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JOHN C. TAYLOR

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ON THE EXISTENCE OF RESOLVENTS ¹⁾

J.C. Taylor

Introduction. In [6] a measure-theoretic theorem is proved which gives a sufficient condition for a proper kernel V on a measure space $(E, \underline{E})$ to be the kernel V_0 defined by a sub-Markovian resolvent.

The theorem of Lion is an immediate corollary of this result. It states that, for E locally compact and σ -compact, if V satisfies the complete maximum principle and maps $\underline{C}_c(E)$ into $\underline{C}_0(E)$ then there is a sub-Markovian resolvent (V_λ) on $\underline{C}_0(E)$ with $V = V_0$ on $\underline{C}_c(E)$.

F. Hirsch in [2], by using different methods from those of Lion in [3], has shown that the theorem of Lion is valid for arbitrary locally compact spaces. The purpose of this note is to obtain a generalisation of theorem 2 in [6] which has as corollary 3.1 the above result of Hirsch. In addition, it has as corollary 3.3 the result of Mokobodzki and Sibony in [5] (without the restriction, imposed in [6], that E be σ -compact) and as corollary 3.2 an extension of Hirsch's result to kernels satisfying the domination principle.

The generalised theorem. Let E be a set and let $\underline{E}$ be a σ -ring of subsets of E (i.e. $\underline{E}$ is closed under countable unions and relative complements). A function $f : E \rightarrow \bar{\mathbb{R}}^+$ is measurable if for all $\alpha > 0$, $\{f > \alpha\} \in \underline{E}$. It will be said to be locally measurable if for each $X \in \underline{E}$, $f|X$ is measurable in the usual

¹⁾ This work was done while the author was a visiting professor at the Mathematisches Institut der Universität Erlangen-Nürnberg.

sense with respect to the σ -field of sets of the form $A \cap X$, $A \in \underline{E}$. A kernel on $(E, \underline{E})$ is an additive, increasing, positively homogeneous map $V : \underline{E}^+ \rightarrow \underline{E}^+$ such that if $(f_n) \subset \underline{E}^+$ is decreasing to f then (Vf_n) decreases to Vf (here $\underline{E}^+$ denotes the convex cone of non-negative measurable functions).

Let E be locally compact. In [1] the σ -ring $\underline{B}_0$ of Baire sets is defined to be the smallest σ -ring containing $\underline{H}$, the class of compact G_δ -subsets of E .

It is shown in [1] that, every $A \in \underline{B}_0$ is a subset of a countable union of sets from $\underline{H}$.

Remark. It is not hard to see that if $K_1, K_2 \in \underline{H}$ then $K_1 \setminus K_2$ is a countable union of sets from $\underline{H}$ i.e. is in $\underline{H}_\sigma$. Consequently, the class of $\underline{H}$ -Borelian sets $\underline{B}(\underline{H})$ coincides with $\underline{B}_0(\underline{B}(\underline{H}))$ is the smallest class containing $\underline{H}$ and closed under countable unions and countable intersections).

A convex cone $\underline{C} \subset \underline{C}^+(E)$ is said to be adapted if for each $f \in \underline{C}$ there exists $g \in \underline{C}$ with $f \in o(g)$ i.e. for all $\epsilon > 0$ there exists a compact $K = K(\epsilon)$ with $f(x) \leq \epsilon g(x)$ if $x \notin K$.

Proposition 1. Let E be locally compact and denote by M a positive linear map of $\underline{C}_c(E)$ into $\underline{C}(E)$ with $M(\underline{C}_c^+(E)) \subset \underline{C}$, where $\underline{C} = \underline{C}_0^+(E)$ or is an adapted cone. Then there is a unique kernel N on $(E, \underline{B}_0)$ such that N agrees with M on $\underline{C}_c(E)$.

The kernel N satisfies the principle of domination (respectively, the complete maximum principle) if for all $\varphi, \psi \in \underline{C}^+(E)$,

$M \varphi \geq M \psi$ on $\{\psi > 0\}$ implies $M \varphi \geq M \psi$ and for all $x \in E$ there exists $\varphi \in C_c(E)$ with $M \varphi(x) \neq 0$ (respectively, $1 + M \varphi \geq M \psi$ on $\{\psi > 0\}$ implies $1 + M \varphi \geq M \psi$).

Proof: If $\underline{C}$ is an adapted cone then each $f \in \underline{C}$ vanishes outside of a countable union of compact sets. Hence, $f = \sum_n \varphi_n$, $(\varphi_n) \subset \underline{C}_c^+(E)$. Consequently, $M \varphi \in \underline{B}_0^+$ if $\varphi \in \underline{C}_c^+(E)$.

Define the Radon measure μ_x by setting $\langle \mu_x, \varphi \rangle = M \varphi(x)$, $\varphi \in \underline{C}_c(E)$. If $f \in \underline{B}_0^+$ define $N(x, f) = \langle \mu_x, f \rangle$.

Let $K \in \underline{H}$ and let $\underline{F}$ be the σ -field of sets of the form $A \cap K$, $A \in \underline{B}_0$. Denote by $\underline{M}$ the vector space of differences of non-negative bounded Baire measurable functions. Then $\underline{F}_b$ can be naturally identified with a subspace of $\underline{M}$.

Consider $\underline{K} = \{f \in \underline{F}_b \mid Nf \in \underline{M}\}$. This is a subspace of $\underline{F}_b$ closed under monotone limits and containing 1 (since $K \in \underline{H}$). Assume $(f_n) \subset \underline{K}$ and that f_n tends to f uniformly. There is a set $X \in \underline{B}_0$ such that, for all n , $\{|Nf_n| > 0\} \subset X$ and the functions Nf_n are all measurable in the usual sense, when viewed on X , with respect to the σ -field on X induced by $\underline{B}_0$. Hence, Nf_n (which equals $\lim_{n \rightarrow \infty} Nf_n$) is in $\underline{M}$ since when viewed on X it is measurable. It follows from IT20 in [4] that $\underline{K} = \underline{F}_b$. Furthermore, if $f \in \underline{F}_b^+$ then $Nf \in \underline{B}_0^+$. This follows since $Nf \geq 0$ is locally measurable and $\{Nf > 0\} \in \underline{B}_0^+$.

If $f \in \underline{B}_0^+$ then $\{f > 0\} \subset \bigcup_n K_n$, $(K_n) \subset \underline{H}$, increasing. Hence, $Nf = \lim_{n \rightarrow \infty} N(f \cdot 1_{K_n})$ and so is in $\underline{B}_0^+$.

The last two statements follow by the argument used to prove XT4 in [4].

Let N be a kernel on $(E, \underline{E})$. A set $A \in \underline{E}$ will be said to be bounded if $N1_A$ is finite and will be said to be σ -bounded if it is a countable union of bounded sets.

A locally measurable function u will be said to be supermedian if, for all f and $g \in \underline{E}^+$,

$$u + Nf \geq Ng \text{ on } \{g > 0\} \text{ implies } u + Nf \geq Ng.$$

A supermedian function u is said to vanish at the boundary if there exists an increasing sequence (A_n) of bounded sets with $\inf_n R_{A_n}^s u = 0$ (note: it is no longer required as in [6] that $\bigcup_n A_n = E$).

Proposition 2. The following conditions are equivalent:

- (1) every finite potential Nf , $f \in \underline{E}^+$, vanishes at the boundary; and
- (2) the potential $N1_A$ of every bounded set A vanishes at the boundary.

Proof: Let $f \in \underline{E}^+$ have a finite potential Nf and let $\{f > 0\} = \bigcup_n B_n$, with each B_n bounded and (B_n) increasing. It can be assumed further that f is bounded on each set B_n .

Let $N1_{B_n}$ vanish at the boundary relative to (A_m^n) . Then, if $A_p = \bigcup_{n=1}^p \bigcup_{m=1}^n A_m^n$, each $N1_{B_n}$ vanishes at the boundary relative to (A_p) .

Hence, each $N(f 1_{B_n})$ vanishes at the boundary relative to (A_p) and so, by the lemma preceding proposition 1 in [6], Nf vanishes at the boundary relative to (A_p) .

Theorem 3. Let N be a kernel on $(E, \underline{E})$ that satisfies the domination principle and is such that the following conditions are satisfied:

- (1) every set $A \in \underline{E}$ is σ -bounded;
- (2) every finite potential Nf , $f \in \underline{E}^+$, vanishes at the boundary ; and
- (3) if $A \in \underline{E}$ there exists a finite supermedian function u which is strictly positive on A .

Then there exists a unique resolvent $(N_\lambda)_{\lambda > 0}$ of kernels N_λ on $(E, \underline{E})$ with $N_0 = N$. Further, this resolvent is sub-Markovian if N satisfies the complete maximum principle.

Proof: If $g \in \underline{E}^+$ there exists a set $X \in \underline{E}$ with the following properties:

- (1) $\{g > 0\} \subset X$;
- (2) $X = \bigcup_n A_n$, each A_n bounded; and
- (3) for each n , $N1_{A_n}$ vanishes on $\bar{X}$ and vanishes at the boundary relative to (A_n) . First note that if (B_n) is any sequence of bounded sets then there exist sequences (B'_n) and (B''_n) of bounded sets such that (a) each $N1_{B'_n}$ vanishes at the boundary relative to (B'_n) (see the proof of proposition 2) and (b) for each n , $\{N1_{B_n} > 0\} \subset \bigcup_m B''_m$. Consequently, there exists a sequence (C_n) of bounded sets such that, for all n , $N1_{C_n}$ vanishes on $(\bigcup_n C_n)$ and vanishes at the boundary relative to some subsequence (depending on n) of (C_n) . Further, (C_n) can be assumed to contain $\{g > 0\}$ in its union.

Let $X = \bigcup_n C_n$ and $A_n = \bigcup_{i=1}^n C_i$.

Let $\underline{X}$ be the σ -field of sets of the form $A \cap X, A \in \underline{E}$. If $f \in \underline{E}^+$ and $\{f > 0\} \subset X$ then (3) implies $\{Nf > 0\} \subset X$. Hence, N induces a proper kernel R on $(X, \underline{X})$ which satisfies the domination principle.

Let u_0 be a finite supermedian function on E which is strictly positive on X . Define $Vf = (1/u_0) R(fu_0)$, $f \in \underline{X}^+$.

Then the kernel V satisfies the hypotheses of theorem 2 in [6] since u supermedian implies u/u_0 restricted to X is supermedian relative to V . Let $(V_\lambda)_{\lambda > 0}$ be the sub-Markovian resolvent on $(X, \underline{X})$ with $V = V_0$. Then, if $R_\lambda h = u_0 V_\lambda (h/u_0)$, $h \in \underline{X}^+$, $(R_\lambda)_{\lambda > 0}$ is the resolvent of kernels on $(X, \underline{X})$ with $R_0 = R$. Hence, $Ng = (I + \lambda N) R_\lambda g$.

Denote by X' another set in E satisfying (1), (2) and (3) and let R' be the kernel induced on $(X', \underline{X}')$ by N . Let $(R'_\lambda)_{\lambda > 0}$ be the resolvent on $(X', \underline{X}')$ with $R'_0 = R'$. Since $Ng = (I + \lambda N) R'_\lambda g$ it then follows from XT 7 in [4] that, for all $\lambda > 0$, $R_\lambda g = R'_\lambda g$.

Define $N_\lambda g$ to be $R_\lambda g$, where $(R_\lambda)_{\lambda > 0}$ is the resolvent defined by a set $X \in \underline{E}$ satisfying (1), (2) and (3) and the kernel R induced by N .

It follows that (i) each N_λ is a kernel, (ii) $(N_\lambda)_{\lambda > 0}$ is a resolvent family and (iii) $N = N_0$.

Application to locally compact spaces. Let E be a locally compact and denote by $\underline{B}_0$ the σ -ring of Baire subsets of E .

Corollary 3.1. (F. Hirsch [2]). Let V be a positive linear map of $\underline{C}_c(E)$ into $\underline{C}_0(E)$ such that for all $\varphi, \psi \in \underline{C}_c^+(E)$

$$1 + V \varphi \geq V \psi \text{ on } \{\psi > 0\} \text{ implies } 1 + V \varphi \geq V \psi.$$

Then there is a resolvent family $(V_\lambda)_{\lambda > 0}$ of sub-Markovian operators V_λ on $\underline{C}_0(E)$ with $V \varphi = \lim_{\lambda \rightarrow 0} V_\lambda \varphi$, for all $\varphi \in \underline{C}_c(E)$.

Proof: From proposition 1 and theorem 3 it follows that there is a sub-Markovian resolvent $(V_\lambda)_{\lambda > 0}$ of sub-Markovian kernels V_λ on $(E, \underline{B}_0)$ with $V_0 = V$. Note that by the lemma preceding proposition 1 in [6] every finite potential Vf vanishes at the boundary because $\{f > 0\} \subset \bigcup_n K_n$, $(K_n) \subset \underline{H}$ and each $V 1_{K_n}$ clearly vanishes at the boundary.

Let $\varphi \in \underline{C}_0^+(E)$. Then there exists an open set $X \in \underline{B}_0$ such that (1) $\{\varphi > 0\} \subset X$; (2) $X = \bigcup_n K_n$, $(K_n) \subset \underline{H}$; and (3) for each n , $V 1_{K_n}$ vanishes on X and vanishes at the boundary relative to (K_n) . It suffices to note that the sets B_n , B_n' and B_n'' in the proof of the theorem can all be assumed to be open and relatively compact under the hypotheses of this corollary.

Consequently, there exists a $a \in \underline{C}^+(E)$ with (1) $X = \{a > 0\}$ and (2) Va bounded. The argument given in the remark following corollary 2.4 in [6] then implies $V_\lambda \varphi$ is continuous for all $\lambda > 0$. Hence, each V_λ leaves $\underline{C}_0(E)$ invariant.

Corollary 3.2. Let M be a positive linear map of $\underline{C}_c(E)$ into $\underline{C}_0(E)$ such that, for all $\varphi, \psi \in \underline{C}_c^+(E)$

$$M \varphi \geq M \psi \text{ on } \{\psi > 0\} \text{ implies } M \varphi \geq M \psi.$$

Assume M is non-degenerate, i.e. for all $x \in E$ there exists $\varphi \in \underline{C}_c(E)$ with $M \varphi(x) \neq 0$ and that there is a supermedian function u with $\overline{\{u < 1\}}$ compact. Then there is resolvent family $(M_\lambda)_{\lambda > 0}$ of unbounded operators M_λ on $\underline{C}_0(E)$ such that $M \varphi = \lim_{\lambda \rightarrow 0} M_\lambda \varphi$, for all $\varphi \in \underline{C}_c(E)$.

Proof: Let N be the kernel on $(E, \underline{B}_0)$ determined by M . It satisfies the domination principle since M is non-degenerate. If $X \in \underline{B}_0$ then there exists $(\varphi_n) \subset \underline{C}_c^+(E)$ with $u = \sum_n M(\varphi_n) \in \underline{C}^+(E)$ and $\{u > 0\} \supset X$. The argument of proposition 3 in [6] shows that u is a supermedian function.

Because there exists a supermedian function u with $\overline{\{u < 1\}}$ compact, every finite potential Nf vanishes at the boundary and so there is a resolvent $(N_\lambda)_{\lambda > 0}$ of kernels N_λ on $(E, \underline{B}_0)$ with $N_0 = N$.

Because each $X \in \underline{E}$ is contained in $\{u > 0\}$, for some u continuous and supermedian, it follows that, if $\varphi \in \underline{C}_c^+(E)$ and $X \in \underline{E}$ is open and satisfies the conditions in the proof of corollary 3.1, then the kernel V on $(X, \underline{X})$ induced by N and u maps $\underline{C}_c(X)$ into $\underline{C}_0(X)$. Hence $R_\lambda \varphi \in \underline{C}_0(X)$ and so N_λ leaves $\underline{C}_0(E)$ invariant. Define $M_\lambda \varphi = N_\lambda \varphi$, if $\varphi \in \underline{C}_0(E)$.

Application to adapted cones

Let E be locally compact and denote by $\underline{C} \subset \underline{C}^+(E)$ an adapted cone. Let M be a positive linear map of $\underline{C}_c(E)$ into $\underline{C}(E)$ with $M(\underline{C}_c^+(E)) \subset \underline{C}$. Assume the following conditions satisfied:

$A_2)$ for each $x \in E$ there exists $u \in \underline{C}$, $u(x) > 0$; and

$A_4)$ if $u \in \underline{C}$ and $\varphi \in \underline{C}_c^+(E)$ then

$$u \geq M \varphi \text{ whenever } u \geq M \varphi \text{ on } \{\varphi > 0\}.$$

Corollary 3.1 (Mokobodzki-Sibony [5]). Let N be the kernel

on $(E, \underline{B}_0)$ determined by M . Then there is a resolvent

$(N_\lambda)_{\lambda > 0}$ of kernels on $(E, \underline{B}_0)$ with $N_0 = N$.

Proof: Let $\underline{C}_\sigma$ denote the set of continuous functions

u on E of the form $u = \sum_n u_n$, $(u_n) \subset \underline{C}$. Each $u \in \underline{C}_\sigma$

is supermedian (see proposition 3 in [6]) and for each

$X \in B_0$ condition A_2) implies that there exists $u \in \underline{C}_\sigma$ with $\{u > 0\} \supset X$.

If $\varphi \in \underline{C}_c^+(E)$ the fact that $\underline{C}$ is adapted and contains only supermedian functions implies $N\varphi$ vanishes at the boundary. Hence, as in the proof of corollary 3.1, each finite potential Nf vanishes at the boundary. The result then follows immediately.

References

- [1] Halmos, P. Measure Theory, Van Nostrand Inc. Princeton, N.J., 1965
- [2] Hirsch, F. Familles résolvantes, générateurs, cogénérateurs, potentiels, Ann. Inst. Fourier 22(1), 89-210 (1972).
- [3] Lion, G. Familles d'opérateurs et frontières en théorie du potentiel, Ann. Inst. Fourier 16 (2) (1966), 389-453
- [4] Meyer, P.A. Probability and potentials, Blaisdell Publishing Company, Waltham, Mass., 1966

- [5] Mokobodzki, G. and Sibony, D. Cônes de fonctions et théorie du potentiel II. Résolvantes et semi-groupes subordonnés à un cône de fonctions. Séminaire de théorie du potentiel (Brelot, Choquet, Deny), Institut Henri Poincaré, Paris, 11^e années, 1966/67, exposé 9, 29 pages.
- [6] Taylor, J.C. On the existence of sub-Markovian resolvents, Invent. Math. 17, 85-93 (1972).

McGill University
 Montreal, Que.
 Canada