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Existence of Small Oscillations at Zeros of Brownian Motion

by Frank B. Knight

0. Introduction. Let $X(t)$, $X(0) = 0$, be a standard one-dimensional Brownian motion, with zero-set $Z = \{0 \leq t \leq 1 : X(t) = 0\}$. Many properties of Z are known, in the sense that they hold with probability 1. For example, Z is a closed uncountable set of Hausdorff dimension $\frac{1}{2}$ [2, 2.5]. If one asks, however, for the conditional behavior of $X(t+h)$ given that $0 < t \in Z$ one encounters the difficulty that, since $P\{t \in Z\} = 0$, the conditioning has no meaning. To be sure if $t = T(w) \in Z$ is a stopping time, then the strong Markov property implies various results, of which the most relevant here is the well-known Local Law of the Iterated Logarithm [6, VI, 51.1]: Set $\Delta_h X(t) = X(t+h) - X(t)$ and $\varphi_2(h) = (h \log \log 1/h)^{1/2}$. Then $P\{\limsup_{h \to 0+} \Delta_h X(T)(\sqrt{2}\, \varphi_2(h))^{-1} = \limsup_{h \to 0+} - \Delta_h X(T)(\sqrt{2}\, \varphi_2(h))^{-1} = 1\} = 1$. There are, however, many (random) $t \in Z$ at which this behavior does not hold, and such t we shall term "exceptional." The object of this paper is to study one type of exceptionality which occurs with probability 1.(1)

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(1) It will be noted that such a type of exceptionality will be represented not only in Z but also at all x in the range of $X(t)$ outside a random set of Lebesgue measure 0. It then follows directly from P. Lévy's modulus of continuity for $X(t)$ [6, VII, 52] that the overall exceptional set has Hausdorff dimension $\geq \frac{1}{2}$ (for this observation I am indebted to Professor N. Jain).

Our main result is to show that there exist $t \in Z$ with $\limsup_{h \to 0+} |\Delta_h X(t)|(\sqrt{2}\,\varphi_2(h))^{-1} < 1$. This gives a partial answer to a question of A. Dvoretzsky [1] which remained unanswered in [9] (without the added information that $t \in Z$). To do this we rely upon a result of B. Mandelbrot [7] and L. Shepp [10]. At the same time, our analysis seems to indicate that there do not exist $t \in Z$ for which $\limsup_{h \to 0+} |\Delta_h X(t)|(\varphi_2(h))^{-1} = 0$. Consequently, if such t exist they must be sought elsewhere than in the set where $X(t)$ has a prescribed value.

Before turning to this result, let us remark upon a type of exceptionality which is quite well understood. A time $t \in Z$ is said to be the starting time of an excursion of $X(t)$ if $X(t+h) \neq 0$ for $0 < h < \varepsilon$ sufficiently small. There are countably many such t, and for all of them the behavior of $\Delta_h X(t)$ is adequately covered by [2, 2.10]. Assuming, as we may, that $X(t+h) > 0$, we have $\limsup_{h \to 0+} \Delta_h X(t)(\sqrt{2}\,\varphi_2(h))^{-1} = 1$ as in the unexceptional case, but also $\liminf_{h \to 0+} \Delta_h X(t) h^{-\frac{1}{2}}(\log 1/h)^{(1+\varepsilon)} > 1$ for $\varepsilon > 0$, in radical contrast with the normal behavior for $-\Delta_h X(t)$. We see immediately that there cannot exist a stopping time T which equals the starting time of an excursion with positive probability.(2)

(2) Another set of exceptional times is of course the set of local maxima and minima. Being countable, however, it does not intersect Z. The behavior of X(t) following such an extremum is entirely analogous to that at the start of an excursion. This is easily seen from P. Levy's equivalence $|X(t)| = M(t) - X(t)$ where $M(t) = \max_{s \leq t} X(s)$. Moreover, by an evident reversal of time most of this exceptional behavior holds in both time directions. In short, the path exhibits a dense set of spine-like projections of sharpness exceeding $\sqrt{|h|}\,(\log \frac{1}{|h|})^{-(1+\varepsilon)}$ for every $\varepsilon > 0$.

1. Exceptional Small Oscillations at $t \in Z$.

We introduce the standard local time $f(t)$ of $X(t)$ at 0 using the indicator function $I_{(-\infty,x)}$ of $(-\infty,x)$:

$$(1.1) \qquad f(t) = \frac{1}{2}\frac{d}{dx}\left[\int_0^t I_{(-\infty,x)}(X(s))ds\right]_{X=0}.$$

The existence and continuity in t of $f(t)$ is a well-known result of P. Lévy (see [2]). The exact statement of our result is as follows.

Theorem 1.1. $P\{\exists t_0 \in Z: \lim_{h\to 0+}\sup|X(t_0+h)|(\varphi_2(h))^{-1} < k\} = 1$, for all $k > 2^{-\frac{1}{2}}$.

Proof. The key to the proof lies in the observation that if the oscillations of $|X(t)|$ above 0 are recorded as a function of the local time $f(t)$ they generate a homogeneous Poisson point process of the type considered in [10].

Definition 1.1. Let $f^{(-1)}(\alpha) = \inf\{t: f(t) > \alpha\}$ be the right-continuous inverse local time at 0, and let $\mathcal{A}(\alpha,\alpha+\varepsilon) = \max_{f^{(-1)}(\alpha)<t<f^{(-1)}(\alpha+\varepsilon)} |X(t)|$, $0 \leq \alpha < \alpha + \varepsilon$.

Lemma 1.1. The random set $\Gamma = \{(\alpha,y): \lim_{\varepsilon\to 0+} \mathcal{A}(\alpha-\varepsilon,\alpha) = y > 0\}$ is a homogeneous Poisson point process with parameter $\alpha \geq 0$ and expectation measure $\lambda \times \mu$ where λ is Lebesgue measure and $\mu(A) = \int_A 2y^{-2}dy$ on $\{y: 0 < y < \infty\}$.

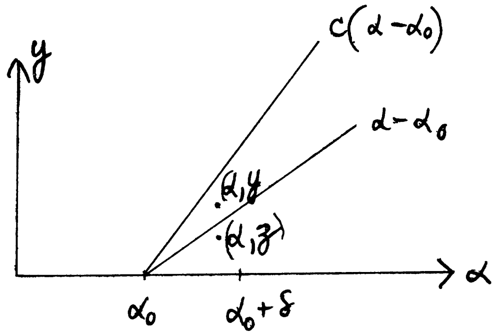
Proof. Since $f^{(-1)}(\alpha)$ is a homogeneous process with independent increments and a stopping time of $X(t)$ for each α, with $X(f^{(-1)}(\alpha)) = 0$, it is clear that Γ is a homogeneous Poisson point process. Taking into account the independence of the local times for $x > 0$ and for $x < 0$ up to time $f^{(-1)}(\alpha)$ and the fact that $P\{M(f^{(-1)}(\alpha)) < z\} = \exp - \frac{\alpha}{z}$, known from [3, Theorems 1.2 and 2.2, or 11, Proposition 2.4], we have $P\{\mathcal{A}(0,\alpha) < z\} = \exp - \frac{2\alpha}{z}$. In view of $\frac{2\alpha}{z} = \alpha \int_z^\infty 2y^{-2}dy$ this implies the result.

The following lemma is now a direct consequence of [7].

Lemma 1.2. $P\{\exists\alpha_0$ with $f^{(-1)}(\alpha_0) \leq 1$ and $\mathcal{A}(\alpha_0,\alpha_0+\varepsilon) < c\varepsilon$, $0 < \varepsilon < \delta$ for some $\delta > 0\} = 1$ for $c > 2$.

Proof. The property $\mathcal{A}(\alpha_0,\alpha_0+\varepsilon) < c\varepsilon$, $0 < \varepsilon < \delta$, may be stated as saying that α_0 is not covered by the union of open intervals $(\alpha-z,\alpha)$ generated by the truncated Poisson process $\{(\alpha,z): z = \frac{y}{c} \wedge \delta, (\alpha,y) \in \Gamma\}$ where $\wedge$ denotes minimum.

Fig. 1



This process has mean density $2c^{-1}y^{-2}$; $y < c\delta$, with a point mass at $y = \delta$ of size $2(c\delta)^{-1}$. The result of [10, (40)] or [7], states that such an α_0 exists with positive probability if and only if $\int_0^\delta (\exp \int_x^\delta 2(cy)^{-1}dy)dx < \infty$, where

$2(cy)^{-1} = \int_y^\delta 2c^{-1}z^{-2}dz + 2(c\delta)^{-1}$ is the expectation measure in $[y,\infty)$. The condition is equivalent to $\int_0^1 x^{-\frac{2}{c}}\,dx < \infty$, i.e., to $c > 2$. Routine use of the scale change $X(t) \equiv k^{-\frac{1}{2}} X(kt)$ shows that $\mathcal{A}(\alpha,\alpha+\varepsilon) \equiv k^{-\frac{1}{2}}\mathcal{A}(k^{\frac{1}{2}}\alpha,k^{\frac{1}{2}}(\alpha+\varepsilon))$, and letting $k \longrightarrow 0$ we may allow $\alpha \longrightarrow \infty$ and apply the 0-1 Law to get the probability 1 as required.

The exceptional t_0 of Theorem 1.1 is essentially $t_0 = f^{(-1)}(\alpha_0)$, but to derive the result directly would involve giving a meaning to the process $X(f^{(-1)}(\alpha_0)+h)$, which is problematical. Instead, we introduce the space $\Omega' = [0,\infty) \times \Omega$, where Ω is the sample space of $X(t)$, and define the conditioning sequentially in such a way that it may be applied at a constant $\alpha = 0$. We then argue that the projection of the limit set in Ω' has positive probability in Ω, and therefore the additional condition of Theorem 1.1 is met at some $t_0 = f^{(-1)}(\alpha_0)$.

Turning to the details, let δ, c_0 and $\rho < 1$ be positive constants, $0 < r < s$ and n be integers, and consider the subset of Ω'

$$S'(n,r,s) = S'(n) \cap M'(r,s); \tag{1.2}$$

$$S'(n) = \{(\alpha,w): f^{(-1)}(\alpha) \leq 1\, \mathcal{A}(\alpha,\alpha+k2^{-n}\delta) \leq ck2^{-n}\delta, 1 \leq k < 2^n\}$$

$$M'(r,s) = \{(\alpha,w): \max_{f^{(-1)}(\alpha)<t<f^{(-1)}(\alpha)+\rho^m\delta} |X(t)| \leq c_0\varphi_2(\rho^m\delta), r \leq m \leq s\}.$$

Furthermore, let $\Phi(S') = \{w:(\alpha,w) \in S'$ for some $\alpha \geq 0\}$ denote the projection onto Ω. The proof rests in showing that, for suitable c_o, c, δ, and r,

(1.3) (a) $$\lim_{s\to\infty} \lim_{n\to\infty} P(\Phi(S'(n,r,s))) > 0, \text{ and}$$

(b) $$\lim_{s\to\infty} \lim_{n\to\infty} \Phi(S'(n,r,s)) = \Phi(\lim_{s\to\infty} \lim_{n\to\infty} S'(n,r,s)) .$$

Indeed, it is clear that

(1.4) $$\lim_{s\to\infty} \lim_{n\to\infty} S'(n,r,s) = \{(\alpha,w) \in \Omega' : f^{(-1)}(\alpha) \leq 1;$$

$$\mathcal{A}(\alpha,\alpha+\varepsilon) \leq c\ \varepsilon,\ 0 < \varepsilon < \delta, \text{ and}$$

$$\max_{f^{(-1)}(\alpha)<t<f^{(-1)}(\alpha)+\rho^m\delta} |X(s)| \leq c_o\varphi_2(\rho^m\delta),\ r \leq m < \infty\} .$$

Since φ_2 is increasing this will imply the result when $c_o \sim 2^{-\frac{1}{2}}$ and $(1-\rho) \sim 0$, for in view of (b) the set of w for which there exists an exceptional $t_o = f^{(-1)}(\alpha_o)$ will have positive probability (the scale change used in Lemma 1.2 again shows easily that the probability must be 0 or 1).

The first step in proving (a) is

<u>Lemma 1.3</u>. $P\{\Phi(S'(n,r,s))\} \geq P\{\Phi(S'(n))\} \times$
$P\{(0,w) \in M'(r,s) | (0,w) \in S'(n)\}$.

Proof. We set $\alpha_n = \inf\{\alpha:(\alpha,w) \in S'(n)\}$ if this is non-null and $\alpha_n = f(1) + 1$ otherwise. Although α_n is not a stopping time, we can reduce it to stopping time on the set $\{\alpha_n \leq f(1)\} = \{(\alpha_n,w) \in S'(n)\}$. On this set, either $\alpha_n = 0$ or else α_n is the local time of an excursion of $X(t)$ such that $\mathcal{A}(\alpha_n-,\alpha_n) > c2^{-n}\delta$. To see this, note that if $0 < \alpha_n \leq f(1)$ then for $\alpha < \alpha_n$ we have $\mathcal{A}(\alpha,(\alpha+k2^{-n}\delta) \wedge \alpha_n) > ck2^{-n}\delta$ for some $k < 2^n$, and the assertion follows as α increases to α_n. The set of $f^{(-1)}(\alpha)$ with $\mathcal{A}(\alpha-,\alpha) > c2^{-n}\delta$ is contained in the sequence $T_1,\ldots,T_n,\ldots$ of stopping times $T_1 = \inf\{t:X(t) = 0$ and $\max_{0<s<t} |X(s)| > c2^{-n}\delta\}$, $T_{n+1} = T_n + T_1 \circ \theta_{T_n}$, where θ_t is the usual translation operator. Setting $T_o = 0$ and using the strong Markov property, we have

$$P\{\Phi(S'(n,r,s))\} \geq \sum_{k=0}^{\infty} P\{(\alpha_n,w) \in S'(n,r,s),\alpha_n = f(T_k)\}$$

$$= P\{\Phi(S'(n))\}P\{(0,w) \in M'(r,s)|(0,w) \in S'(n)\},$$

as required.

The next step is to obtain an estimate of the above conditional probability. The analytical content is contained in

Lemma 1.4. For $\beta > 0$, $x > 0$, $K > 0$ and large r,

$$\lim_{n\to\infty} P\{f^{(-1)}(\beta\varphi_2(\rho^m\delta)) < x\rho^m\delta|(0,w) \in S'(n)\}$$
$$\leq |\log \rho^m\delta|^{Kx-2\beta\sqrt{2K}} (c\beta\sqrt{2K} \log|\log \rho^m\delta|)^{\frac{c}{2}}.$$

Proof. Given $(0,w) \in S'(n)$ the increments $f^{(-1)}(j2^{-n}\delta) - f^{(-1)}((j-1)2^{-n}\delta)$ remain independent, $1 \leq j < n$, and their conditional distribution is the same as that of $f^{(-1)}(2^{-n}\delta)$ given that $\max_{0<t<f^{(-1)}(2^{-n}\delta)} |X(t)| < cj2^{-n}\delta$. The Laplace transform of this conditional distribution is readily obtained from [4, Theorem 2.1], in which we set $\alpha = 2^{1-n}\delta$, $a = cj2^{-n}$, and square the result since the f of [4] is twice the present f and the sojourns in $(0,a)$ and $(-a,0)$ are independent.(3) Multiplying from $j = 1$ to k we obtain

$$(1.5) \qquad E(\exp - \lambda f^{(-1)}(k2^{-n}\delta) \,|\, (0,w) \in S'(n))$$
$$= \exp \sum_{j=1}^{k} \left(\frac{2}{cj} - 2^{1-n}\delta\sqrt{2\lambda}\, \operatorname{cotanh}(cj2^{-n}\delta\sqrt{2\lambda})\right).$$

Letting $k2^{-n}\delta = \varepsilon$ remain fixed as $n \longrightarrow \infty$ the exponent becomes

$$\lim_{n\to\infty} 2^{-n}\delta \sum_{j=1}^{k} \left(\frac{2^{n+1}}{cj\delta} - 2\sqrt{2\lambda}\, \operatorname{cotanh} cj2^{-n}\delta\sqrt{2\lambda}\right)$$
$$= \lim_{\varepsilon'\to 0} \int_{\varepsilon'}^{\varepsilon} \left(\frac{2}{cx} - 2\sqrt{2\lambda}\, \operatorname{cotanh} c\sqrt{2\lambda}\, x\right)dx$$
$$= -\frac{2}{c}(\log(\varepsilon^{-1}\sinh \varepsilon c\sqrt{2\lambda}) - \lim_{\varepsilon'\to 0}(\log c\sqrt{2\lambda} + o(\varepsilon')))$$
$$= -\frac{2}{c}\log((\varepsilon c\sqrt{2\lambda})^{-1}\sinh \varepsilon c\sqrt{2\lambda}),$$

and so the Laplace transform converges to

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(3)The "check" on p. 179 of [4] has a mistaken integrand. It should be $\exp - (\alpha\sqrt{2\lambda}\, \operatorname{cotanh} a\sqrt{2\lambda})$.

$$(1.6) \qquad ((\varepsilon c\sqrt{2\lambda})(\sinh \varepsilon c\sqrt{2\lambda})^{-1})^{+\frac{2}{c}} .$$

Accordingly, the conditional distributions converge weakly, and the limits may be bounded by using $P\{R < k\} < e^{\lambda k} Ee^{-\lambda R}$, valid for any positive random variable R and $\lambda > 0$. Setting $\varepsilon = \beta\varphi_2(\rho^m\delta)$, $k = x\rho^m\delta$, and $\lambda = K(\rho^m\delta)^{-1}\log|\log \rho^m\delta|$, we have $\lambda k = Kx \log|\log \rho^m\delta|, \varepsilon c\sqrt{2\lambda} = c\beta\sqrt{2K} \log|\log \rho^m\delta|$, and using the fact that for large values of the argument we may replace $\sinh(\cdot)$ by $\frac{1}{2}\exp(\cdot)$ in (1.6), for large r and $m > r$ we obtain the required upper bound of Lemma 1.4.

To continue, let $\varepsilon' > 0$ be fixed and note that $\lim_{n\to\infty} P\{|X(t)| \leq cf(t) + \varepsilon', 0 < f(t) < \delta | (0,w) \in S'(n)\} = 1$, as follows from $|X(t)| \leq \mathcal{A}(0,f(t))$ in view of (1.4). On the other hand, since the limit distribution with transform (1.6) is concentrated near 0 for small ε, we have for any $\delta' > 0$ and $m > r$ sufficiently large, $\lim_{n\to\infty} P\{\rho^m\delta < f(\delta) | (0,w) \in S'(n)\} > 1-\delta'$. Therefore

$$(1.7) \qquad \lim_{n\to\infty} P\{\bigcup_{m=r}^{s}(\max_{0<t<\rho^m\delta} |X(t)| > c_0\varphi_2(\rho^m\delta)) | (0,w) \in S'(n)\}$$

$$\leq \lim_{n\to\infty} P\{\bigcup_{m=r}^{s}(cf(t) + \varepsilon' \geq |X(t)|, 0 < t < \rho^m\delta,$$

$$\text{and } \max_{0<t<\rho^m\delta} |X(t)| > c_0\varphi_2(\rho^m\delta)) | (0,w) \in S'(n)\} + \delta' .$$

Thus if we set $T(m) = \inf\{t : cf(t) > c_o' \varphi_2(\rho^m \delta)\}$ for $c_o' < c_o$, and let $\varepsilon' \longrightarrow 0$, (1.7) will be bounded by

$$\lim_{n \to \infty} P\{\bigcup_{m=r}^{s} (T(m) < \rho^m \delta \text{ and } \max_{T(m) < t < \rho^m \delta} |X(t)| > c_o \varphi_2(\rho^m \delta)) \,|\, (0,w) \in S'(n)\} + \delta'.$$

Next, since $T(m)$ is a stopping time and $X(T(m)) = 0$, this limit is seen to be bounded by

$$(1.8) \qquad \lim_{n \to \infty} \sum_{m=r}^{s} \int_0^1 P\{\max_{0 < t < \rho^m \delta (1-x)} |X(t)| > c_o \varphi_2(\rho^m \delta)\} dF_{m,n}(x) + \delta',$$

where $F_{m,n}(x)$ is the conditional distribution function of $T(m)(\rho^m \delta)^{-1}$ given $\{(0,w) \in S'(n)\}$. Here we have $F_{m,n}(x) \leq P\{cf(x\rho^m \delta) > c_o' \varphi_2(\rho^m \delta) \,|\, (0,w) \in S'(n)\} \leq P\{f^{(-1)}(\frac{c_o'}{c} \varphi_2(\rho^m \delta)) < x\rho^m \delta \,|\, (0,w) \in S'(n)\}$. In applying Lemma 1.4 we may simply set $c_o' = c_o$ since the bound is continuous. Moreover, for large m the last factor may be absorbed by an arbitrarily small increase in the exponent $Kx - 2\beta\sqrt{2K}$, where $\beta = \frac{c_o}{c}$.

As for the integrand in (1.8), we use the standard inequality

$$(1.9) \qquad P\{\max_{0 < s < t} |X(s)| > k\} \leq 4P\{X(t) > k\} \leq \left(\frac{4}{k}\sqrt{\frac{t}{2\pi}}\right) \exp - \frac{k^2}{2t},$$

where the first factor on the right will be small for large m and may be replaced by unity. It follows from this and the weak convergence of the distributions in Lemma 1.4 that (1.8) is bounded by

$$(1.10)\quad \sum_{m=r}^{s} \int_0^1 \exp - \frac{c_o^2\varphi_2^2(\rho^m\delta)}{2\rho^m\delta(1-x)}\, d_x(|\log \rho^m\delta|^{Kx - \frac{2c_o}{c}\sqrt{2K}})$$

$$= \sum_{m=r}^{s} K \log|\log \rho^m\delta| \int_0^1 |\log \rho^m\delta|^{(-\frac{c_o^2}{2(1-x)} + Kx - \frac{2c_o}{c}\sqrt{2K})} dx + \delta'.$$

Now for given K the exponent is maximized at $x = 1 - c_o(\frac{1}{2K})^{+\frac{1}{2}}$, where it becomes $K - K^{\frac{1}{2}}c_o(\sqrt{2} + \frac{2^{3/2}}{c})$. We can easily minimize this over $K > 0$ to obtain the value $E(c_o) = -\frac{c_o^2}{4}(2 + \frac{8}{c} + \frac{8}{c^2})$. If we choose c_o to make this less than -1, then the integrals in (1.8) are of the order $m^{E(c_o)}$, which is the general term of a convergent series.

By choosing $c - 2$ small, this may be accomplished for any $c_o > \frac{1}{\sqrt{2}}$. Recalling that δ' in (1.7) does not depend on s we can then let $s \longrightarrow \infty$ and (1.7) will be strictly less than 1 if r is large. In view of Lemmas 1.3 and 1.2 this proves property (1.3), (a): $\lim_{s\to\infty} \lim_{n\to\infty} P(\Phi(S'(n,r,s))) > 0$, for any $c_o > \frac{1}{\sqrt{2}}$ when c and r are suitably chosen.

It remains only to prove (1.3), (b). The inclusion from right to left is obvious. Conversely, let $w \in \lim_{s\to\infty} \lim_{n\to\infty} \Phi(S'(n,r,s))$, and let $(w,\alpha_{n,s}) \in S'(n,r,s)$ for each (n,s). Keeping s fixed and choosing a subsequence we may assume that $\lim_{n\to\infty} \alpha_{n,s} = \alpha_s$ exists. We will show that $\lim_{n\to\infty} f^{(-1)}(\alpha_{n,s}) = f^{(-1)}(\alpha_s)$. In the contrary case, α_s would be the local time of an excursion of $X(t)$, and $\alpha_{n,s} < \alpha_s$ would hold for infinitely many n. This would contradict the definition of $S'(n,r,s)$ since $\mathcal{A}(\alpha_s-,\alpha_s) > 0$ is impossible when $\mathcal{A}(\alpha_{n,s},\alpha_{n,s} + k2^{-n}\delta) \leq ck2^{-n}\delta$, $1 \leq k < 2^n$, for $0 < \alpha_s - \alpha_{n,s}$ sufficiently small. It thus follows from the definitions that $(w,\alpha_s) \in \lim_{n\to\infty} S'(n,r,s)$. Similarly, let $\lim_{s\to\infty} \alpha_s = \alpha$ exist along a subsequence. Then $\lim_{s\to\infty} f^{(-1)}(\alpha_s) = f^{(-1)}(\alpha)$ in view of (1.4), and so $(w,\alpha) \in \lim_{s\to\infty} \lim_{n\to\infty} S'(n,r,s)$. This implies the result.

A very slight change in this proof also shows the existence of two-sided exceptional times.

Corollary 1.2. $P\{\exists t_0 \in Z: \limsup_{h\to 0+} |X(t_0+\varepsilon_1)-X(t_0-\varepsilon_2)|(\varphi_2(h))^{-1} < k$, $0 < \varepsilon_1,\varepsilon_2;\ \varepsilon_1 + \varepsilon_2 = h\} = 1$ for all $k > \frac{4}{3}$.

Remark. It is shown in [12] that for $t > 0$ $P\{\limsup_{h\to 0+} |X(t+\varepsilon_1)-X(t-\varepsilon_2)|(\varphi_2(h))^{-1} = \sqrt{2}\} = 1$ for t fixed. Since $\frac{4}{3} < \sqrt{2}$ the t_0 obtained above is exceptional.

Proof. The argument of Lemma 1.2 also shows that $P\{\exists\alpha_0$ with $f^{(-1)}(\alpha_0) \leq 1$, and both $\mathcal{A}(\alpha_0,\alpha_0+\varepsilon) < c\varepsilon$ and $\mathcal{A}(\alpha_0-\varepsilon,\alpha_0) < c\varepsilon$, $0 < \varepsilon < \delta\} = 1$ for $c > 4$. Indeed, this is equivalent to α_0 not being covered by the intervals $(\alpha-z,\alpha+z)$, and by the homogeniety of the Poisson process this is equivalent to replacing z

by $2z$. The mean density is then $4c^{-1}y^{-2}$ and the integral converges for $c > 4$. Since the problem only involves the increments of $X(t)$ we can assign to $X(0)$ a uniform initial measure on $(-\infty,\infty)$ and obtain a stationary process, $-\infty < t < \infty$. Then the same proof given above, but with $c > 4$, applies both to $X(t_o+\varepsilon_1) - X(t_o)$ and to $X(t_o-\varepsilon_2) - X(t_o)$. The condition that $E(c_o) < -1$ becomes $c_o > \frac{2}{3}\sqrt{2}$, and since $\varphi_2(\varepsilon_1) + \varphi_2(\varepsilon_2) < \sqrt{2}\,\varphi_2(h)$ when $h = \varepsilon_1 + \varepsilon_2$ is small (as is not difficult to show) we obtain the constant $\frac{4}{3}$. The Corollary is proved.

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