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NOTE ON PASTING OF TWO MARKOV PROCESSES

By Masao NAGASAWA

The author remarked (e.g. in his lectures at UCSD in 1967/68 and at Erlangen in 1973/74) that Courrege-Priouret's theorem [1] of pasting two continuous strong Markov processes is a simple corollary of the theorem of piecing out (or revival) of a Markov process which was given by the author (cf. [3]). Since this remark was published nowhere, it would be of use to explain the procedure, though Meyer gave similar one recently in [4], which contains other interesting applications.

Let U and V be open subsets of a Polish space and adjoin ∂ as an extra point to U and V as usual. Suppose we have continuous strong Markov processes $(W^U, \underline{E}_t^U, x_t^U, P_x^U)$ on U and $(W^V, \underline{E}_t^V, x_t^V, P_x^V)$ on V which are killed at each boundary. Suppose $U \cap V \neq \emptyset$ and moreover

- (1) x_t^U and x_t^V coincide on $U \cap V$ (the consistency condition),

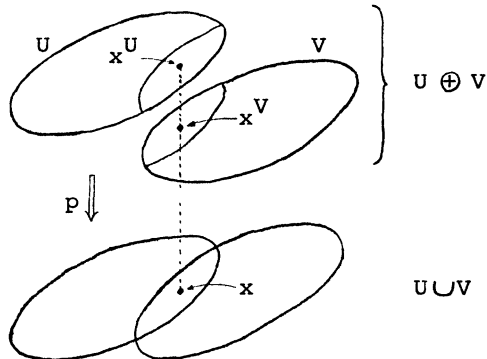
that is, if we kill the two processes at the first leaving time T from $U \cap V$, we get the same Markov process on $U \cap V$. The problem is to construct a continuous strong Markov process x_t on $U \cup V$ pasting the two processes, in other words, to obtain a process x_t which moves on x_t^U in U and on x_t^V in V . This pasting technique is an important tool, for example, when we construct diffusions on a Riemannian Manifold.

(i) Let $U \oplus V$ be the direct sum of U and V . When $x \in U \cap V$, denote x^U the corresponding point in U and x^V in V as a point in $U \oplus V$ (as is clear from Figure 1). The natural identification mapping p from $U \oplus V$ onto $U \cup V$ is defined by

$$p(x^U) = p(x^V) = x,$$

and we call it the pasting operator (not projection!).

Fig.1.



First we construct a process on $U \oplus V$ (treating U and V as separated sheets) and get the one on $U \cup V$ (pasting U and V). Let us define a continuous strong Markov process $(W^\circ, \underline{B}_t^\circ, x_t^\circ, P_x^\circ)$ on $U \oplus V$, putting

$$W^\circ = W^U \oplus W^V, \quad \underline{B}_t^\circ = \underline{B}_t^U \oplus \underline{B}_t^V,$$

$$P_x^\circ = \begin{cases} P_x^U, & x \in U, \\ P_x^V, & x \in V, \end{cases} \quad x_t^\circ = \begin{cases} x_t^U, & \text{on } W^U, \\ x_t^V, & \text{on } W^V. \end{cases}$$

Next, let us define a kernel on $(U \oplus V) \cup \{\partial\}$ by

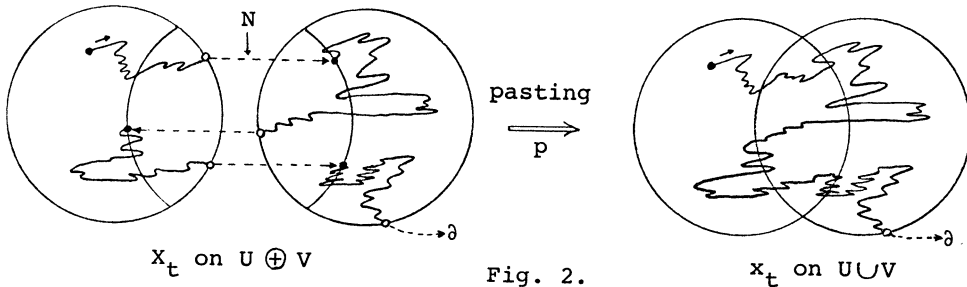
$$n(x, \cdot) = \begin{cases} \epsilon_{x^V} & \text{if } x = x^U \in (\partial U) \cap V, \\ \epsilon_{x^U} & \text{if } x = x^V \in (\partial V) \cap U, \\ \epsilon_\partial & \text{if otherwise,} \end{cases}$$

where ϵ_a denote the point mass at a , and put

$$(2) \quad N(w^\circ, \cdot) = \begin{cases} n(x_{\zeta^-}^U(w^U), \cdot) & \text{if } w^\circ = w^U \in W^U, \\ n(x_{\zeta^-}^V(w^V), \cdot) & \text{if } w^\circ = w^V \in W^V. \end{cases}$$

Then N is a "revival kernel" (it was called "instantaneous distribution" in [3]). Now apply the piecing out (or revival) theorem to x_t° and N , obtaining a right continuous strong Markov process $(\bar{W}, F_t, X_t, \bar{P}_x)$ on $U \oplus V$ (cf. figure 2). To obtain a continuous strong Markov process on $U \cup V$, that we are looking for, what we need to do is to apply the pasting operator p :

$$(3) \quad x_t = p(X_t), \quad P_x = \bar{P}_p^{-1}(x), \quad x \in U \cup V.$$



Since the pasting operator is a transformation of the state space (cf. [2], p.325), to be well-defined it is enough to prove

$$(4) \quad \bar{E}_{x^U}[f \circ p(X_t)] = \bar{E}_{x^V}[f \circ p(X_t)], \quad x \in U \cap V,$$

for f on $U \cup V$. The right hand side of (4) is equal to

$$(5) \quad \bar{E}_{x^U}[f \circ p(X_t); t < T] + \bar{E}_{x^U}[\bar{E}_{X_T}[f \circ p(X_{t-s})] |_{s=T}; t \geq T],$$

where T is the first leaving time from $U \cap V$. But from the way of construction of X_t , the first term of (5) is equal to

$$E_x^U[f(x_t^U); t < T],$$

which is equal to

$$E_x^V[f(x_t^V); t < T],$$

by the consistency condition (1), therefore is equal to

$$\bar{E}_x^V[f \circ p(X_t); t < T].$$

The second term of (5) is equal to

$$\bar{E}_x^V[\bar{E}_{X_T} [f \circ p(X_{t-s})] |_{s=T; t \geq T}],$$

because

$$(6) \quad \bar{P}_x^U[X_T \in \cdot] = \bar{P}_x^V[X_T \in \cdot].$$

Thus we have (4). It is clear that $x_t = p(X_t)$ is continuous.

(ii) Now, let x_t^U and x_t^V be strong Markov process on U and V , resp. We don't assume any continuity of paths, but assume $\exists x_{\zeta-}^U \in \bar{U}$ and $\exists x_{\zeta-}^V \in \bar{V}$, and the consistency condition (1). Let $n(x, dy)$ be a probability kernel on $(\bar{U} \oplus \bar{V}) \cup \{\emptyset\}$ satisfying

$$(7) \quad \begin{aligned} &\text{the support of } n(x^U, \cdot) \text{ is contained in } (V \setminus U) \cup \{\emptyset\}, \\ &\text{the support of } n(x^V, \cdot) \text{ is contained in } (U \setminus V) \cup \{\emptyset\}. \end{aligned}$$

Define a revival kernel $N(w^\circ, \cdot)$ by (2) using this kernel $n(x, dy)$. To keep consistency for jumps of paths of the process which will be constructed, we assume the second consistency condition: When $x \in U \cap V$,

$$(8) \quad \begin{aligned} P_x^U[n(x_{\zeta-}^U, \cdot); T = \zeta] &= P_x^V[x_T^V \in \cdot; T < \zeta], \\ P_x^V[n(x_{\zeta-}^V, \cdot); T = \zeta] &= P_x^U[x_T^U \in \cdot; T < \zeta]. \end{aligned}$$

Let T_U and T_V be the first leaving times from U and V , resp., then

$x_t = p(X_t)$ is a strong Markov process on $U \cup V$ satisfying that x_t killed at T_U is equal to x_t^U , x_t killed at T_V is equal to x_t^V , and

$$(9) \quad \begin{aligned} P_z[x_{T_U} \in \cdot | x_{T_U-} = x] &= n(x, \cdot), \quad \text{if } x \in U, \\ P_z[x_{T_V} \in \cdot | x_{T_V-} = x] &= n(x, \cdot), \quad \text{if } x \in V. \end{aligned}$$

If x_t^U and x_t^V are right continuous, x_t on $U \cup V$ is also right continuous. The equality (4) is proved in the same way. The equality (6) in this

case is verified as follows: Putting $a = x^U$ and $b = x^V$, $x \in U \cap V$,

$$\begin{aligned} \bar{P}_a[X_T \in \cdot] &= \bar{P}_a[X_T \in \cdot; T=r_1] + \bar{P}_a[X_T \in \cdot; T < r_1] \\ &= P_x^U[n(x_{\zeta-}^U, \cdot); T=\zeta] + P_x^U[x_T^U \in \cdot; T < \zeta] \\ &= P_x^V[x_T^V \in \cdot; T < \zeta] + P_x^V[n(x_{\zeta-}^V, \cdot); T=\zeta] \\ &= \bar{P}_b[X_T \in \cdot; T < r_1] + \bar{P}_b[X_T \in \cdot; T=r_1] \\ &= \bar{P}_b[X_T \in \cdot], \end{aligned}$$

where r_1 is the first jumping (or revival) time of X_t . The second and the fourth equalities are by the revival theorem, and the third is by the second consistency condition (8). (9) is clear because

$$P_z[x_{r_1}^U \in \cdot | x_{T_U}^- = x] = \bar{P}_z[X_{r_1} \in \cdot | X_{r_1}^- = x] = n(x, \cdot).$$

(iii) Appendix.

DEFINITION. A probability kernel $N(w, dy)$ on $W \times \underline{B}(S)$ is revival kernel (or instantaneous distribution) if it satisfies

- (i) $N(\theta_t w, \cdot) = N(w, \cdot)$, if $t < \zeta(w)$, and
- (ii) $N(w, \cdot) = \varepsilon_\emptyset$, if $\zeta(w) = 0$.

THEOREM (of revival or piecing out). Let $(W, \underline{B}_t, x_t, P_x)$ $x \in S$ be a Markov process and N be a revival kernel. Then there exists a Markov process $(\bar{W}, \underline{F}_t, X_t, \bar{P}_x)$ on S satisfying

- (i) X_t , $t < r_1$, is equivalent to x_t ,
- (ii) $\bar{P}_x[X_{r_1} \in \cdot | \underline{F}_{r_1}^-] = N(w_1, \cdot)$,

where $\bar{W} = W^{(1,2,3,\dots)} \ni \bar{w} = (w_1, w_2, \dots)$ and $r_1(\bar{w}) = \zeta(w_1)$. If x_t is strong Markov, so is X_t , and if x_t is right continuous, so is X_t . If x_t is continuous and if $N(w, \cdot) = \varepsilon_{x_{\zeta-}(w)}$, then X_t is continuous.

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