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A CHARACTERIZATION OF BMO-MARTINGALES

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Let (Ω, \mathcal{F}, P) be a probability space with a non-decreasing right continuous family (\mathcal{F}_t) of sub \mathcal{A} -fields of \mathcal{F} such that \mathcal{F}_0 contains all P -null sets. In this note we deal only with continuous martingales X over (\mathcal{F}_t) such that $X_0 = 0$. A martingale X belongs to the class BMO if $\|X\|_{\text{BMO}}^2 = \sup_t \text{ess. sup}_{\omega} E[\langle X \rangle_{\infty} - \langle X \rangle_t | \mathcal{F}_t] < \infty$. Our aim is to prove the following.

THEOREM. Assume that M is an L^2 -bounded martingale. Then M belongs to the class BMO if and only if $Z_t = \exp(M_t - \frac{1}{2} \langle M \rangle_t)$ satisfies the condition :

$$(A_p) \quad \sup_t \text{ess. sup}_{\omega} Z_t E[(\frac{1}{Z_{\infty}})^{\frac{1}{p-1}} | \mathcal{F}_t]^{p-1} < \infty$$

for some $p > 1$.

The condition (A_p) has already appeared many times in the literature in connection with several different questions (see B.Muckenhoupt [2]).

PROOF. Generally, if $\|X\|_{\text{BMO}} < 1$, then $E[e^{\langle X \rangle_{\infty}}] \leq \frac{1}{1 - \|X\|_{\text{BMO}}^2}$ so that for each stopping time T $E[e^{\langle X \rangle_{\infty} - \langle X \rangle_T} | \mathcal{F}_T] \leq \frac{1}{1 - \|X\|_{\text{BMO}}^2}$. This is the John-Nierenberg type inequality (see R.K.Gettoor and M.J.Sharpe [1]). It is clear that the process Z_t is a continuous local martingale. As M is L^2 -bounded, we have $\langle M \rangle_{\infty} < \infty$ and so $Z_{\infty} > 0$.

Suppose firstly that $\|M\|_{\text{BMO}} < \infty$, and choose $p > 1$ such that $\|\frac{\sqrt{p+1}}{p-1} M\|_{\text{BMO}} < 1$. Then we get

$$\begin{aligned}
& Z_t E \left[\left(\frac{1}{Z_\infty} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_t \right]^{p-1} \\
&= E \left[\exp \left(-\frac{1}{p-1} (M_\infty - M_t) + \frac{1}{2(p-1)} (\langle M \rangle_\infty - \langle M \rangle_t) \right) \middle| \mathcal{F}_t \right]^{p-1} \\
&\leq E \left[\exp \left(-\frac{1}{p-1} (M_\infty - M_t) - \frac{1}{(p-1)^2} (\langle M \rangle_\infty - \langle M \rangle_t) \right) \right. \\
&\quad \times \exp \left(\frac{p+1}{2(p-1)^2} (\langle M \rangle_\infty - \langle M \rangle_t) \right) \middle| \mathcal{F}_t \left. \right]^{p-1} \\
&\leq E \left[\exp \left(-\frac{2}{p-1} (M_\infty - M_t) - \frac{2}{(p-1)^2} (\langle M \rangle_\infty - \langle M \rangle_t) \right) \middle| \mathcal{F}_t \right]^{\frac{p-1}{2}} \\
&\quad \times E \left[\exp \left(\frac{p+1}{(p-1)^2} (\langle M \rangle_\infty - \langle M \rangle_t) \right) \middle| \mathcal{F}_t \right]^{\frac{p-1}{2}} \\
&\leq \frac{1}{\left\{ 1 - \frac{p+1}{(p-1)^2} \|M\|_{\text{BMO}}^2 \right\}^{\frac{p-1}{2}}}
\end{aligned}$$

by using the John-Nirenberg type inequality. Thus Z_t satisfies (A_p) .

On the other hand, for every $p > 1$, by the Jensen inequality

$$\begin{aligned}
& Z_t E \left[\left(\frac{1}{Z_\infty} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_t \right]^{p-1} \\
&= \left\{ \exp \left(\frac{1}{p-1} M_t - \frac{1}{2(p-1)} \langle M \rangle_t \right) E \left[\exp \left(-\frac{1}{p-1} M_\infty + \frac{1}{2(p-1)} \langle M \rangle_\infty \right) \middle| \mathcal{F}_t \right] \right\}^{p-1} \\
&\geq \left\{ \exp \left(\frac{1}{p-1} M_t - \frac{1}{2(p-1)} \langle M \rangle_t - \frac{1}{p-1} M_t + \frac{1}{2(p-1)} E[\langle M \rangle_\infty | \mathcal{F}_t] \right) \right\}^{p-1} \\
&= \exp \left(\frac{1}{2} E[\langle M \rangle_\infty - \langle M \rangle_t | \mathcal{F}_t] \right),
\end{aligned}$$

from which we get $\|M\|_{\text{BMO}} < \infty$ if Z_t satisfies the condition (A_p) for some p .

This completes the proof.

In a forthcoming paper [3] we shall study another properties on the condition (A_p) .

References

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