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SKOROKHOD STOPPING IN DISCRETE TIME

by David HEATH¹

I. Introduction. This note presents a more general version, for discrete-time processes, of a construction presented in [1] which generalized the construction presented by Skorokhod in [3] for stopping Brownian motion to achieve a given distribution. Many of the ideas presented here are due to Mokobodzki -- in particular the integral representation of one excessive measure in terms of another (3) is simply the version for measures of the theorem of Mokobodzki presented in [1].

II. Statement Of The Theorem. We use basically the same notation as Watanabe [4]. Let N be a sub-Markov kernel on the measurable space (E, \mathcal{E}) and let $(\Omega, \mathcal{F}, X_k, \mathcal{F}_k, P_x, x \in E)$ be a realization of the Markov chain on E with transition kernel N . We shall suppose that on this space there is also a random variable S with distribution uniform on $[0, 1]$ independent of $(X_k, k \geq 0)$ and measurable with respect to each \mathcal{F}_k . Let $G = \sum_{n \geq 0} N^n$ be the potential associated with N ; we suppose that $G1$ is bounded. We then have the following:

THEOREM. Suppose μ_0 and μ_1 are (sub-) probability measures on (E, \mathcal{E}) with $\mu_0 G \geq \mu_1 G$. There is then an increasing collection $(A(s), s \in [0, 1])$ of sets in \mathcal{E} such that if T is defined by $T = \inf \{k \geq 0 : X_k \in A(S)\}$, then for every $B \in \mathcal{E}$, $P^{\mu_0}(X_T \in B) = \mu_1(B)$.

REMARK. It is easy to show that if there is any stopping time T satisfying the condition stated for the (sub-) probability measures μ_0 and μ_1 , then $\mu_0 G \geq \mu_1 G$.

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III. Proof Of The Theorem. For $t \in [0,1]$ let $\bar{t} = 1-t$ and define:

$$\nu_t = (\mu_1 - \bar{t}\mu_0)G, \quad \sigma_t = \nu_t L_E, \quad \text{and} \quad \beta_t = \sigma_t - \nu_t.$$

Clearly β_t is a (positive) measure; since $\nu_t \leq \bar{t}\mu_0 G$ and

$\sigma_t \leq \mu_1 G \leq \mu_0 G$, β_t is absolutely continuous with respect to $\mu_0 G$.

Let $A^\circ(t) = \{x \in E : \frac{d\beta_t}{d\mu_0 G} = 0\}$ where any version of the Radon-Nikodym derivative is used; $A^\circ(t)$ is then unique up to $\mu_0 G$ -equivalence. Moreover we have $\sigma_t L_{A^\circ(t)} = \sigma_t$; this follows from the easy-to-prove result for measures corresponding to Corollary 6 of Mokobodzki [2].

We wish to show now that $(A^\circ(s), s \in [0,1])$ is "almost increasing": since for $s < t$, $\sigma_s + (t-s)\mu_0 G$ is excessive and dominates $\nu_s + (t-s)\mu_0 G = \nu_t$ we clearly have $\sigma_s + (t-s)\mu_0 G \geq \sigma_t$ which implies $\sigma_s - \nu_s \geq \sigma_t - \nu_t$, so (β_s) is a decreasing family. Thus if $s < t$, $\mu_0 G(A^\circ(s) \setminus A^\circ(t)) = 0$. We thus obtain that for $s < t$, $\sigma_s L_{A^\circ(t)} = \sigma_s$.

Since $\nu_t = \nu_s + (t-s)\mu_0 G$, we obtain $\sigma_t \leq \sigma_s + (t-s)\mu_0 G$ and applying $L_{A^\circ(t)}$ gives $\sigma_t \leq \sigma_s + (t-s)\mu_0^{GL} L_{A^\circ(t)}$ which implies:

$$(1) \quad \frac{\sigma_t - \sigma_s}{t-s} \leq \mu_0^{GL} L_{A^\circ(t)}.$$

In the other direction, $\sigma_t \geq \nu_t = \nu_s + (t-s)\mu_0 G$ which, on $A^\circ(s)$, is equal to $\sigma_s + (t-s)\mu_0 G$, so, by the additivity of $L_{A^\circ(s)}$ on excessive measures (see Watanabe [4]) we obtain

$\sigma_t \geq \sigma_t L_{A^\circ(s)} \geq \sigma_s L_{A^\circ(s)} + (t-s)\mu_0^{GL} L_{A^\circ(s)} = \sigma_s + (t-s)\mu_0^{GL} L_{A^\circ(s)}$, which implies

$$(2) \quad \mu_0^{GL} L_{A^\circ(s)} \leq \frac{\sigma_t - \sigma_s}{t-s}.$$

Combining (1) and (2) we conclude:

$$\sigma_1 - \sigma_0 = \int_0^1 \mu_0^{GL} L_{A^\circ(s)} ds.$$

We now modify the collection $(A^\circ(s), s \in [0,1])$ to make it monotone: Let \mathbb{Q} be the set of rationals in $[0,1]$; for $s \in [0,1]$ define

$$A(s) = \bigcap_{\substack{r \geq s \\ r \in \mathbb{Q}}} A^{\circ}(r).$$

Clearly $(A(s), s \in [0, 1])$ is increasing, and $\mu_0^G(A^{\circ}(r) \setminus A(r)) = 0$ for every rational r , so $\mu_0^{\text{GL}} A(r) = \mu_0^{\text{GL}} A^{\circ}(r)$ for each $r \in \mathbb{Q}$. Since any two positive monotone functions on $[0, 1]$ which agree on \mathbb{Q} have the same integral on $[0, 1]$, we obtain:

$$(3) \quad \mu_1^G = \int_0^1 \mu_0^{\text{GL}} A(s) \, ds.$$

Now let T be defined as in the statement of the theorem; clearly the distribution of X_T (when the process is started according to μ_0) is given by $\int_0^1 \mu_0^H A(s) \, ds$; we wish to show that this measure is μ_1 .

Clearly the potential of this measure is $(\int_0^1 \mu_0^H A(s) \, ds)^G = \int_0^1 \mu_0^H A(s)^G \, ds =$ (see (2.12) of [4]) $\int_0^1 \mu_0^{\text{GK}} A(s) \, ds =$ (by Theorem 1 of [4]) $\int_0^1 \mu_0^{\text{GL}} A(s) \, ds$, which, according to (3) is the potential of μ_1 . Applying (I-N) we obtain the desired conclusion.

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