Hans Föllmer

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PHASE TRANSITION AND MARTIN BOUNDARY

by Hans Föllmer

The theory of stochastic fields, and in particular of Markov fields in the sense of DOBRUSHIN [1] and NELSON [10], has some close analogies to probabilistic potential theory. Our purpose here is to discuss the analogy between phase transition and the existence of non-constant harmonic functions of a Markov process. In particular we derive a general integral representation of stochastic fields as mixtures of phases using the Martin boundary technique. A similar approach based on the "harmonic" martingales of (1.7) below has already been suggested in [6]. But it turns out that DYNKIN's construction of the boundary in [4] is a much more convenient method. Moreover it allows to drop the Markov property, and this is desirable in Statistical Mechanics (example 2a below) and also of interest in Quantum field theory (cf. [13]).

Sections 1 and 2 introduce some basic notions and examples, and sections 3 and 4 are a straightforward modification of DYNKIN [4] adapted to stochastic fields. We refer to PRESTON [12] for a different approach based on Choquet simplex theory which leads to very similar results.

1. STOCHASTIC FIELDS

Let $(\Omega, \underline{\underline{F}})$ be a standard Borel space (cf. [11] p. 133), $\underline{\underline{V}}$ an index set ordered by a relation $\subseteq$, and $(\hat{\underline{\underline{F}}}_V)_{V \in V}$ a decreasing family of sub-σ-fields of $\underline{\underline{F}}$. For each $V \in \underline{\underline{V}}$ let π_V be a probability kernel on $(\Omega, \underline{\underline{F}})$ such that

(1.1) $\qquad \pi_V(\cdot, A)$ is $\hat{\underline{\underline{F}}}_V$ -measurable $(A \in \underline{\underline{F}})$,

and suppose that the collection $\Pi = (\pi_V)_{V \in \underline{\underline{V}}}$ satisfies the consistency condition

(1.2) $\qquad \pi_W \, \pi_V = \pi_W \quad (V, W \in \underline{\underline{V}} \; ; \; V \subseteq W).$

Any probability measure P on $(\Omega,\underline{\underline{F}})$ which is compatible with Π in the sense that

(1.3) $$E[I_A|\hat{\underline{\underline{F}}}_V] = \pi_V(.,A) \quad \text{P-a.s.} \quad (A\in\underline{\underline{F}})$$

will be called a stochastic field with local characteristics Π. The set $C = C(\Pi)$ of all such fields is obviously convex, and ist extreme points will be called the phases of Π. Let us assume $C \neq \emptyset$; a sufficient condition for the existence of some $P\in C$ will be given in (3.2) below. The case $|C| > 1$, where the stochastic field is not uniquely determined by its local characteristics, is often called a phase transition.

In many applications, e.g. in lattice gas models of type (2.1) below, C is a metrizable simplex, and so the Choquet integral representation theory implies that each stochastic field is a mixture of phases. In our general context, without compactness assumptions,we will derive such an integral representation using the Martin boundary technique. There is in fact a close analogy between the stochastic fields of Π and the harmonic functions of a Markov process. To make it more explicit, let us first introduce the notion of a Markov field; cf. DOBRUSHIN [1] and NELSON [10].

Suppose we have, in addition to the decreasing family $(\hat{\underline{\underline{F}}}_V)_{V\in\underline{\underline{V}}}$, an increasing family $(\underline{\underline{F}}_V)_{V\in\underline{\underline{V}}}$ such that $\underline{\underline{F}} = \bigvee \underline{\underline{F}}_V$ and $\underline{\underline{F}}_V \vee \hat{\underline{\underline{F}}}_V = \underline{\underline{F}}$ $(V\in\underline{\underline{V}})$. Let us write $\partial\underline{\underline{F}}_V = \underline{\underline{F}}_V \cap \hat{\underline{\underline{F}}}_V$, and let us say that Π has the Markov property if

(1.4) $$\pi_V(.,A_V) \text{ is } \partial\underline{\underline{F}}_V\text{ - measurable} \quad (A_V \in \underline{\underline{F}}_V).$$

In that case, any $P \in C(\Pi)$ will be called a Markov field.

Let us now fix $P\in C$ and let us look at the class $C_P = C_P(\Pi)$ of all fields $Q\in C$ which are "locally absolutely continuous" with respect to P in the sense that

(1.5) $$Q_V << P_V \quad (V\in\underline{\underline{V}})$$

where Q_V denotes the restriction of Q to $\underline{\underline{F}}_V$. Clearly, any $Q\in C_P$ induces a non-negative martingale $X = (X_V)_{V\in\underline{\underline{V}}}$ over $(\Omega,F,(\underline{\underline{F}}_V),P)$ which is normalized to $E[X_V] = 1$: just take

(1.6) $$X_V = \frac{dQ_V}{dP_V} \quad (V \in \underline{\underline{V}}) .$$

The following proposition gives an intrinsic characterization of all those martingales which arise in that manner from a stochastic field in C_P. Here we assume that $(\underline{\underline{F}}_V)_{V\epsilon\underline{\underline{V}}}$ is a "standard system" in the sense of [11] p. 143: each $(\Omega,\underline{\underline{F}}_V)$ is standard Borel, and any decreasing sequence of atoms $A_n \epsilon \underline{\underline{F}}_{V_n}$ $(V_1 \subseteq V_2 \subseteq \ldots)$ has a non-void intersection. Cf. [8] Theorem VII.2 for a similar result in Quantum field theory.

(1.7) PROPOSITION. - Suppose that Π has the Markov property. Then (1.6) establishes a 1-1 correspondence between $C_P(\Pi)$ and the set of all those non-negative normalized martingales $X = (X_V)_{V\epsilon\underline{\underline{V}}}$ over $(\Omega,\underline{\underline{F}},(\underline{\underline{F}}_V)_{V\epsilon\underline{\underline{V}}},P)$ which are adapted to $(\partial\underline{\underline{F}}_V)_{V\epsilon\underline{\underline{V}}}$ in the sense that

(1.8) $\quad X_V$ is $\partial\underline{\underline{F}}_V$ - measurable P-a.s. $\quad (V\epsilon\underline{\underline{V}})$

PROOF. -

1) Let $X = (X_V)$ be the martingale associated to a field $P^X \epsilon C_P$. For any F-measurable bounded function φ_V we have

$$E[X_V\varphi_V] = E^X[\varphi_V] = E^X[E^X[\varphi_V|\underline{\underline{F}}_{\partial V}]] = E^X[E[\varphi_V|\underline{\underline{F}}_{\partial V}]]$$

$$= E[X_V E[\varphi_V|\underline{\underline{F}}_{\partial V}]] = E[E[X_V|\underline{\underline{F}}_{\partial V}]\varphi_V]$$

and this implies (1.8).

2) Let $X = (X_V)$ be a non-negative martingale with $E[X_V] = 1$. Then $P^X[A_V] = E[X_V;A_V]$ $(A_V \epsilon \underline{\underline{F}}_V)$ yields a consistent definition of P^X on $\bigcup\underline{\underline{F}}_V$, and since $(\underline{\underline{F}}_V)$ is a standard system we may view P^X as a probability measure on $\underline{\underline{F}}$; cf. [11] Th. 4.2. Now assume (1.8) and let us verify (1.3). Since $\underline{\underline{F}} = \underline{\underline{F}}_V \vee \hat{\underline{\underline{F}}}_V$ it is enough to consider the case $A\epsilon\underline{\underline{F}}_V$. Take any $W \epsilon \underline{\underline{V}}$ with $W \subseteq V$ and a $\hat{\underline{\underline{F}}}_V \cap \underline{\underline{F}}_W$ -measurable bounded φ. Since $\underline{\underline{F}} = \bigvee\underline{\underline{F}}_W$ and Π is Markovian, it is enough to check $E^X[I_A\varphi] = E^X[E[I_A \mid \partial\underline{\underline{F}}_V]\varphi]$. In fact we have

$$E^X[I_A\varphi] = E[X_W\, I_A\varphi] = E[E[X_W\, I_A|\hat{\underline{\underline{F}}}_V]\varphi] \quad ,$$

and since $\partial\underline{\underline{F}}_W \subseteq \hat{\underline{\underline{F}}}_W \subseteq \hat{\underline{\underline{F}}}_V$, (1.8) allows to continue

$$= E[X_W\, E[I_A|\hat{\underline{\underline{F}}}_V]\varphi] = E^X[E[I_A|\partial\underline{\underline{F}}_V]\varphi] .$$

2. EXAMPLES

a) **LATTICE GAS MODELS.** - Let E be a countable set, $T = Z^d = \{t = (t_1,\ldots,t_d)|t_i$ integer$\}$ with $d \geq 1$, and consider the space $\Omega = E^T$ of configurations ω: $T \longrightarrow E$. For any $V \subseteq T$ define $\hat{\underline{\underline{F}}}_V$ as the σ-field generated by the maps $\omega \longrightarrow \omega(t)$ $(t \notin V)$, and take $\underline{\underline{V}}$ = the finite subsets of T.

In the lattice gas models of Statistical Mechanics the local characteristics are given in terms of a <u>potential function</u> U , that is, a collection of maps

$$U_V : E^V \longrightarrow R \qquad (V \in \underline{\underline{V}}) .$$

If $U_V(.) = 0$ as soon as the diameter of A is greater than some fixed integer r then U is said to be of <u>finite range</u> r . If this is the case with r = 1 then U is called a <u>nearest neighbor potential</u>. The energy of a configuration $\xi \in \Omega$ on $V \in \underline{\underline{V}}$ is defined as

$$E_V(\xi) = \sum_{W \in \underline{\underline{V}}, W \cap V \neq \emptyset} U_W(\xi_W)$$

where ξ_W is the restriction of ξ to W , and where the assumptions on U are such that the right side is always absolutely convergent. For $\omega \in \Omega$ the measure $\pi_V(\omega,.)$ is concentrated on the finite set of all those $\xi \in \Omega$ which coincide with ω on T - V , and any such ξ carries the weight

(2.1) $$\pi_V(\omega, \{\xi\}) = Z_V^{-1}(\omega) \exp(-E_V(\xi))$$

with

$$Z_V(\omega) = \sum_{\xi : \xi = \omega \text{ on } T-V} \exp(-E_V(\xi)) .$$

$\Pi = (\pi_V)_{V \in \underline{\underline{V}}}$ is then consistent, and any $P \in C(\Pi)$ is called a <u>Gibbs field</u>.

If U is of finite range r then Π has the Markov property (1.4) if we define $\underline{\underline{F}}_V$ as the σ-field generated by all maps $\omega \longrightarrow \omega(t)$ with $t \in T$ and dist $(t,V) \leq r$.

Let us now consider the case where E is finite and where U is a potential function with range r = 1 . Then there is no phase transition for d = 1 (cf. [14]). For $d \geq 2$ the <u>Ising model</u> ($E = \{-1,1\}, U_V(\xi) = J\, \xi(s)\, \xi(t)$ if $V = \{s,t\}$ with $\|s-t\| = 1$ and $U_V(.) = 0$ else) yields examples of phase transition as soon as $|J|$ is large enough (cf. [14]). Let us however emphasize that even for d = 1 we may encounter phase transitions as soon as either the Markov property (cf. DYSON [5]) or the condition that E is finite (cf. example c) below is dropped.

b) QUANTUM FIELDS. - Take the space $\Omega = \mathcal{D}'$ of distributions on Euclidean space R^d and $\underline{\underline{V}}$ = the relatively compact open subsets of R^d . For any open $V \subseteq R^d$ define $\underline{\underline{F}}_V$ as the σ-field generated by the maps $\omega \longrightarrow \omega(f)$

($f \in \mathfrak{D}$, supp (f) $\subseteq$ V), and for any closed $A \subseteq R^d$ define $\underline{\underline{F}}_A$ as the intersection of the fields $\underline{\underline{F}}_V$ (V open, $V \subseteq A$). For $V \in \underline{\underline{V}}$ we set $\partial \underline{\underline{F}}_V = \underline{\underline{F}}_{\partial V}$ and $\hat{\underline{\underline{F}}}_V = \underline{\underline{F}}_{T-V}$.

The "free Markov field" in NELSON [10] is a probability measure on $(\Omega, \underline{\underline{F}})$ such that

$$E[I_A | \hat{\underline{\underline{F}}}_V] = E[I_A | \partial \underline{\underline{F}}_V] \qquad (A \in \underline{\underline{F}}_V) .$$

The question of phase transition arises if the free Markov field is "perturbed" by a multiplicative functional; this induces a Markovian collection $\Pi = (\pi_V)_{V \in \underline{\underline{V}}}$ of local characteristics as indicated in DOBRUSHIN-MINLOS [2] (the kernels can be constructed consistently in the sense of (1.2) at least for a countable base of $\underline{\underline{V}}$).

c) MARKOV CHAINS. - Take $\Omega = E^T$ where E is some countable state space and $T = \{0,1,2,\dots\}$, $\underline{\underline{V}}$ = the sets $\{0,\dots,n\}$, and for $V \in \underline{\underline{V}}$ define the "past" $\underline{\underline{F}}_V$, the "future" $\hat{\underline{\underline{F}}}_V$ and the "present" $\partial \underline{\underline{F}}_V$ as in example a) (with r = 1). Any Markov chain on E with transition matrix P(.,.) and fixed initial distribution μ , considered as a probability measure on Ω , is a Markov field with local characteristics

$$\pi_V(\omega, \{\xi\}) = \frac{\mu(\xi(0))\ P(\xi(0), \xi(1)) \dots P(\xi(n-1), \omega(n))}{\mu P^n(\omega(n))}$$

($V = \{0,\dots,n-1\}$; $\omega, \xi \in \Omega$; $\xi(t) = \omega(t)$ for $t \geq n$). Proposition (1.7) shows that the fields $P \in C(\Pi)$ may be identified with the non-negative normalized martingales $X = (X_V)$ such that each X_V is $\partial \underline{\underline{F}}_V$ -measurable, that is, of the form $X_{\{0,\dots,n-1\}}(\omega) = h(\omega(n), n)$ for some space-time function $h : E \times \{1,2,\dots\} \to R$. The martingale property of the process X is equivalent to the space-time harmonicity of the function h. Thus we have a phase transition as soon as P(.,.) admits space-time harmonic functions other than the constants. In particular we get many examples of a Markovian phase transition in one dimension as soon as $|E| = \infty$.

3. THE MARTIN BOUNDARY OF Π

Let $\Pi = (\pi_V)_{V \in \underline{\underline{V}}}$ be a collection of local characteristics over $(\Omega, \underline{\underline{F}})$ and suppose that $\underline{\underline{V}}$ has a countable base. We fix a polish topology on Ω compatible with $\underline{\underline{F}}$, and thereby a polish topology on the set $M(\Omega)$ of all probability measures on $(\Omega, \underline{\underline{F}})$ (cf. 11 Th. 6.5). Let $C_\infty = C_\infty(\Pi)$ be the set of all limits

(3.1) $$\lim_n \pi_{V_n}(\omega_n, .)$$

where (V_n) is some countable base of $\underline{\underline{V}}$ and (ω_n) some sequence in Ω. C_∞ is complete in $M(\Omega)$, and thus a polish space whose Borel field we denote by $\underline{\underline{C}}_\infty$; as shown below, our assumption $C \neq \emptyset$ will imply $C_\infty \neq \emptyset$. Let us call $(C_\infty, \underline{\underline{C}}_\infty)$ the <u>Martin boundary</u> of Π .

(3.2) REMARK. - In many applications there is a natural topology on Ω and the kernels π_V have the Feller property, that is $\pi_V \phi$ is continuous for bounded continuous ϕ . In that case it is easy to show $C_\infty \subseteq C$. If, as in (2.1), Ω is in addition compact then we have $C_\infty \neq \emptyset$ (without a priori assumptions on C), and in particular $C \neq \emptyset$. Moreover C_∞ is a compact space.

Let (V_n) be a countable base of $\underline{\underline{V}}$ and take $P \in C$. Then we have for any $\varphi \in L^1(P)$

(3.3) $$E[\varphi|\hat{\underline{\underline{F}}}_\infty] = \lim_n E[\varphi|\hat{\underline{\underline{F}}}_{V_n}] = \lim_n \int \pi_{V_n}(\cdot, d\omega)\varphi(\omega) \quad \text{P-a.s.}$$

where $\hat{\underline{\underline{F}}}_\infty$ denotes the <u>tail field</u> $\bigcap_V \hat{\underline{\underline{F}}}_V = \bigcap_n \hat{\underline{\underline{F}}}_{V_n}$. Letting φ run through a suitable sequence we obtain, P-a.s. , the existence of

(3.4) $$\rho(\omega) = \lim_n \pi_{V_n}(\omega, \cdot) \in C_\infty$$

(cf. [11] Th. 6.6). Denoting by P_δ resp. E_δ the measure resp. the expectations corresponding to $\delta \in C_\infty$, we can combine (3.3) and (3.4) into

(3.5) $$E[\varphi|\hat{\underline{\underline{F}}}_\infty](\omega) = E_{\rho(\omega)}[\varphi] \quad \text{P-a.s.}$$

For any bounded functions $\hat{\varphi}$, φ , $\hat{\varphi}_V$ which are, respectively, $\hat{\underline{\underline{F}}}_\infty$-, $\underline{\underline{F}}$-, $\hat{\underline{\underline{F}}}_V$-measurable we have

$$E[\hat{\varphi}E_{\rho(\cdot)}[\varphi\hat{\varphi}_V]] = E[\hat{\varphi}\varphi\hat{\varphi}_V] = E[\hat{\varphi}E[\varphi|\hat{\underline{\underline{F}}}_V]\hat{\varphi}_V]$$
$$= E[\hat{\varphi}E_{\rho(\cdot)}[E[\varphi|\hat{\underline{\underline{F}}}_V]\hat{\varphi}_V]]$$

and this implies

(3.6) $$P_{\rho(\cdot)} \in C \quad \text{P-a.s.}$$

Since

$$E[\hat{\varphi}f(\rho(\cdot))] = E[\hat{\varphi}E_{\rho(\cdot)}[f(\rho(\cdot))]]$$

for any bounded measurable function f on C_∞ , we also have

(3.7) $$P_{\rho(\omega)}[\rho(\cdot) = \rho(\omega)] = 1 \quad \text{P-a.s.}$$

In view of (3.6) and (3.7) let us call

$$\Delta := \{\delta \in C_\infty \mid \delta \in C\ , \quad P_\delta[\rho(.) = \delta] = 1\}$$

the essential part of the boundary. It is easy to see that Δ is a Borel set in C_∞ (if Π has the Feller property then it is even a G_δ-set), and thus $(\Delta, \underline{\underline{\Delta}})$ with $\underline{\underline{\Delta}} = \underline{\underline{C}}_\infty \cap \Delta$ is again a standard Borel space. From now on let us view $\rho(.)$ as a measurable map from Ω to Δ (due to (3.6) and (3.7) we can redefine it arbitrarily on $\{\rho(.)\notin\Delta\}$). By (3.5) we have

$$(3.8) \qquad \rho^{-1}(\underline{\underline{\Delta}}) = \hat{\underline{\underline{F}}}_\infty \qquad P\text{-a.s.}$$

Defining a probability measure μ^P on Δ through

$$(3.9) \qquad \mu^P(A) = P[\rho(.) \in A] \qquad (A \in \underline{\underline{\Delta}})$$

we may use (3.5) to write

$$(3.10) \qquad E[\varphi] = E[E_{\rho(.)}[\varphi]] = \int_\Delta E_\delta[\varphi]\ \mu^P(d\delta).$$

It is easy to check that, conversely, any probability measure μ^P on Δ induces a stochastic field $P \in C$ via (3.10). This implies in particular that P is extremal in C if and only if $P = P_\delta$ for some $\delta \in \Delta$. We have thus shown that C admits an integral representation

$$(3.11) \qquad P(.) = \int_\Delta P_\delta(.)\mu^P(d\delta)$$

which is coupled to the tail field in the sense that μ^P is obtained from the restriction of P to $\hat{\underline{\underline{F}}}_\infty$ via $\rho(.)$. To summarize:

(3.12) THEOREM. - The relations (3.9) and (3.11) establish a 1-1 correspondence between the stochastic fields in $C(\Pi)$ and the probability measures on the essential part Δ of the Martin boundary of Π, and in particular between Δ and the set of extreme points of $C(\Pi)$.

As corollaries we get two results which have been obtained independently by GEORGII [7] and PRESTON [12] . The first characterizes the phases of Π by a 0-1 law on the tail field. The second says that any phase can be approximated by measures of the form $\pi_V(\omega,.)$ - in analogy to the well known fact in potential theory that extremal harmonic functions can be approximated by normalized Green functions.

(3.13) COROLLARY. - A stochastic field $P \in C$ is an extreme point of C if and only if

$$P[A] \in \{0,1\} \qquad (A \in \hat{\underline{\underline{F}}}_\infty).$$

PROOF. - We have seen that $P \in C$ is a phase if and only if μ^P is a one-point measure, and this is equivalent to a 0-1 law for P on $\hat{\underline{\underline{F}}}_\infty$ due to (3.8).

(3.14) COROLLARY. - <u>Any extreme point of</u> C <u>belongs to</u> C_∞.

PROOF. - Immediate from our construction of Δ .

4. SPECIAL INTEGRAL REPRESENTATIONS

Let $\Pi = (\pi_V)_{V \in \underline{\underline{V}}}$ be a collection of local characteristics and assume that $\underline{\underline{V}}$ has a countable base. In many applications the real object of interest is not so much the set $C(\Pi)$ of all stochastic fields with local characteristics Π , but rather a suitable convex subset of $C(\Pi)$. We will shortly discuss two examples.

a) LOCAL ABSOLUTE CONTINUITY. - Suppose as in (1.7) that we have an increasing family of σ-fields $(\underline{\underline{F}}_V)_{V \in \underline{\underline{V}}}$ such that $\underline{\underline{F}} = \bigvee \underline{\underline{F}}_V$. Let us fix some $P \in C(\Pi)$ and consider the convex set $C_P(\Pi)$ of all those $Q \in C(\Pi)$ which are locally absolutely continuous with respect to P in the sense of (1.5). We assume that Π admits <u>reference measures</u> in the following sense: For each $V \in \underline{\underline{V}}$ there is a σ-finite measure m_V on $\underline{\underline{F}}_V$ such that

$$\text{(4.1)} \qquad \pi_W(\omega,.) \ll m_V \quad \text{on } \underline{\underline{F}}_V \qquad (\omega \in \Omega)$$

for some $W \in \underline{\underline{V}}$ with $W \subseteq V$. Since $P(.) = \int P(d\omega)\, \pi_W(\omega,.)$ we have

$$\text{(4.2)} \qquad P_V \ll m_V \quad (V \in \underline{\underline{V}}).$$

Let φ_V be a density of P_V with respect to m_V and define $A_V := \{\varphi_V = 0\}$.

(4.3) LEMMA. - <u>For any</u> $Q \in C(\Pi)$ <u>we have</u>

$$Q \in C_P(\Pi) \iff Q[A_V] = 0 \quad (V \in \underline{\underline{V}})$$

PROOF. - " $\Rightarrow$ " is clear since $P[A_V] = 0$. Now take a set $B \in \underline{\underline{F}}_V$ with

$$P[B] = \int_B \varphi_V \, dm_V = 0.$$

Then $B \subseteq A_V$ m_V-a.s., due to (4.1) we get

$$Q[B] = \int \pi_W(.,B)\,dQ \leq \int \pi_W(.,A_V)\,dQ = Q[A_V] ,$$

and this yields " $\Leftarrow$ " .

The lemma shows

$$Q \in C_P(\Pi) \iff \int \mu^Q(d\delta)\; P_\delta[A_V] = 0 \qquad (V \in \underline{\underline{V}})$$
$$\iff \mu^Q[\Delta - \Delta_P] = 0$$

where

$$\Delta_P := \{\delta\varepsilon\Delta \mid P_\delta[A_V] = 0 \quad (V \in \underline{V})\}$$

is $\underline{\underline{\Delta}}$ -measurable since $\underline{\underline{V}}$ has a countable base. To summarize:

(4.4) THEOREM. - The relations (3.9) and (3.11) establish a 1-1 correspondence between $C_P(\Pi)$ and the set of probability measures on Δ_P, and in particular between Δ_P and the set of extreme points of $C_P(\Pi)$.

b) INVARIANCE. - Suppose that we have a group $(\theta_t)_{t\in T}$ of measurable bijections $\theta_t\colon \Omega \longrightarrow \Omega$ which is compatible with Π in the sense that

$$(4.5) \qquad P\,\theta_t \in C(\Pi) \qquad (P \in C(\Pi),\quad t \in T)$$

and

$$(4.6) \qquad \underline{\underline{I}} \subseteq \underline{\underline{F}} \quad \text{mod } P \qquad (P \in I)$$

where $\underline{\underline{I}}$ is the σ-field of (θ_t)-invariant sets, I the set of (θ_t)-invariant probability measures and where $P\,\theta_t$ denotes the image of P under θ_t.

(4.7) REMARK. - If for $t \in T$ and $V \in \underline{\underline{V}}$ there is a $V_t \in \underline{\underline{V}}$ such that

$$(4.8) \qquad \hat{\underline{\underline{F}}}_{V_t} = \theta_t^{-1}(\hat{\underline{\underline{F}}}_V)$$

and

$$(4.9) \qquad \pi_V(\theta_t(\omega),.) = \pi_{V_t}(\omega,.)\,\theta_t$$

then it is easy to check (4.5). If, in addition, $\underline{\underline{F}} = \bigvee \underline{\underline{F}}_V$ as in a) and if for V, W $\in \underline{\underline{V}}$ there is a $t \in T$ such that

$$(4.10) \qquad \theta_t^{-1}(\underline{\underline{F}}_W) \subseteq \underline{\underline{F}}_V$$

then (4.6) follows as well; cf. [7] for a proof. In the situation of example 2a) above it is natural to take $T = Z^d$ and θ_t as the shift transformation defined through $(\theta_t\omega)(s) = \omega(t+s)$. Then (4.8) and (4.10) are satisfield, and (4.9) boils down to the condition that the potential function $U = (U_V)$ is translation invariant in the sense that $U_V(\omega_V) = U_{V+t}(\omega_{V+t})\circ\theta_t \quad (V \in \underline{\underline{V}},\ t \in T,\ \omega\varepsilon\Omega)$.

We denote by $C_o = C_o(\Pi)$ the convex set $C(\Pi)$ I. Let us assume $C_o \neq \emptyset$; for a sufficient condition combine (4.7) with (3.2). For any phase P_δ the measure $P_\delta\theta_t$ is again a phase, and so it corresponds to some point $\theta_t(\delta)$. Thus (θ_t) induces a group of transformations on Δ

which we denote again by (θ_t). For each $P \varepsilon C(\Pi)$ with representing measure μ^P we may then write

$$P\theta_t[A] = P[\theta_t^{-1}(A)] = \int \mu^P(d\delta) P_\delta[\theta_t^{-1}(A)]$$
$$= \int \mu^P(d\delta) P_{\theta_t(\delta)}[A] = \int \mu^P \theta_t(d\delta) P_\delta[A] \qquad (A \varepsilon \underline{\underline{F}}).$$

Thus $\mu^P \theta_t$ is the unique representing measure for $P\theta_t$ in the sense of (3.12). In particular we have $P \varepsilon C_o$ if and only if μ^P is (θ_t)-invariant. We are now going to invoke the ergodic theorem, and so we have to be more specific about $(\theta_t)_{t \varepsilon T}$. In order to simplify the exposition, let us just consider the case where $T = Z^d$ and $\theta_t \circ \theta_s = \theta_{t+s}$ as in example 2a above; cf. (4.7). Writing $T_n = \{t \varepsilon T \mid \|t\| \leq n\}$ we can apply WIENER's version of the ergodic theorem and obtain

$$(4.11) \qquad E[\varphi \mid \underline{\underline{I}}] = \lim_n \frac{1}{|T_n|} \sum_{t \varepsilon T_n} \varphi \circ \theta_t \qquad P\text{-a.s.}$$

for any bounded $\underline{\underline{F}}$-measurable φ and for any $P \varepsilon I$ ([3]VIII.6.9). Now we assume $P \varepsilon C_o(\Pi)$ and use (4.6) and (3.5) to conclude

$$E[\varphi \mid \underline{\underline{I}}] = E[E[\varphi \mid \underline{\underline{I}}] \mid \hat{\underline{\underline{F}}}_\infty] = \lim_n E[\frac{1}{|T_n|} \sum_{t \varepsilon T_n} \varphi \circ \theta_t \mid \hat{\underline{\underline{F}}}_\infty]$$
$$= \lim_n \frac{1}{|T_n|} \sum_{t \varepsilon T_n} E_{\theta_t(\rho(.))}[\varphi] \qquad P\text{-a.s.}$$

This implies the existence of

$$(4.12) \qquad \mu(\delta) := \lim \frac{1}{|T_n|} \sum_{t \varepsilon T_n} P_{\theta_t(\delta)}$$

in $M(\Omega)$ for μ^P-almost all $\delta \varepsilon \Delta$. Proceeding in exact analogy to the proof of (3.12) we can now construct a polish space $(\Delta_o, \underline{\underline{\Delta}}_o)$ parametrizing a certain subset of $C_o(\Pi)$ and redefine λ as a measurable map from Δ to Δ_o such that the map $\rho_o = \lambda \circ \rho$ satisfies

$$(4.13) \qquad \rho_o^{-1}(\underline{\underline{\Delta}}_o) = \underline{\underline{I}} \quad \text{mod } P$$

$$(4.14) \qquad E[\varphi \mid \underline{\underline{I}}] = E_{\rho_o(.)}[\varphi] \qquad P\text{-a.s.}$$

for any $P \varepsilon C_o(\Pi)$. Writing

$$E[\varphi] = E[E[\varphi \mid \underline{\underline{I}}]] = E[E_{\rho_o}[\varphi]] = \int_{\Delta_o} \mu_o^P(d\delta_o) E_{\delta_o}[\varphi]$$

with

$$(4.15) \qquad \mu_o^P[A] := \mu^P[\lambda^{-1}(A)] = P[\rho_o^{-1}(A)] \qquad (A \varepsilon \underline{\underline{\Delta}}_o)$$

we obtain an <u>integral representation</u>

(4.16) $$P(.) = \int_{\Delta_o} \mu_o^P (d\delta_o) P_{\delta_o}(.)$$

for $C_o(\Pi)$ which is coupled to the invariant σ-field $\underline{\underline{I}}$ via the map ρ_o . To summarize:

(4.17) THEOREM. - (4.15) and (4.16) establish a 1-1 correspondence between stochastic fields in $C_o(\Pi)$ and probability measures on Δ , and in particular between Δ and the set of extreme points of $C_o(\Pi)$.

As corollaries we obtain two more results of GEORGII [7] resp. PRESTON [12] .

(4.18) COROLLARY. - A stochastic field $P \in C_o$ is an extreme point of C_o if and only if

$$P[A] \in \{0,1\} \qquad (A \in \underline{\underline{I}}) ,$$

that is, if and only if P is ergodic.

PROOF. - As for (3.13) with $\underline{\underline{I}}$ instead of $\hat{\underline{\underline{F}}}_\infty$.

(4.19) COROLLARY. - Any extreme point P of C_o can be represented in the form

$$P = \lim_n \frac{1}{|T_n|} \sum_{t \in T_n} \pi_{V_n}(\omega,.)\theta_t$$

for P-almost all ω and for any countable base (V_n) of $\underline{\underline{V}}$.

PROOF (GEORGII [7]). - If $P \in C_o$ is extremal and φ $\underline{\underline{F}}$-measurable and bounded then

$$\begin{aligned} E[\varphi] &= E[\varphi | \underline{\underline{I}}] = E[E[\varphi | \underline{\underline{I}}] | \hat{\underline{\underline{F}}}_\infty] \\ &= \lim_n E[\frac{1}{|T_n|} \sum_{t \in T_n} \varphi \circ \theta_t | \hat{\underline{\underline{F}}}_{V_n}] \\ &= \lim_n \frac{1}{|T_n|} \sum_{t \in T_n} \int \varphi \, d\pi_{V_n}(\omega,.)\theta_t \end{aligned}$$

where we use (4.18) in the first step, (4.6) in the second, and (4.11) combined with HUNT's extension of the martingale convergence theorem (cf. [9]) in the third. This implies (4.19).

P.S. The whole point of this paper was to show that DYNKIN's approach to Martin boundary theory applies very naturally to stochastic fields. In fact DYNKIN has just pointed out to me that part of the argument in two of his articles subsequent to [4] applies directly, even without the straightforward modification required by [4]; cf. Uspehi Mat. Nauk vol.26, no.4 (1971) and vol.27, no.1 (1972). C. PRESTON has taken up the above situation in chapter 10 of his "Notes on Random Fields" (to appear). In particular he clarifies in terms of tightness the existence problem mentioned in 3.2.

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