0)$ and $A = \Sigma_X(\overline{D})$ gives (1.4). To show (1.5) implies (1.4) we use $G' = \Sigma_X(\overline{D})$, $B = F(X:D)$, $G = F(X:0)$ and $A = \Sigma_X(D^c)$ in Lemma 0.2(b).

1.6 Remark: Condition (b) was used by Pitt [12] in his definition of Markov property and condition (c) was used by Nelson [9].

2. Markov property on relatively compact open sets: We associate with a stochastic process $\{X_t, t \in T\}$ on (Ω,F,P) the following family of sub-σ-fields of F. Let O_∂ denote the family of all open subsets of T containing the boundary ∂D of an open subset D of T. $G_1(\partial D) = \bigcap_{0\in O_\partial} F(X:0 \cap D)$, $G_2(\partial D) = \bigcap_{0\in O_\partial} F(X:0 \cap \overline{D}^c)$, $G_3(\partial D) = \bigcap_{0\in O_\partial} F(X:0 \cap \overline{D})$, $G_4(\partial D) = \bigcap_{0\in O_\partial} F(X:0 \cap D^c)$, $G_5(\partial D) = \Sigma_X(\partial D)$. Also we introduce "past" and "future" fields $F_1(D) = F(X:D)$ and $F_i(D) = F(X:\overline{D})$ $(2 \leq i \leq 5)$, $F^2(D) = F(X:\overline{D}^c)$ and $F^i(D) = F(X:D^c)$ $(i = 1,3,4,5)$.

2.1 Definition: We say that $\{X_t, t \in T\}$ has "germ-field" Markov property (i) (for short, GFMP(i)) if $F_i(D)$ and $F^i(D)$ are conditionally independent givwn $G_i(D)$ $(1 \leq i \leq 5)$.

We note that GFMP(5) is equivalent to the Markov property over each D.

2.2 Theorem. Let C denote the family of relatively compact open sets. Then the following are equivalent

(i) $\{X_t, t \in T\}$ has GFMP(i) on each $D \in C$ $(i = 1,2,3,4,5)$.

Proof: (1) implies conditional independence $F(X:D)$ and $F(X:D^c)$ given $F(X:0 \cap D)$ for all open sets 0 containing ∂D by Lemma 0.2(a). Since for all open sets 0 containing ∂D, $F(X:0 \cap D) \subseteq F(X:0 \cap \overline{D}) \subseteq F(X:0) \subseteq \sigma(F(X:0 \cap D) \cup F(X:D^c)$, by Lemma 0.2(a) and Martingale convergence theorem we get $F(X:D)$ and $F(X:D^c)$ are conditionally independent given $G_3(\partial D)$ and $G_5(\partial D)$. Using Lemma 0.2(b) we get that $F(X:\overline{D})$ and $F(X:D^c)$ are conditionally independent, i.e. (1) $\Rightarrow$ (3) or (5). Now (3) implies $F(X:D)$ and $F(X:D^c)$ are conditionally indpeendent given $G_3(0 \cap \overline{D})$ for all open $0 \supseteq \partial D$ and hence by Lemma 0.2(a) and Martingale convergence theorem (5) follows. Similarly, we can prove that (2) $\Rightarrow$ (4) $\Rightarrow$ (5). We now prove (5) $\Rightarrow$ (1). Suppose (5) holds. As in the proof of Theorem (1.2) we observe that $F(X:D)$ and $F(X:D^c)$ are conditionally independent given $F(X:0_\varepsilon)$ for all 0_ε containing ∂D where $0_\varepsilon = \{x:\rho(x,\partial D) \leq \varepsilon\}$ for all $D \in C$ $(\varepsilon > 0)$ where ρ denotes the Euclidean distance. Denote by $D_\varepsilon = D \cap (D^c \cup 0_\varepsilon)^c$. Then D_ε lies in C for $\varepsilon \leq \varepsilon_0$ and hence $F(X:D_\varepsilon)$ and $F(X:D_\varepsilon^c)$ are conditionally independent given $F(X:\tilde{0}_\varepsilon)$ where $\tilde{0}_\varepsilon = \{x:\rho(x,\partial D_\varepsilon) < \varepsilon\}$. Since $F^5(D) \subseteq F^5(D_\varepsilon)$ this gives $F_5(D_\varepsilon)$ and $F^5(D)$ are conditionally independent given $F_X(\tilde{0}_\varepsilon)$ $\varepsilon > 0$. Hence $F_5(D_\varepsilon)$ and $F^5(D)$ are conditionally independent given $F(X:\tilde{0}_\delta)$ $\delta < \varepsilon$ since $F_5(D_\varepsilon) \subseteq F_5(D_\delta)$. But $\bigcap_{\delta<\varepsilon} F(X:\tilde{0}_\delta) = G_1(\partial D)$ giving $F_5(D_\varepsilon)$ and $F^5(D)$ conditionally independent given $G_1(\partial D)$. Therefore $\sigma(\bigcup_\varepsilon F_5(D_\varepsilon))$ and $F^5(D)$ are conditionally independent given $G_1(\partial D)$ giving the result.

3. <u>Markov Property for Gaussian Processes</u>: Let $\{X_t, t \in T\}$ be a Gaussian stochastic process[2)] ([10]) defined on a complete probability space. Throughout this section we assume all σ-fields involved contain all sets of measure zero. We denote by $H(X:0)$ the linear subspace of $L_2(\Omega,F,P)$ generated by $\{X_t, t \in T\}$ for an open subset 0. For a closed subset C of T, $H(X:C) = \cap H(X:0)$ where the intersection is over all open subsets 0 containing C. In [3], Markov property for $\{X_t, t \in T\}$ on D was defined by

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2) See [10] for definition of Gaussian subspace also.

(3.1) $$Q_{H(X:\overline{D})}Q_{H(X:D^c)} = Q_{H(X:\partial D)}$$

where Q_M denotes the orthogonal projection on $H(X:T)$ onto its subspace M.

In view of Lemma 5 ([2], p. 69) we get that the condition (4.1) is equivalent to conditional independence of $\sigma\{H(X:\overline{D})\}$ and $\sigma(H(X:D^c))$ given $\sigma(H(X:\partial D))$. However our Markov property on D is equivalent to

(3.2) $\Sigma_X(\overline{D})$ and $\Sigma_X(D^c)$ are conditionally independent given $\Sigma_X(\partial D)$.

In this case, $\Sigma_X(\overline{D}) = \cap\sigma\{H(X:0)\}$ where intersection is over all open subsets containing $\overline{D}$. Similar expressions are possible for $\Sigma_X(D^c)$ and $\Sigma_X(\partial D)$. Thus (3.1) and (3.2) are equivalent if $\Sigma_X(\overline{D}) = \sigma(H(X:\overline{D}))$, $\Sigma_X(D^c) = \sigma(H(X:D^c))$ and $\Sigma_X(\partial D) = \sigma(H(X:\partial D))$. We achieve this through the following Lemma.

3.3 <u>Lemma</u>: Let (Ω,F,P) be a complete probability space.

(a) If H_1,H_2 are two subspaces of a Gaussian subspace H of $L_2(\Omega,F,P)$ then $\sigma(H_1 \cap H_2) = \sigma(H_1) \cap \sigma(H_2)$.

(b) If $\{H_i,\ i \in I\}$ are Gaussian subspaces of a Gaussian subspace H of $L_2(\Omega,F,P)$ then $\sigma(\cap_i H_i) = \cap_i \sigma(H_i)$.

<u>Proof</u>: (a) Let $Y \in H$ then $E^{\sigma(H_1)\cap\sigma(H_2)} Y = \lim_{n\to\infty} (p_1p_2)^n Y$ where p_i is the projection onto $L_2(\Omega,\sigma(H_i),P)$ $(i = 1,2)$ by alternating projection theorem ([13], p. 56). But for each i, $p_iY = Q_{H_i}Y$ by ([10], p. 24-25). Hence $E^{\sigma(H_1)\cap\sigma(H_2)} Y = \lim_{n\to\infty} (Q_{H_1}\cdot Q_{H_2})^n Y = Q_{H_1\cap H_2} Y \in H_1 \cap H_2$; by alternating projection theorem.

Now it suffices to prove that for each g bounded $\sigma(H)$-measurable $E^{\sigma(H_1)\cap\sigma(H_2)} g$ is measurable $\sigma(H_1 \cap H_2)$. Since g is bounded $g \in L_2(\Omega,\sigma(H),P)$. In view of Wiener's chaos expansion $g = \lim_{L_2(\Omega,\sigma(H),P)}$ of polynomials in elements of H. It therefore suffices to show that

$E^{\sigma(H_1)\cap\sigma(H_2)} Y_1^{\gamma_1} \ldots Y_n^{\gamma_n}$ is $\sigma(H_1 \cap H_2)$ measurable. Let $Y_i = X_i + Z_i$ with $X_i = E^{\sigma(H_1)\cap\sigma(H_2)} Y_i$ and Z_i independent of $\sigma(H_1) \cap \sigma(H_2)$. Then $Y_1^{\gamma_1} \ldots Y_n^{\gamma_n} = \prod_{i=1}^{n} (X_i + Z_i)^{\gamma_i}$, i.e. sum of polynomials in $X_1 \ldots X_n$ with coefficients in $Z_1 \ldots Z_n$. Hence $E^{\sigma(H_1)\cap\sigma(H_2)} Y_1^{\gamma_1} \ldots Y_n^{\gamma_n}$ = polynomial in $X_1 \ldots X_n$. Thus completing the proof of (a).

(b) In view of (a) we can assume I is directed set. Hence for $Y \in H$, $E^{\cap_i \sigma(H_i)} Y = \lim_i E^{\sigma(H_i)} Y = Q_{H_i} Y = \lim_i Q_{\cap_i H_i} Y \in \cap_i H_i$. Repeating argument as in (a) we get $E^{\cap_i \sigma(H_i)} Y_1^{\gamma_1} \ldots Y_n^{\gamma_n}$ is $\cap_i \sigma(H_i)$ measurable.

4.4 Remarks. In [12], equality $G_i(\partial D)$ (Section 2) was assumed for $i = 1,2,5$ for validity of Markov property. In stationary case this condition is always satisfied by simple adaptation of the proof in [11]. In most of the standard examples ([3]) it can be shown that this condition is satisfied. In fact equality of these three fields in necessary and sufficient for $G_5(\partial D)$ being a minimal splitting field in the sense of [12] for $F(X:D)$ and $F(X:\overline{D}^c)$. In a subsequent paper we shall present a class of (not necessarily) stationary processes for which this happens.

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