WILHELM VON WALDENFELS

Interval partitions and pair interactions

Séminaire de probabilités (Strasbourg), tome 9 (1975), p. 565-588
<http://www.numdam.org/item?id=SPS_1975__9__565_0>

INTERVAL PARTITIONS AND PAIR INTERACTIONS

Wilhelm von Waldenfels

Introduction.

The paper has two different objects, only the second one appears in the title. § 1, 2, 3 treat the algebraic aspect of cluster expansions. A cluster expansion is related to a hierarchy of functions, i.e., a sequence f_0, $f_1(x_1)$, $f_2(x_1, x_2), \dots$. These functions may be connected in different ways. The most known example is

$$f_n(x_1, \dots, x_n) = f_k(x_1, \dots, x_k)\, f_{n-k}(x_{k+1}, \dots, x_n)$$

if dist $\left(\{x_1, \dots, x_k\}, \{x_{k+1}, \dots, x_n\}\right)$ is large. Then new functions $f[x_1, \dots, x_n]$ which are polynomials in the old ones may be formed, with the property $f[x_1, \dots, x_n] = 0$ if $\{x_1, \dots, x_n\}$ may be divided into two subsets whose distance is large. This kind of condition determines $f[x_1, \dots, x_n]$ in a unique way. This fact is of order theoretic character and is explained in § 1. In § 2 three different orderings are presented which are related to three different kinds of hierarchies in § 3.

In § 4 the results of the preceding sections are applied to the problem of cluster expansion of pair interaction. We deal with a variant of Ursell-Maier's treatment of pair interaction in statistical mechanics. The essential difference is that interval partition instead of general partitions and linear graphs instead of general graphs come in. The result is somewhat weaker than the results in statistical mechanics. I have to assume that the function $|\exp i\phi(t) - 1|$ is exponentially bounded.

§ 1. General algebraic considerations

We assume that P is a partially ordered set with the property that for any given $x \in P$ there exist only finitely many y with $y \leq x$. Call Δ the lower triangel of $P \times P$ i.e., the set $\{(y, x) : y, x \in P, y \leq x\}$. A convolution can be defined for real functions on Δ by

$$\varphi * \psi(y, x) = \sum_{y \leq z \leq x} \varphi(y, z)\, \psi(z, x).$$

The algebra of all real functions on Δ with convolution as product is called the incidence algebra of P. Distinguished elements are the unity $\delta(y, x)$ (Kronecker's δ) and the ζ -function $\zeta(y, x) = 1$ and its inverse the Moebius function μ. The existence of μ can be easily proved by using induction and one of the defining relations

$$\sum_{y \le z \le x} \mu(y,z) = \delta(y,x)$$

$$\sum_{y \le z \le x} \mu(z,x) = \delta(y,x).$$

Hence $\mu(x,x) = 1$ and $\mu(y,x) = -\sum_{y<z\le x}\mu(z,x)$ for $y < x$.

If α is a function $P \to V$ where V is some real vector space, then the elements φ of the incidence algebra may be applied to α by the formula

$$\alpha * \varphi(x) = \sum_{y \le x} \alpha(y)\,\varphi(y,x)$$

So

$$\beta = \alpha * \zeta \qquad \text{if} \quad \beta(x) = \sum_{y\le x} \alpha(y)$$

and hence

$$\alpha = \beta * \mu$$

or

$$\alpha(x) = \sum_{y \le x} \beta(y)\,\mu(y,x).$$

Define the vector space $V(P)$ of all formal real linear combinations of elements of P. For any $x \in P$ define in an inductive way the elements $[x] \in V(P)$ by

$$x = \sum_{y\le x} [y]$$

Hence $[x] = x$ if x is minimal and $[x] = x - \sum_{y<x} [y]$ if x is not minimal.

The considerations above show that

$$[x] = \sum_{y \le x} \mu(y,x)\, y$$

This is another way of defining μ.

Assume now that two elements x, y in P have a lowest upper bound $x \wedge y$ and extend this operation to $V(P)$.

<u>Theorem 1:</u> The element $[x] \in V(P)$ satisfies the relation

$$[x] \wedge y = \begin{cases} [x] & \text{if} \quad x \wedge y = x \\ 0 & \text{if} \quad x \wedge y \neq x \end{cases}$$

and this relation determines $[x]$ up to a factor.

<u>Proof:</u> Consider the subset $P_x = \{z \in P : z \le x\}$. By the function $z \mapsto z \wedge y$ the set P_x is mapped onto $Q = \{u \in P_x : u \wedge y = u\}$.

Hence

$$[x] \wedge y = \sum_{z \le x} \mu(z,x)\,(z \wedge y)$$

$$= \sum_{u \in Q} \Big[\sum_{\{z : z \le x,\, z \wedge y = u\}} \mu(z,x) \Big] u.$$

If u equals $x \wedge y$ the maximal element of Q then $\{z : z \leq x,\ z \wedge y = x \wedge y\} = \{z : x \wedge y \leq z \leq x\}$ and the coefficient of $x \wedge y$ vanishes if $x \wedge y \neq x$. Assume now that the coefficient of $v \in Q,\ v > u$ vanishes. Then the sum of $\mu(z,x)$ over the set $\{z : z \leq x,\ z \wedge y = u\}$ equals the sum over the set $\{z : z \leq x,\ z \wedge y \geq u\} = \{z : u \leq z \leq x\}$ and this sum vanishes by the definition of μ. So $[x] \wedge y = 0$ if $x \wedge y \neq x$. This part of the proof is nothing else than a special case of |1|, theorem 2.

If $x \wedge y = x$ then for $z \leq x$ one has $z \wedge y = (z \wedge x) \wedge y = z$ and hence $[x] \wedge y = [x]$.

We have still to prove the uniqueness. The relation $[x] \wedge x = [x]$ shows that $[x]$ is a linear combination of elements $z \leq x$, hence $[x] = \sum_{z \leq x} \alpha(z)\, x$. If $y < x$ then the coefficient of y in $[x] \wedge y$ is equal to $\sum_{y \leq z \leq x} \alpha(z) = 0$. Hence $\alpha(z) = \mu(z,x)\, \alpha(x)$ where $\alpha(x)$ may be chosen arbitrarily.

§ 2. Applications to three different partially ordered sets

First example: Subsets. Consider the partially ordered set of all finite subsets of $\mathbb{N}$. It is well-known (cf. |1|) that the Möbius function $\mu(S', S) = (-1)^{|S-S'|}$ for $S' \subset S$.

So
$$[\emptyset] = (\emptyset)$$
$$[1] = (1) - (\emptyset)$$
$$[12] = (12) - (1) - (2) + (\emptyset)$$

and generally
$$[S] = \sum_{S' \subset S} (-1)^{|S-S'|}\, S'$$

It is easily checked that
$$[S] \wedge S' = \sigma$$

if $S \cap S' \neq \emptyset$. This establishes by § 1, theorem 1 again a proof for the values of the Möbius function.

Second example: Partitions. The partially ordered set consists of all finite subsets $S \subset \mathbb{N}$ and of all their partitions $\pi = \{S_1, \dots, S_k\} = S_1 \cdots S_k$ for short. Two partitions $\pi = S_1 \cdots S_k$, $\pi' = S_1' \cdots S_\ell'$ are in the relation $\pi' \leq \pi$ if $S_1 \cup \dots \cup S_k = S_1' \cup \dots \cup S_\ell'$ and if every set S_j, $j = 1, \dots, k$ is a union of sets S_i'.

We introduce a product in $V(P)$ by the definition

$$\pi = S_1 \cdots S_k,\ \pi' = S_1' \cdots S_\ell' \longmapsto \pi\pi' = \begin{cases} \sigma & \text{if } (S_1 \cup \dots \cup S_k) \cap (S_1' \cup \dots \cup S_\ell') \neq \emptyset \\ S_1 \cdots S_k S_1' \cdots S_\ell' & \text{otherwise.} \end{cases}$$

With this product the set of all final linear combinations of elements of P gets an associative and commutative algebra called A(P).

If $\pi = S_1 \cdots S_k$ then the partially ordered set $\{\pi' : \pi' \leq \pi\}$ is the product of the partially ordered sets $\{\pi_j : \pi_j \leq S_j\}$ for $j = 1, \dots, k$. By proposition 5, p. 345 of |1| one gets immediately

$$[S_1 \cdots S_k] = [S_1] \cdots [S_k].$$

and by proposition 3, p. 359 of |1|

$$[S] = S - \sum_{S_1 S_2 \leq S} S_1 S_2 + 2! \sum_{S_1 S_2 S_3 \leq S} S_1 S_2 S_3 - 3! \sum_{S_1 S_2 S_3 S_4 \leq S} S_1 S_2 S_3 S_4 + \cdots$$

The last equation can be easily derived by some algebraic considerations. If $\mathcal{A}$ is an associative algebra and f and g are functions of the set of all finite subsets $S \subset \mathbb{N}$ into $\mathcal{A}$, then a convolution can be defined by

$$f \star g\,(S) = \sum_{S_1 + S_2 = S} f(S_1)\, g(S_2)$$

where $S_1 + S_2$ means disjoint union. The identity with respect to convolution is

$$\delta_\emptyset \; : \; \delta_\emptyset(S) = \begin{cases} 1 & \text{if } S = \emptyset \\ 0 & \text{if } S \neq \emptyset. \end{cases}$$

Take now for $\mathcal{A}$ the algebra $A(P)$ and define

$$\iota \; : \; \iota(S) = \begin{cases} S & \text{for } S \neq \emptyset \\ 0 & \text{for } S = o \end{cases}$$

and

$$\gamma \; : \; \gamma(S) = \begin{cases} [S] & \text{for } S \neq \emptyset \\ 0 & \text{for } S = \emptyset \end{cases}$$

Then by definition of $[\pi]$ one has

$$\begin{aligned} S &= \sum_{\pi \leq S} [\pi] \\ &= \sum_{k \geq 1} \sum_{S_1 \cdots S_k \leq S} [S_1] \cdots [S_k] \\ &= \sum_{k \geq 1} \frac{1}{k!} \sum_{S_1 + \cdots + S_k = S,\; S_j \neq \emptyset} [S_1] \cdots [S_k] \end{aligned}$$

So

$$\delta_\emptyset + \iota = \exp_\star \gamma$$

and

$$\gamma = \log_\star(\delta_\emptyset + \iota) = \iota - \tfrac{1}{2}\,\iota \star \iota + \tfrac{1}{3}\,\iota \star \iota \star \iota - \cdots$$

Taking on both sides the values on the set S , one gets $[S]$ and hence μ,

$$\mu(S_1 \cdots S_k ; S) = (-1)^{k-1}(k-1)! \quad \text{if } S_1 \cdots S_k \leq S .$$

Third example: Interval partitions. Here P is the set of all finite intervals $I \subset N$ and all partitions π of intervals into subintervals. If $I = \{k, k+1, \dots, \ell\}$ then a partition $\pi = I_1 \cdots I_m$ has the elements $I_j = \{k_{j-1}, \dots, k_j - 1\}$ with

$$k_0 = k < k_1 < \cdots < k_{m-1} < k_m = \ell + 1.$$

Remark that the order of the I_j in π is fixed by the order of N. If $\pi' = I_1' \cdots I_\ell'$ then $\pi \leq \pi'$ if $I_1 \cup \cdots \cup I_m = I_1' \cup \cdots \cup I_\ell'$ and any subinterval I_j, $j = 1, \dots, m$ is a union of subintervals of π'. We define the algebra $A(P)$ of formal linear combinations of elements of P by the product

$$\pi = I_1 \cdots I_m,\ \pi' = I_1' \cdots I_\ell' \mapsto \pi\pi' = \begin{cases} I_1 \cdots I_m I_1' \cdots I_\ell' \\ \text{if } I_1' \text{ follows directly to } I_m \\ \sigma \qquad \text{otherwise.} \end{cases}$$

With this definition $A(P)$ gets an associative non-commutative algebra.

As in the second example

$$[I_1 \cdots I_m] = [I_1] \cdots [I_m]$$

and one has

$$[I] = I - \sum_{I_1 I_2 \leq I} I_1 I_2 + \sum_{I_1 I_2 I_3 \leq I} I_1 I_2 I_3 - \sum_{I_1 I_2 I_3 I_4 \leq I} I_1 I_2 I_3 I_4 + \cdots$$

For the ordered set of interval partitions of $\{1, 2, \dots, n\}$ is isomorphic to the ordered set of subsets of $\{1, 2, \dots, n-1\}$ with respect to inclusion.

We prove again this last formula by algebraic methods which show the connection between interval partitions and the inverse function.

If $I_1, \dots, I_m$ are finite intervals one following the other we write $I_1 \cup \cdots \cup I_m = I_1 \circ I_2 \circ \cdots \circ I_m$. The empty interval $\emptyset$ has the property $I \circ \emptyset = \emptyset \circ I = I$. Define a convolution on the set of all functions of the finite subintervals of N to an associative algebra $\mathcal{A}$ by

$$f * g\ (I) = \sum_{I_1 \circ I_2 = I} f(I_1)\, g(I_2).$$

Take for $\mathcal{A}$ the algebra $A(P)$ and consider

$$\iota : \iota(I) = I,\ I \neq \emptyset;\ \iota(\emptyset) = \sigma$$

$$\gamma : \gamma(I) = [I],\ I \neq \emptyset;\ \gamma(\emptyset) = \sigma$$

Then by the definition of |1|

$$I = \sum_{\pi \leq I} [\pi]$$
$$= \sum_{k \geq 1} \sum_{I_1 \circ \cdots \circ I_k = I;\ I_j \neq \emptyset} [I_1] \cdots [I_k]$$

Hence

$$\delta_\emptyset + \iota = \delta_\emptyset + \gamma + \gamma * \gamma + \gamma * \gamma * \gamma + \cdots$$
$$= (\delta_\emptyset - \gamma)^{*(-1)} = \frac{\delta_\emptyset}{\delta_\emptyset - \gamma}$$

and

$$\gamma = \frac{\iota}{\delta_\emptyset + \iota} = \iota - \iota * \iota + \iota * \iota * \iota - \ldots$$

This yields immediately the expression for $[I]$.

§ 3. Hierarchies

We are going to apply the results of § 1 and § 2 to three examples of hierarchies correspondeing to the three examples of § 2. A hierarchy of functions is given by two sets X and Y (where Y carries usually an algebraic structure, say $Y = \mathbb{R}$ or Y is a real associative algebra) and a sequence of functions

$$f_0 = \text{const} \in Y$$
$$f_n : X^n \to Y$$

Assume furthermore a sequence of variables $x_1, x_2, \ldots$ taking values in X and define $f(\emptyset) = f_0$ and for a finite subsequence $(i_1, \ldots, i_n)$ the function $f(i_1, \ldots, i_n)$ by $f(i_1, \ldots, i_n)(x) = f_n(x_{i_1}, \ldots, x_{i_n})$ where $x = (x_1, x_2, \ldots)$ So $f(i_1, \ldots, i_n)$ is a function on $X^{\mathbb{N}}$ depending only on $x_{i_1}, \ldots, x_{i_n}$ and f appears as an application of the set of finite subsequences of $\mathbb{N}$ into the set of all functions $X^{\mathbb{N}} \to Y$. Between the elements f_n of a hierarchy different relations may hold.

<u>First example:</u> Assume the functions f_n to be symmetric and to have the property: there exists $K \subset X$ such that $f_n(x_1, \ldots, x_n) = f_{n-1}(x_1, \ldots, x_{n-1})$ if $x_n \notin K$. If $S = \{i_1, \ldots, i_n\} \subset \mathbb{N}$ is a finite set define

$$f(S) = f(i_1, \ldots, i_n).$$

Assume that $\mathcal{P}$ is the partially ordered set of all finite subsets of $\mathbb{N}$. Then f can be considered as a function of $\mathcal{P}$ to the set of all functions $X^{\mathbb{N}} \to Y$. Assume Y to be a real vector space and extend f in a linear way to $V(\mathcal{P})$.

Then

$$f[S] = \sum_{S' \subset S} (-1)^{|S-S'|} f(S')$$

has the property

$$f[S](x) = \sigma$$

if there exists $i \in S$ such that $x_i \notin K$. Call $T = \{ i \in S : x_i \in K \}$. Then for any $S' \subset S$ one has $f(S')(x) = f(S' \cap T)(x)$ and thus

$$f([S])(x) = f([S] \cap T)(x)$$

As $S \cap T \neq S$ one has by theorem 1 of § 1 that $[S] \cap T = \sigma$ and so $f[S](x) = 0$

These developments may explain why the functions $f[S]$ play such an important role in the Taylor expression of a Poisson measure |5|. There one was looking for functions with the related property $f_n(x_1, \dots, x_n) = f_{n-1}(x_1, \dots, x_{n-1})$ if $x_n \to \infty$. So the functions $f[S]$ have the greatest chance of all linear combinations of f to be integrable as $f[S](x) \to 0$ if one of the x_i, $i \in S$ goes to infinity.

<u>Second example.</u> We are commenting the algebraic method of deriving cluster expansions in statistical mechanics as stated in Ruelle's book |2|. Assume again the functions f_n to be symmetric. So f can be considered as a function of finite subsets in $\mathbb{N}$. Assume that $Y = \mathbb{R}$ that X is a metric space and that there exists a constant $\rho \geqslant 0$ such that

$$f(S)(x) = f(S_1)(x)\, f(S_2)(x)$$

if $\{S_1, S_2\} = S_1 S_2$ is a partition of S such that $\operatorname{dist}(x_{S_1}, x_{S_2}) > \rho$ with $x_S = \{ x_i : i \in S \}$.

Let $\mathcal{P}$ be the set of all finite subsets $S \subset \mathbb{N}$ and all their partitions $\pi = S_1 \cdots S_k$. Extend f from all finite subsets of $\mathbb{N}$ to $A(\mathcal{P})$ as an algebra homomorphism by

$$f(\pi) = f(S_1) \cdots f(S_k) .$$

Then

$$f[S] = f(S) - 1! \sum_{S_1 S_2 \leqslant S} f(S_1) f(S_2) + 2! \sum_{S_1 S_2 S_3 \leqslant S} f(S_1) f(S_2) f(S_3) - + \cdots$$

has the property

$$f[S](x) = 0$$

if there exists a partition $\pi = S_1 S_2$ of S such that dist $(x_{S_1}, x_{S_2}) > \rho$. The proof is based again on the theorem 1 of § 1.

Assume a real symmetric function $\Phi : X \times X \to \mathbb{R}$ and define

$$f(\emptyset) = 1$$

$$f_n(x_1, \ldots, x_n) = \exp - \beta \sum_{1 \leq i < j \leq n} \Phi(x_i, x_j) .$$

The famous theorem of Ursell and Maier |2| states

$$f[1,2,\ldots,n](x) = \sum_{\gamma} \prod_{(i,j) \in \gamma} \left[\left(\exp - \beta \Phi(x_i, x_j)\right) - 1 \right]$$

where the sum runs over all connected graphs γ with vertices 1, 2, ..., n. So $f[1,2,\ldots,n](x)$ vanishes if $\Phi(x,y)$ vanishes for dist $(x,y) > \rho$ and if $\{1,2,\ldots,n\}$ can be split into two subsets S_1, S_2 such that dist $(x_{S_1}, x_{S_2}) \geqslant \rho$. The condition for Φ can be weakened in various ways, $\Phi(x,y) \to 0$ if dist $(x,y) \to \infty$ or $x \mapsto \Phi(x,y)$ is integrable, etc.

<u>Third example.</u> This example is quite similar to the preceding one, the main difference is that $X = \mathbb{R}$ and that the linear ordering of $\mathbb{R}$ is heavily utilized. The use of interval partitions has been used in the theory of pressure broadening of spectral lines |3|, |4|.

Denote the variables $t_1, t_2, \ldots$ instead of $x_1, x_2, \ldots$ and assume that Y is an associative real algebra and that the f_n are symmetric and that

$$f_n(t_1, \ldots t_n) = f_k(t_1, \ldots, t_k)\, f_{n-k}(t_{k+1}, \ldots, t_n)$$

if $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$ and $t_{k+1} - t_k > \tau$ where τ is a constant $\geqslant 0$.

Let $\mathcal{P}$ be the set of all finite intervals in $\mathbb{N}$ and all their interval partitions $\pi = I_1 \cdots I_m$ Extend f from all finite subintervals to an algebra homomorphism $A(\mathcal{P}) \to Y$ by

$$f(\pi) = f(I_1) \cdots f(I_n).$$

Then

$$f[I] = f(I) - \sum_{I_1 \circ I_2 = I} f(I_1) f(I_2) + \sum_{I_1 \circ I_2 \circ I_3 = I} f(I_1) f(I_2) f(I_3) - + \cdots$$

and $f[I]$ has as in the preceding two examples the property

$$f[I](t) = 0$$

if there exists a partition $I = I_1 \circ I_2$ such that $t_1 \leq \cdots \leq t_n$; $t_{k+1} - t_k > \tau$ where k is the last element of I_1 and $k+1$ the first element of I_2.

Assume now that $Y = \mathbb{C}$

$$f(\emptyset) = 1$$
$$f_1(t_1) = 1$$
$$f_n(t_1, \ldots, t_n) = \exp i \sum_{1 \leq j < k \leq n} \Phi(t_k - t_j)$$

where $\Phi(-t) = \Phi(t)$ Then a theorem similar to the theorem of Ursell and Maier of the preceding example holds.

Theorem. One has

$$f[1, \ldots, n](t) = \sum_{\gamma \in \Lambda^c(1, \ldots, n)} \prod_{(j,k) \in \gamma} \left[\exp i \Phi(t_j - t_k) - 1 \right]$$

where Λ^c is the set of all linearly connected graphs with vertices $1, \ldots, n$.

We say that a graph γ with vertices $1, 2, \ldots, n$ is linearly connected if there does not exist a partition $I_1 \circ I_2 = (1, 2, \ldots, n)$ into intervals I_1, I_2 such that I_1 and I_2 are not connected by γ or, in other words, if there does not exist a k, $1 \leq k < n$ such that any interval $[i, j]$, $i, j \in \gamma$ is contained either in $[1, k]$ or in $[k+1, n]$.

Proof. The proof is quite similar to that of Ursell and Maier's theorem. Write

$$\begin{aligned} f_n(t_1, \ldots, t_n) &= f(1, \ldots, n)(t) = f(I)(t) \\ &= \prod_{1 \leq j < k \leq n} \left[1 + \left(\exp i \Phi(t_j - t_k) - 1 \right) \right] \\ &= \sum_{\gamma} \prod_{(j,k) \in \gamma} \left(\exp i \phi(t_j - t_k) - 1 \right) = \sum_{\gamma} \Psi(\gamma), \end{aligned}$$

where the sum runs over all graphs with vertices $1, 2, \ldots, n$; i.e. over all sets of pairs (j, k), $j < k$.

If γ decomposes into its linear components $\gamma_1, \gamma_2, \ldots, \gamma_m$ then $\Psi(\gamma) = \Psi(\gamma_1) \cdots \Psi(\gamma_m)$. To the decomposition of γ corresponds a decomposition of I into $I_1 \circ \cdots \circ I_m$ and γ_j is a linearly connected graph on I_j for $j = 1, \ldots, m$. On the other hand every graph on I can be obtained by choosing a partition $I = I_1 \circ \cdots \circ I_m$ and then by choosing a linearly connected graph γ_j on I_j for every $j = 1, \ldots, m$. Thus

$$f(I) = \sum_{I_1 \circ \cdots \circ I_m = I} \prod_{j=1}^{m} g(I_j)$$

with
$$g(I) = \sum_{\gamma \in \Lambda^c(I)} \bar{\Psi}(\gamma).$$

As this system of equations determines the $g(I)$ in a recursive way by the $f(I)$ and as this system is formally the same as that combining I and $[I]$ one gets $f[I] = g(I$

§ 4. An analytical problem related with interval partitions

Throughout this section we assume a hierarchy $(f_n)_{n=0,1,2,\ldots}$ of complex-valued symmetric Borel functios on $\mathbb{R}$ bounded in modulus by 1 and $f_0 = 1$. We assume translational invariance, i.e., $f_n(t_1+h, \ldots, t_n+h)$ is independent of h, so $f_1(t)$ = const = f_1. Define

$$(1) \quad F(T) = e^{-cT}\left(1 + \sum_{n=1}^{\infty} \frac{c^n}{n!} \int_0^T \!\!\cdots\! \int f_n(t_1, \ldots, t_n)\, dt_1 \ldots dt_n\right)$$

This formula has a probabilistic interpretation. If $\tau_1, \ldots, \tau_N$ are the jumping points of a Poisson process on $\mathbb{R}_+$ with parameter c in the interval $[0, T]$ then

$$F(T) = E\, f_N(\tau_1, \ldots, \tau_N).$$

We are interested in the Laplace transform of $F(T)$

$$(2) \quad \hat{F}(p) = \int_0^\infty e^{-pT} F(T)\, dT$$

for $\mathcal{R}e\, p > 0$.

A simple case is clearly $f_n(t_1, \ldots, t_n) = f_1^n$. Then $F(T) = \exp c(f_1 - 1)T$ and

$$(1a) \quad \hat{F}(p) = \frac{1}{p - c(f_1 - 1)}.$$

As f_n is symmetric

$$\int_0^T \!\!\cdots\! \int = n! \int_{0 \le t_1 \le \cdots \le t_n \le T} \!\!\cdots\! \int$$

One introduces the new variables $u_0, \ldots, u_n$ by

$$\begin{aligned}
(3) \quad t_1 &= u_0 \\
t_2 &= u_0 + u_1 \\
&\vdots \\
t_n &= u_0 + u_1 + \cdots + u_{n-1} \\
T &= u_0 + u_1 + \cdots + u_{n-1} + u_n
\end{aligned}$$

and obtains

(4) $$\hat{F}(p) = \frac{1}{c+p}\Big[1 + \sum_{n=1}^{\infty} c^n \int_0^{\infty}\!\!\cdots\!\int du_0 \cdots du_{n-1}\; e^{-(c+p)(u_0+\cdots+u_{n-1})} f_n(u_0, u_0+u_1, u_0+\cdots+u_{n-1}),$$

We generalize and introduce a complex Borel measure ρ on $\mathbb{R}_+$ with $|\rho|(\mathbb{R}_+) < 1$. Define the formal power series in the indeterminate z by

(5) $$\langle \rho, f\rangle(z) = f(\emptyset) + \sum_{n=1}^{\infty} z^n \int\cdots\int \rho(du_0)\cdots\rho(du_{n-1})\; f_n(u_0, u_0+u_1, \ldots, u_0+\cdots+u_{n-1}).$$

As the f are bounded by 1 this power series converges for $|z| < \left(|\rho|(\mathbb{R}_+)\right)^{-1}$, hence in a neighborhood of the closed circle $|z| \leq 1$. Using the translational invariance of f_n and remembering the convolution defined in § 2, third example, one obtains

$$\langle \rho, f * g\rangle = \langle \rho, f\rangle \langle \rho, g\rangle$$

if g is a hierarchy similar to f.

Remind the definition of the $A(P)$- valued functions ι and γ of § 2, third example. As $f(\iota(I)) = f(I)$ and $f(\gamma(I)) = f[I]$ for $I \neq \emptyset$ one gets setting $g(I) = f[I]$ for $I \neq \emptyset$, $g(\emptyset) = 0$ the equation

(7) $$f = (\delta_{\emptyset} - g)^{*(-1)}$$

As $f \mapsto \langle \rho, f\rangle$ is a homomorphism with respect to convolution

(8) $$\langle \rho, f\rangle = \frac{1}{1 - \langle \rho, g\rangle}.$$

If $\langle \rho, g\rangle(z)$ converges for $z = 1$ then

(9) $$\langle \rho, f\rangle(z) = \frac{1}{1 - \langle \rho, g\rangle(z)}$$

for $|z| < 1$ and $z = 1$ For the equation $\langle \rho, f\rangle(1 - \langle \rho, g\rangle) = 1$ holds for $|z| < 1$ and $z = 1$ by Abel's lemma.

We apply this formalism to equation (4) and put

(10) $$\rho(p, du) = c\, e^{-(c+p)u}\, du$$

So

(10a) $$\hat{F}(p) = \frac{1}{c+p}\langle \rho(p), f\rangle(1).$$

Then $|\rho|(\mathbb{R}_+) = c/(c+\mathcal{Re}\,p) < 1.$

Now using translational invariance

(11) $$\int\cdots\int \rho(p,du_0)\cdots\rho(p,du_{n-1})\, g_n(u_0,\ldots,u_0+\cdots+u_{n-1}) = \frac{c}{c+p}\, G_n(p)$$

(12) $$G_1 = g_1 = f_1$$

$$G_n(p) = \int\cdots\int \rho(p,du_1)\cdots\rho(p,du_{n-1})\, g_n(0,u_1,\ldots,u_1+\cdots+u_{n-1})$$

for $n \geq 2$.

Proposition 1. If $\langle \rho, g\rangle(z)$ converges for $z=1$ and $\mathcal{Re}\,p>0$

(13) $$\hat{F}(p) = \frac{1}{p - c\,(G(p)-1)}$$

where

$$G(p) = G_1 + G_2(p) + G_3(p) + \cdots$$

converges and is holomorphic in $\mathcal{Re}\,p > 0$.

Of course, formula (13) is a generalization of formula (1a). For in the simple case $f_n = f_1^{\,n}$ one has $G(p) = f_1$.

Proof. For any p with $\mathcal{Re}\,p>0$ one has by (9)

$$\hat{F}(p) = \frac{1}{c+p}\,\frac{1}{1-\frac{c}{c+p}G(p)} = \frac{1}{p+c-c\,G(p)}\;.$$

The next lemma shows a special case in which $\langle\rho, g\rangle$ converges for $z=1$. Let us assume for the moment that ρ obeys only the general condition $|\rho|(\mathbb{R}_+)<1$.

Lemma 1. If there exists a constant $\tau \geq 0$ such that

$$f_n(t_1,\ldots,t_n) = f_k(t_1,\ldots,t_k)\, f_{n-k}(t_{k+1},\ldots,t_n)$$

if $t_1 \leq t_2 \leq \cdots \leq t_n$ and $t_{k+1}-t_k > \tau$ and if $|\rho|[0,\tau] < \frac{1}{2}$ then $\langle \rho, g\rangle(1)$ converges.

Proof. By the discussions of § 3, third example, the function $g(t_1,\ldots,t_n) = f[1,2,\ldots,n](t)$ has the property of vanishing if $t_1 \leq \cdots \leq t_n$ and if there exists a k, $1\leq k<n$ such that $t_{k+1}-t_k > \tau$. Hence $g(u_0, u_0+u_1,\ldots u_0+\cdots+u_{n-1})$ vanishes unless $u_k \leq \tau$ for $k=1,\ldots,n-1$.

On the other hand g_n consists of 2^{n-1} terms of modulus ≤ 1. Therefore

$$\int\cdots\int \rho(du_0)\cdots\rho(du_{n-1})\, g_n(u_0,\ldots,u_0+\cdots+u_{n-1})$$

is bounded in modulus by $(2\,|\rho|\,[0,\tau])^{n-1}$ This proves the lemma.

Theorem 1. If there exists a constant $\tau \geq 0$ such that

$$f_n(t_1,\ldots,t_n) = f_k(t_1,\ldots,t_k)\, f_{n-k}(t_{k+1},\ldots,t_n)$$

if $t_1 \leq \cdots \leq t_n$ and if $t_{k+1} - t_k > \tau$ and if $c\tau <$ log 2, then for any p with $\mathcal{R}e\, p > 0$ the equation

$$\hat{F}(p) = \frac{1}{p - c(G(p)-1)}$$

holds where $G(p)$ is holomorphic in $\mathcal{R}e\, p > 0$ and is given by

$$G = G_1 + G_2(p) + G_3(p) + \cdots$$

$$G_1 = f_1$$

$$G_2(p) = c \int_0^\infty e^{-(c+p)u}\, du \, \left(f_2(0,u) - f_1^2 \right)$$

$$G_3(p) = c^2 \iint_0^\infty e^{-(c+p)(u_1+u_2)}\, du_1\, du_2 \left[f_3(0, u_1, u_1+u_2) - f_1 f_2(0, u_2) - f_2(0, u_1) f_1 + f_1^3 \right]$$

Proof. After the preparations one has only to prove that $|\rho(p)|\,[0,\tau] < \frac{1}{2}$ for $\rho(p, du)$ given by (10). But this is immediate.

We treat now a more natural example. Assume a real Borel function Φ on $\mathbb{R}$ with $\Phi(t) = \Phi(-t)$ and define

$$f(\emptyset) = 1, \; f_1 = 1, \; f_n(t_1, \ldots, t_n) = \exp i \sum_{1 \leq j < k \leq n} \Phi(t_j - t_k)$$

Then, as has been pointed out in § 3, third example, one has

$$(14) \quad g_n(t_1, \ldots, t_n) = \sum_{\gamma \in \Lambda^c(1,2,\ldots,n)} \Psi(\gamma)$$

where Λ^c is the set of all linearly connected graphs with vertices $1, 2, \ldots, n$ and

$$(15) \quad \Psi(\gamma)(t_1, \ldots, t_n) = \prod_{(j,k) \in \gamma} \left(\exp i\, \Phi(t_j - t_k) - 1 \right)$$

In the following we identify a graph γ on an interval I with the set of intervals $\{ [j,k] \subset I : (j,k) \in \gamma \}$. Then $I = (1, 2, \ldots, n)$ is linearly connected with respect to γ if there does not exist a $k, 1 \leq k < n$ such that any interval of γ is a subinterval of one of the intervals $[1, k]$ or $[k+1, n]$.

We say that a graph γ with vertices $1, \ldots, n$ has a (linear) knot $l, 1 < l < n$ if every interval $[j, k] \in \gamma$ is contained either in $[1, l]$ or in $[l, n]$. We call γ twice linearly connected of it does not contain a knot.

We define the hierarchy g^o on $\mathbb{R}^+$ by

$$(16) \quad g_n^o(u_1, \ldots, u_n) = g_{n+1}(0, u_1, u_1 + u_2, \ldots, u_1 + \cdots + u_n)$$

of course

$$g^{\circ}(\emptyset) = g_1(\sigma) = 1.$$

Proposition 2. Define the hierarchy h on $\mathbb{R}_+$ by $h(\emptyset) = \sigma$ and $h(I) = g^{\circ}[I]$ for $I \neq \emptyset$. Then

$$(17) \quad h_n(u_1, \dots, u_n) = \sum_{\gamma \in \Lambda^{cc}(0,1,\dots,n)} \bar{\Psi}_\gamma(0, u_1, \dots, u_1 + \dots + u_n)$$

where $\bar{\Psi}_\gamma$ was defined in (15) and Λ^{cc} signifies the set of all twice connected linear graphs with vertices $0, 1, \dots, n$.

Proof. Let γ be a connected graph on $0, 1, \dots, n$, let $0 < \ell_1 < \dots < \ell_{k-1} < n$ be the knots of γ and γ_j be the restrictions of γ to $I_j = [\ell_{j-1}, \ell_j]$ for $j = 1, \dots, k$ with $\ell_0 = 0$ and $\ell_k = n$. The graphs γ_j are twice connected. Any interval $[p, q] \in \gamma$ is contained in one of the γ_j. So

$$\bar{\Psi}_\gamma = \prod_{j=1}^{k} \prod_{(p,q) \in \gamma_j} [\exp i\, \phi(t_p - t_q) - 1]$$

with $t_0 = 0, \dots, t_n = u_1 + \dots + u_n$ The product $\prod_{(p,q)\in\gamma_j} \cdots$ depends only on the differences $t_p - t_q$; $p, q \in I_j$ hence only on $u_{\ell_{j-1}+1}, \dots, u_{\ell_j}$. As in the proof of Prop. 1 one concludes

$$(18) \quad g^{\circ}(1, 2, \dots, n) = \sum_{I_1 \circ \dots \circ I_k = (1,2,\dots,n)} h(I_1) \dots h(I_k)$$

and hence the proposition.

Define the formal power series

$$(19) \quad g^{\circ}(p, z) = 1 + \sum_{n \geq 1} z^n \int \cdots \int \rho(p, du_1) \cdots \rho(p, du_n)\, g_n^{\circ}(u_1, \dots, u_n)$$

If it converges for $z = 1$ then

$$(20) \quad g^{\circ}(p, 1) = G(p)$$

Defining in an analogous way

$$(21) \quad h(p, z) = \sum_{n \geq 1} z^n \int \cdots \int \rho(p, du_1) \cdots \rho(p, du_n)\, h_n(u_1, \dots, u_n)$$

then by (18) one gets

$$(22) \quad g^{\circ}(p, z) = \frac{1}{1 - h(p, z)}$$

and as

(23) $$\langle \rho(p), g \rangle (z) = \frac{cz}{c+p}\, g^{\circ}(p,z)$$

one gets

(24) $$\frac{1}{c+p} \langle \rho(p), f \rangle (z) = \frac{1-h(p,z)}{(c+p)(1-h(p,z))-cz}$$

Theorem 2. If $h(p,z)$ converges for $z=1$ to $h(p,1)=H(p)$, then

(25) $$\hat{F}(p) = \frac{1-H(p)}{p-(c+p)H(p)}$$

and $H(p)$ is holomorphis for $\mathcal{R}e\, p > 0$.

Proof. For $|z|<1$ one has

$$\frac{1}{c+p} \langle \rho(p), f \rangle (z) \left[(c+p)(1-h(p,z))-cz\right] = 1-h(p,z)$$

Going to the limit $z \to 1$ on both sides one gets (25) by Abel's lemma. As $\hat{F}$ is holomorphic in $\mathcal{R}e\, p > 0$ one concludes that H(p) is holomorphic where it is finite, hence $H(p)$ is holomorphic everywhere.

In the rest of the section we discuss the convergency of $H(p)$.

We considered a graph γ on $(1,2,\ldots,n)$ as set of subintervals of $(1,2,\ldots,n)$ These intervals are ordered by inclusion. Denote by $\chi(\gamma)$ the maximal elements of this partial ordering. $\chi(\gamma)$ is called the characteristic of γ. Clearly γ is connected iff $\chi(\gamma)$ is connected and γ is twice connected iff $\chi(\gamma)$ is twice connected. Denote by $\Lambda(0,1,\ldots,n)$ the set of all graphs on $(0,1,\ldots,n)$.

Lemma 2. Be $\gamma_0 \in \Lambda(0,1,\ldots,n)$ such that $\chi(\gamma_0)=\gamma_0$. Then

(26) $$\sum_{\gamma \in \Lambda(0,1,\ldots,n)\,:\,\chi(\gamma)=\gamma_0} \Psi_\gamma = \Psi_{\gamma_0} \prod_{(j,k)\in\gamma_1} \exp i\,\Phi_{jk}$$

where

$$\gamma_1 = \{(j,k) : 0 \le j < k \le n,\ \exists\, I \in \gamma_0 : [j,k] \subsetneqq I\}$$

and $\phi_{jk}(t) = \Phi(t_j - t_k)$.

Proof. Every γ with $\chi(\gamma)=\gamma_0$ contains γ_0 and a subset of γ_1. Hence

$$\sum_{\gamma\,:\,\chi(\gamma)=\gamma_0} \Psi_\gamma = \Psi_{\gamma_0} \sum_{\gamma\subset\gamma_1} \Psi_\gamma .$$

Now for any graph λ on $0,1,\ldots,n$ one has

$$\sum_{\gamma\subset\lambda} \Psi_\gamma = \prod_{(j,k)\in\lambda} \exp i\,\Phi_{jk}$$

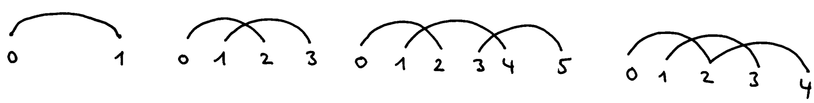
as one proves easily by induction.

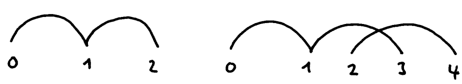
Assume a graph γ on $0, 1, \ldots, n$ and $\chi(\gamma) = \{ [\alpha_j, \beta_j]; j = 1, \ldots, \ell \}$. Consider the set $\{ \alpha_j, \beta_j : j = 1, \ldots, k \}$ and order it with respect to the natural order, so $\{ \alpha_j, \beta_j : j = 1, \ldots, k \} = \{ \xi_j : j = 0, \ldots, m \}$ and $0 \leq \xi_0 < \xi_1 < \cdots < \xi_m \leq n$ and $m \leq 2(\ell - 1)$. Then

$\alpha_j = \xi_{a_j}, \beta_j = \xi_{b_j}$ and we call the graph

$$\omega(\gamma) = \{ [a_1, b_1], \ldots, [a_\ell, b_\ell] \}$$

on $0, 1, \ldots, m$ the <u>reduced characteristic</u> of γ. γ is connected iff $\xi_0 = 0$ and $\xi_m = n$ and $\omega(\gamma)$ is connected. The same holds for γ being twice connected. We draw some simple reduced characteristics for a) twice connected and b) connected but not twice connected graphs.

a) 

b) 

A graph which is equal to its reduced characteristic is said to be <u>reduced</u>.

<u>Lemma 3.</u> Let $\gamma_0 = \omega(\gamma_0)$ be a reduced characteristic and denote by $\Lambda(0, \ldots, n)(\gamma_0)$ the set of graphs γ with vertices $0, \ldots, n$ such that $\xi_0 = 0$ and $\xi_m = n$ and that $\omega(\gamma) = \gamma_0$. Then

$$(27) \quad \sum_{n \geq 1} \Big| \sum_{\gamma \in \Lambda(0, \ldots, n)(\gamma_0)} \int \cdots \int \rho(p, du_1) \cdots \rho(p, du_n) \, \tilde{\Psi}_\gamma(0, u_1, \ldots, u_1 + \cdots + u_n) \Big|$$

$$\leq \int_0^\infty \cdots \int \tilde{\rho}(p, du_1) \cdots \tilde{\rho}(p, du_m) \, \Psi_{\gamma_0}(0, u_1, \ldots, u_1 + \cdots + u_m)$$

with

$$\tilde{\rho}(p, du) = c \, e^{-\mathcal{R}e \, p \, u} \, du$$

and

$$\Psi_{\gamma_0} = \prod_{(i,j) \in \gamma_0} \varphi_{ij}$$

and

$$\varphi_{ij}(t) = \varphi(t_i - t_j), \quad \varphi(u) = | e^{i \Phi(u)} - 1 |$$

with

$$t_0 = 0,\ t_1 = u_1, \ldots, t_n = u_1 + \cdots + u_n \ ; \ t = (t_0, t_1, \ldots, t_n).$$

Proof. Let $\gamma_0 = \{[a_1, b_1], \ldots, [a_l, b_l]\}$ with $a_1 = 0$ and $b_l = m$. Then the characteristic γ_1 of a graph γ with $\omega(\gamma) = \gamma_0$ can be obtained by choosing intergers $0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_m = n$. Then by lemma 2

$$\left| \sum_{\gamma \in \Lambda(0,\ldots,n)(\gamma_0) \ : \ \chi(\gamma) = \gamma_1} \bar{\Psi}_\gamma \right| \leq \psi_{\gamma_1} = \prod_{(i,j)\in\gamma_0} \varphi_{\xi_i \xi_j}$$

and

$$\left| \sum_{\gamma \in \Lambda(0,\ldots,n)(\gamma_0) \ : \ \chi(\gamma) = \gamma_1} \int \cdots \int \rho(p, du_1) \cdots \rho(p, du_n) \bar{\Psi}_\gamma (0, u_1, \ldots, u_1 + \cdots + u_n) \right|$$

$$\leq \int_0^\infty c^{n_1} \frac{v_1^{n_1 - 1}}{(n_1 - 1)!} e^{-(c + \mathcal{R}e\, p) v_1} dv_1 \cdots \int_0^\infty c^{n_m - 1} \frac{v_m^{n_m - 1}}{(n_m - 1)!} e^{-(c + \mathcal{R}e\, p) v_m} dv_m \prod_{(i,j)\in\gamma_0} \varphi(v_{i+1} + \cdots + v_j)$$

with

$$\xi_1 = n_1$$
$$\vdots$$
$$\xi_m = n_1 + \cdots + n_m$$

and

$$v_1 = u_1 + \cdots + u_{\xi_1}$$
$$\vdots$$
$$v_m = u_{\xi_{m-1}+1} + \cdots + u_{\xi_m}$$

Summing up over all values of $n_1 \geq 1, \ldots, n_m \geq 1$ one obtains the right side of (27).

Proposition 3. A sufficient condition in order that $h(p, z)$ converge for $z = 1$ is that

$$(28) \quad \sum_{m \geq 2} \sum_{\gamma} \int \cdots \int \tilde{\rho}(p, du_1) \cdots \tilde{\rho}(p, du_m) \psi_\gamma (0, u_1, \ldots, u_1 + \cdots + u_m)$$

converges where the second sum runs over all reduced twice connected graphs $\gamma = \{[a_1, b_1], \dots, [a_\ell, b_\ell]\}$ with $a_1 = \sigma$, $b_\ell = m$.

Proof. Immediate consequence of lemma 3.

It was impossible for me to prove the convergence if one assumes that $\varphi(t)$ is t-integrable. The Kirkwood-Salsburgh technique fails in this case. I was only able to prove convergence in the case

$$\varphi(t) \leq \mu\, e^{-\lambda t} \tag{29}$$

with $\lambda > 0$. We begin with some topological lemmata.

Lemma 4. Let $\gamma = \chi(\gamma) = \{[\alpha_1, \beta_1], \dots, [\alpha_\ell, \beta_\ell]\}$ with $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_\ell$. Then

$$\alpha_1 < \alpha_2 < \dots < \alpha_\ell$$

and

$$\beta_1 < \beta_2 < \dots < \beta_\ell .$$

Proof. Assume $\alpha_{j+1} = \alpha_j$. Then $\beta_{j+1} \neq \beta_j$. Assume $\beta_j < \beta_{j+1}$. Then $[\alpha_j, \beta_j] \subsetneqq [\alpha_{j+1}, \beta_{j+1}]$ and $[\alpha_j, \beta_j]$ is not a maximal element of the partially ordered set γ. By the same kind of reasoning one excludes $\beta_{j+1} \leq \beta_j$.

Lemma 5. Be $\gamma = \chi(\gamma) = \{[\alpha_1, \beta_1], \dots, [\alpha_\ell, \beta_\ell]\}$ with $\alpha_1 < \alpha_2 < \dots < \alpha_\ell$ twice connected on $0, \dots, m$ and $\ell \geq 2$. Then $\alpha_1 < \alpha_2 < \beta_1$ and $\gamma' = \{[\alpha_2, \beta_2], \dots, [\alpha_\ell, \beta_\ell]\}$ is twice connected on $\alpha_2, \dots, m$ and $\chi(\gamma') = \gamma'$.

Proof. Assume $\alpha_2 > \beta_1$ then between β_1 and α_2 there is a gap in the graph and γ is not connected. If $\alpha_2 = \beta_1$ then α_2 is knot and γ is not twice connected. Thus $\alpha_2 < \beta_1$ Assume there is a knot in γ', say $\beta_k = \alpha_{k+1}$ for one k, $k = 2, \dots, \ell - 1$. As $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$ there follows that $\beta_k = \alpha_{k+1}$ is a knot for γ too. The same reasoning applies to gaps. Hence γ' is twice connected.

Theorem 3. Assume

$$\left| e^{i\phi(t)} - 1 \right| \leq \mu\, e^{-\lambda |t|} \tag{29}$$

and

$$\frac{\mu c}{\lambda} \left(1 + \sqrt{\frac{c}{\lambda}}\right)^2 < 1 \tag{30}$$

Then $h(p, z)$ converges for $z = 1$ and any p, $\Re p \geq 0$. Hence the hypotheses of theorem 2 are fulfilled.

Proof. Denote by G the set of all reduced twice connected graphs. If γ is a graph on $0, 1, \dots, m$ denote

$$A(\gamma) = \int_0^\infty \!\!\cdots \int c^m \, du_1 \cdots du_m \; \tilde{\psi}_\gamma (0, u_1, \dots, u_1 + \dots + u_m)$$

with

$$\tilde{\Psi}_\gamma(t_0, t_1, \dots, t_n) = \prod_{(i,j)\in\gamma} \mu e^{-\lambda|t_i - t_j|}$$

It is enough to prove

$$\sum_{\gamma\in G} A(\gamma) < \infty$$

We finish the proof in several steps.

(i) If $\gamma = \{[a_1, b_1], \dots, [a_\ell, b_\ell]\}$ is a graph on $0, 1, \dots, n$. Then

$$A(\gamma) = c^n \lambda^{-n} \mu^\ell \frac{1}{q_1} \cdots \frac{1}{q_n}$$

where q_j is the number of intervals $I \in \gamma$ such that $[j-1, j] \subset I$. The proof of this statement follows right away from the definition of $A(\gamma)$.

(ii) Denote by G_ℓ the subset of G consisting of all graphs of the form

$$\gamma = \{[a_1, b_1], \dots, [a_\ell, b_\ell]\}$$

Then

$$\eta_\ell : \eta_\ell(\gamma) = \omega\{[a_2, b_2], \dots, [a_\ell, b_\ell]\}$$

is a mapping from G_ℓ into $G_{\ell-1}$

This is an immediate consequence of lemma 5.

(iii) Denote by $G_{\ell,m}$ the subset of G_ℓ of all graphs of the form

$$\gamma = \{[0, m], [a_2, b_2], \dots, [a_\ell, b_\ell]\}.$$

Then $G_{\ell,m} = \emptyset$ for $\ell < m$ and one has: G_1 consists only of the graph $\{[0,1]\}$, G_2 consists only of the graph $\{[0,2], [1,3]\}$. So of all $G_{1,m}$ only $G_{1,1}$ and of all $G_{2,m}$ only $G_{2,2}$ are non-void. Denote by $G_{\ell,m,m'}$ the subset of G of all graphs of the form

$$\gamma = \{[0, m], [1, m'+1], [a_3, b_3], \dots, [a_\ell, b_\ell]\}$$

and by $G'_{\ell,m,m'}$ the subset of $G_{\ell,m,m'}$ where $m \in \{a_2, b_2, \dots, a_\ell, b_\ell\}$ and $G''_{\ell,m,m'}$ the subset with $m \notin \{a_2, b_2, \dots, a_\ell, b_\ell\}$ Then $G_{\ell,m,m'} = \emptyset$ for $m > m'$ and the restriction of η_ℓ to $G'_{\ell,m,m'}$ is a one-to-one mapping onto $G_{\ell-1,m'}$ and the restriction of η_ℓ to $G''_{\ell,m,m'}$ is a one-to-one mapping onto $G_{\ell-1,m'-1}$.

We illustrate a graph γ on $0, 1, \dots, n$ by a scheme of three lines. Write in the top the numbers $0, 1, \dots, n$ into the second line under each number the index q_j which is equal to that number and into the third line the numbers b_j. So, e.g., $\gamma = \{[0,2], [1,3]\}$ gets

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ a_1 & a_2 & & \\ & & b_1 & b_2 \end{array}$$

Then the typical example of an element of $G'_{\ell,m,m'}$ is

$$\text{(a)}\quad \begin{array}{ccccccccccc} 0 & 1 & 2 & \cdots & m-1 & m & m+1 & \cdots & m' & m'+1 & \cdots \\ a_1 & a_2 & a_3 & & a_m & a_{m+1} & a_{m+2} & & a_{m'+1} & 2 & \cdots \\ & & & & & b_1 & & & & b_2 & \cdots \end{array}$$

The reason for this scheme is easily explained. As $b_1 < b_2 < b_3 < \cdots$ one has $m \leq m'$ and in the interval $0, 1+m'$ no b_j, $j \geqslant 3$ can occur. As γ is reduced, under each number $0, 1, 2, \ldots$ there has to stay either an index a_j or an index b_j or both. As there are only b_1 and b_2 under $0, \ldots, m'+1$ the rest has to be filled up by a_j in increasing order. As $m \in \{a_2, b_2, \ldots, a_\ell, b_\ell\}$ one has a_{m+1} under m. Skipping away the interval $[0, m]$ from γ one gets

$$\text{(b)}\quad \begin{array}{cccccccccc} 1 & 2 & \cdots & m-1 & m & m+1 & \cdots & m' & m'+1 & \cdots \\ a_2 & a_3 & \cdots & a_m & a_{m+1} & a_{m+2} & \cdots & a_{m'+1} & 2 & \cdots \\ & & & & & & & & b_2 & \cdots \end{array}$$

As reduction means only changing of numeration by one, one gets $\eta_\ell(\gamma) \in G_{\ell-1,m'}$. This application is one-to-one, as any graph $\gamma' \in G_{\ell-1,m'}$ can be written into form (a) and then a $\gamma \in G'_{\ell,m,m'}$ can be constructed for any $m \leq m'$.

A typical element of $G''_{\ell,m,m'}$ has the form

$$\text{(c)}\quad \begin{array}{ccccccccccc} 0 & 1 & 2 & \cdots & m-1 & m & m+1 & \cdots & m' & m'+1 & \cdots \\ a_1 & a_2 & a_3 & \cdots & a_m & & a_{m+1} & \cdots & a_{m'} & 2 & \cdots \\ & & & & & b_1 & & & & b_2 & \cdots \end{array}$$

Skipping away the first interval one gets

$$\text{(d)}\quad \begin{array}{cccccccccc} 1 & 2 & \cdots & m-1 & m & m+1 & \cdots & m' & m'+1 & \cdots \\ a_2 & a_3 & \cdots & a_m & & a_{m+1} & & a_{m'} & 2 & \cdots \\ & & & & & & & & b_2 & \cdots \end{array}$$

Reduction means here not only changing of numeration by one, but loosing one point: m. These considerations easily prove the statements of step (iii).

(iv) If $\gamma \in G'_{\ell,m,m'}$ then $A(\gamma) = \frac{\mu c}{\lambda m} A(\eta_\ell(\gamma))$ and if $\gamma \in G''_{\ell,m,m'}$ then $A(\gamma) = \dfrac{c^2\mu}{\lambda^2 m(m-1)} A(\eta_\ell(\gamma))$.

For the proof write the values of the q_j into the fourth line of the schemes. Then one gets for $\gamma \in G'_{\ell,m,m'}$

$$\text{(a')}\quad \begin{array}{ccccccccccc} 0 & 1 & 2 & \cdots & m-1 & m & m+1 & \cdots & m' & m'+1 & \cdots \\ a_1 & a_2 & a_3 & & & a_m & a_{m+1} & a_{m+2} \cdots & a_{m'+1} & 2 & \cdots \\ & & & & & b_1 & & & & b_2 & \cdots \\ & 1 & 2 & \cdots & m-1 & m & m & \cdots & m'-1 & m' & \cdots \end{array}$$

and taking away the first interval one gets the graph γ' :

(b')

$$\begin{array}{ccccccccccc} 1 & 2 & \cdots & m-1 & m & m+1 & \cdots & m' & m'+1 & \cdots \\ a_2 & a_3 & \cdots & a_m & a_{m+1} & a_{m+2} & & a_{m'+1} & ? & \cdots \\ & & & & & & & & b_2 & \cdots \\ & 1 & \cdots & m-2 & m-1 & m & \cdots & m'-1 & m' & \cdots \end{array}$$

Hence by step (i) one has

$$A(\gamma) = \frac{\mu c}{\lambda m} A(\gamma').$$

Similarly for $\gamma \in G''_{\ell, m, m'}$

(c')

$$\begin{array}{cccccccccccc} 0 & 1 & 2 & \cdots & m-1 & m & m+1 & m+2 & \cdots & m' & m'+1 & \cdots \\ a_1 & a_2 & a_3 & \cdots & a_m & & a_{m+1} & a_{m+2} & & a_{m'} & ? & \cdots \\ & & & & & b_1 & & & & & b_2 & \cdots \\ & 1 & 2 & \cdots & m-1 & m & m-1 & m & \cdots & m'-2 & m'-1 & \cdots \end{array}$$

and for γ'

(d')

$$\begin{array}{ccccccccccc} 1 & 2 & \cdots & m-1 & m & m+1 & m+2 & \cdots & m' & m'+1 & \cdots \\ a_2 & a_3 & \cdots & a_m & & a_{m+1} & a_{m+2} & \cdots & a_{m'} & ? & \cdots \\ & & & & & & & & & b_2 & \cdots \\ & 1 & \cdots & m-2 & & m-1 & m & & m'-2 & m'-1 & \cdots \end{array}$$

Hence

$$A(\gamma) = \frac{c^2 \mu}{\lambda^2 m (m-1)} A(\gamma')$$

(v) Denote

$$c_{\ell, m} = \sum_{\gamma \in G_{\ell, m}} A(\gamma)$$

Then

$$c_{1,1} = \frac{c\mu}{\lambda} \; ; \quad c_{1,m} = 0 \text{ for } m \neq 1$$

$$c_{2,2} = \frac{c^3 \mu^2}{2\lambda^3} \; ; \quad c_{2,m} = 0 \text{ for } m \neq 2 ,$$

and generally for $\ell \geqslant 2$

$$c_{\ell, m} = \sum_{m' \geqslant m} \frac{c\mu}{m\lambda} c_{\ell-1, m'} + \sum_{m' \geqslant m} \frac{c^2 \mu}{\lambda^2 m (m-1)} c_{\ell-1, m'-1} .$$

For following step (iii) and (iv)

$$c_{\ell,m} = \sum_{m' \geq m} \sum_{\gamma \in G'_{\ell,m,m'}} A(\gamma) + \sum_{m' \geq m} \sum_{\gamma \in G''_{\ell,m,m'}} A(\gamma)$$

$$= \sum_{m' \geq m} \Big[\frac{c\mu}{m\lambda} \sum_{\gamma' \in G_{\ell-1,m'}} A(\gamma') + \frac{c^2\mu}{m(m-1)\lambda^2} \sum_{\gamma' \in G_{\ell-1,m'-1}} A(\gamma') \Big]$$

(vi) Define the operator T on the space of all sequences $(x_m)_{m \geq 2}$ by

$$(Tx)_m = \sum_{m' \geq m} \frac{c\mu}{m\lambda} x_{m'} + \sum_{m' \geq m-1} \frac{c^2\mu}{m(m-1)\lambda^2} x_{m'}$$

Then

(a) $$\sum_{\gamma \in G} A(\gamma) = c_{1,1} + \sum_{\ell \geq 2, m} c_{\ell,m}$$

$$= c_{1,1} + \sum_{\ell \geq 0, m} (T^\ell x_0)(m)$$

$$= c_{1,1} + \sum_{\ell \geq 0} \langle \mathbb{1}, T^\ell x_0 \rangle$$

$$= c_{1,1} + \sum_{\ell \geq 0} \langle x_0, (T^t)^\ell \mathbb{1} \rangle$$

where $\mathbb{1}$ is the vector $(1,1,\ldots)$ and T^t the transposed operator and

$$(x_0)_m = \delta_{2,m} \frac{c^3\mu^2}{2\lambda^3}.$$

The adjoint T^t is given by

$$\eta_m = (T^t \xi)_m = \sum_{m' \leq m} \frac{p}{m'} \xi_{m'} + \sum_{m' \leq m+1} \frac{q}{m'(m'-1)} \xi_{m'}$$

$$\leq \sum_{m' \leq m} \frac{p}{m'} \xi_{m'} + \sum_{m' \geq 2} \frac{q}{m'(m'-1)} \xi_{m'}$$

$$= (S\xi)_m$$

with $p = \frac{c\mu}{\lambda}$, $q = \frac{c^2\mu}{\lambda^2}$, provided all $\xi_{m'}$ are ≥ 0. As this is the case in (a) we have

(a') $$\sum_{\gamma \in G} A(\gamma) \leq c_{1,1} + \sum_{\ell \geq 0} \langle x_0, S^\ell \mathbb{1} \rangle .$$

As there is a one-to-one correspondence between sequences $\xi = (\xi_2, \xi_3, \dots)$ and holomorphic functions $\xi(z) = \sum \xi_m z^{m-2}$ we introduce for $0 < \alpha < 1$ the norm

$$\| \xi \|_\alpha = \sup_{0 \leq |z| \leq 1} (1 - |z|)^{1+\alpha} |\xi(z)| .$$

Then

$$\| \mathbb{1} \|_\alpha = 1$$

the norm of the functional x_0 is

$$\| x_0 \|_\alpha \leq c^3 \mu^2 \lambda^{-3} / 2$$

and the corresponding operator norm of S is

$$\| S \|_\alpha \leq \frac{p}{\alpha} + \frac{q}{1-\alpha} \qquad \text{with} \quad \alpha = \left(1 + \sqrt{\frac{p}{q}}\right)^{-1} .$$

Hence $\| S \|_\alpha < 1$ if $\frac{c\mu}{\lambda} \left(1 + \sqrt{\frac{c}{\lambda}}\right)^2 < 1$.
This proves the theorem.

One has

$$S\xi(z) = \sum_{m' \leq m} \frac{p}{m'} \xi_{m'} z^{m-2} + \frac{1}{1-z} \sum_{m' \geq 2} \frac{q}{m'(m'-1)} \xi_{m'}$$

$$= \frac{1}{1-z} \left[\frac{p}{z^2} \int_0^z t\, \xi(t)\, dt + q \int_0^1 (1-t)\, \xi(t)\, dt \right]$$

If

$$|\xi(z)| \leq (1 - |z|)^{-(1+\alpha)}$$

then

$$(1 - |z|)^{1+\alpha} |S\xi(z)| \leq$$
$$\leq (1 - |z|)^\alpha \left[\frac{p}{|z|^2} \int_0^{|z|} \frac{t\, dt}{(1-t)^{1+\alpha}} + q \int_0^1 \frac{dt}{(1-t)^\alpha} \right]$$
$$\leq \frac{p}{\alpha} + \frac{q}{1-\alpha}$$

after some straightforward estimates. So $\| S \|_\alpha \leq \frac{p}{\alpha} + \frac{q}{1-\alpha}$.

The right side takes its minimum for $\alpha = \left(1+\sqrt{\frac{q}{p}}\right)^{-1}$. For this value of α

$$\| S \|_\alpha \leq \left(\sqrt{p} + \sqrt{q}\right)^2 = \frac{c\mu}{\lambda}\left(1+\sqrt{\frac{c}{\lambda}}\right).$$

Literature

|1| Rota, G.C.
On the Foundations of Combinatorial Theory. I. Theory of Möbius Functions. Z. Wahrscheinlichkeitstheorie verw. Geb. 2, 340-368 (1964).

|2| Ruelle, D.
Statistical Mechanics, W.A. Benjamin, Amsterdam 1969.

|3| Waldenfels, W. von
An Approach to the Theory of Pressure Broadening of Spectral Lines. Probability and Information Theory II, 19-69. Ed. by M. Behara. Lecture Notes in Mathematics 296. Springer-Verlag, Heidelberg, 1973.

|4| Waldenfels, W. von
Stochastic Motion in Hilbert Space Related to a Renewal Process. Unpublished report. Available at the author. Institut für Angewandte Mathematik der Universität Heidelberg, 1972.

|5| Waldenfels, W. von
Taylor Expansion of a Poisson Measure. Séminaire de Probabilités VIII, 344-354. Lecture Notes in Mathematics 381. Springer-Verlag, Heidelberg, 1974.

February 25, 1975

Universität Heidelberg
Institut für Angewandte Mathematik
69 Heidelberg / BRD
Im Neuenheimer Feld 294