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RELAXATION IN INFINITE SPIN SYSTEMS

Hermann Rost

We present in this paper some recent results of DOBRUŠIN and HOLLEY in the field of statistical mechanics.

I. Consider a finite or countable set I , the set of "lattice sites", and associate to each $i \in I$ a "spin" x_i equal to ± 1 ; a spin configuration is a family $x = (x_i)_{i \in I}$; the set S of all spin configurations will be provided with its usual product topology and product σ -field. For each i let X_i be mapping which assigns to $x \in S$ its i -th coordinate.

Further, for each i we are given a "local energy function" $U(.,i)$ on S which we assume to be of the form

$$U(x,i) = \sum_{i \in P} J_P \left(\prod_{j \in P} x_j \right),$$

where J_P is a real number for every finite subset P of I such that for all $i \in I$

$$\sum_{i \in P} |J_P| < \infty.$$

A probability measure μ on S will be called a Gibbs measure iff for every $i \in I$ and $x \in S$

$$(\exp(-U(x,i)) + \exp(-U(\tau_i x, i)))^{-1} \cdot \exp(-U(x,i))$$

is a version of the conditional μ -probability of $\{X_i = x_i\}$ under the hypothesis $X_j = x_j$ for all $j \neq i$. (Here τ_i is the inversion of the spin at site i : $y = \tau_i(x)$ iff $y_i = -x_i$ and $y_j = x_j$ for $j \neq i$.) Under the above assumptions, the existence of a Gibbs measure is always guaranteed; for the question of its uniqueness see [1] or [5].

II. In order to describe relaxation phenomena or time dependent behaviour of a spin system we will look for a stochastic process

(if possible Markovian) with state space S which may serve as a model for the changes which undergoes a spin configuration under the influence of some thermal disorder. From the mathematical point of view there are many ways to define such a process; see for example [6].

Here we try to find a Markov process whose generator A is defined at least on all functions on S which depend only on a finite number of coordinates and is of the form

$$(*) \quad (Af)(x) = \sum_{i \in I} \exp(U(x,i)) \cdot (f(\tau_i x) - f(x)).$$

For finite I there is no problem; in the case of infinite I the construction of a suitable stochastic semigroup acting on $\mathcal{C}(S)$, the space of all continuous functions on S , and hence a Markov Process is given in [2]. Since that construction involves a nice combinatorial argument we will present it here in all essential steps.

Theorem 1. Let for each $i \in I$ the continuous function $V(.,i)$ on S be given, $V(.,i) \geq 0$. For finite $E \subset I$ let $(P_t^E)_{t \geq 0}$ be the semigroup on $\mathcal{C}(S)$ with generator A^E , where

$$(A^E f)(x) = \sum_{i \in E} V(x,i) (f(\tau_i x) - f(x)), \quad f \in \mathcal{C}(S).$$

Set $G(i,j) := \sup_x |V(\tau_j x, i) - V(x, i)|$, $i, j \in I$.

Then, if $\sup_i (\sum_j G(i,j)) < \infty$, there exists for each $f \in \mathcal{C}(S)$

and $t > 0$ the limit (in the uniform sense)

$$P_t f := \lim_{E \nearrow I} P_t^E f$$

and defines a peller semigroup $(P_t)_{t \geq 0}$ on $\mathcal{C}(S)$. (The generator A of (P_t) is of the form $(*)$ if $V = \exp U$.)

The theorem is an easy consequence of the following

Lemma. Let C, D, E be finite subsets of I with $C \subset D \subset E$;

let $f = f(z)$ be an element of $\mathcal{C}(S)$ depending only on z_i ,
 $i \in C$, and satisfying $0 \leq f \leq 1$;

let $x, y \in S$ be fixed and such that $x_i = y_i$ for $i \in D$;

let (Q_t) and (R_t) be the semigroups on $\mathcal{C}(S)$ with gener-
ator, respectively,

$$g \mapsto \sum_{i \in E} V(., i)(g \circ \tau_i - g), \quad g \mapsto \sum_{i \in E} W(., i)(g \circ \tau_i - g),$$

where all $W(., i)$ are continuous and positive and $W(., i) =$
 $= V(., i)$ for $i \in D$.

Then one has for all $t \geq 0$

$$|Q_t(x, f) - R_t(y, f)| \leq \sum_{i \in C} \sum_{j \notin D} (\exp tG)(i, j) .$$

The proof of the lemma is based on an idea of WASSERSTEIN ([7]), who introduces the following measure of a distance between two probability measures on the product space S : we say that $s_i, i \in I$, is a bound for μ and ν iff there exists a probability measure ρ on $S \times S$ having μ and ν as its projections on the first and second factor and such that

$$\rho(X_i \neq Y_i) \leq s_i \text{ for all } i \in I .$$

(Here X_i , resp. Y_i , assigns to each $(x, y) \in S \times S$ the coordinate x_i , resp. y_i .)

It is easy to see that if $s_i, i \in I$, is a bound for μ and ν then we have for a function f on S with values between 0 and 1 and depending only on $z_i, i \in C$,

$$(+)\quad |< \mu - \nu, f >| \leq \sum_{i \in C} s_i .$$

So one tries to find a bound for the measures $Q_t(x, .)$ and $R_t(y, .)$ by constructing a suitable joint realization of the processes $(X(t))_{t \geq 0}$ and $(Y(t))_{t \geq 0}$ having (Q_t) , resp. (R_t) , as

transition semigroup and x , resp. y , as initial point.

A realization of one process, say $(X(t))$, may be found the following way :

$X_i(\cdot)$ remains unchanged equal to x_i for all $i \notin E$;
for each $i \in E$ we consider independently a Poisson point process $p_i = (p_i^1, p_i^2, \dots)$ and construct the path $X(\cdot)$ given p_i , $i \in E$, in a deterministic way, namely

$$X_i(0) = x_i, \quad i \in E ;$$

$X_i(\cdot)$ jumps at time t if and only if $v_i(t)$, defined as $\int_0^t V(X(s), i) ds$, is equal to some point p_i^k , $k \geq 1$, in the sample p_i , $i \in E$.

By the same procedure we construct the process $(Y(t))$ using functionals w_i instead of v_i , where $w_i(t) = \int_0^t W(Y(s), i) ds$. The salient point is that we define both processes by means of the same set of auxiliary Poisson point processes p_i , $i \in E$. Then one has for $i \in D$, $t > 0$:

$$\begin{aligned} P(X_i(t) \neq Y_i(t)) &\leq P(\text{there is a point of } p_i \text{ between } v_i(t) \\ &\quad \text{and } w_i(t)) \leq \\ &\leq \mathbb{E} |v_i(t) - w_i(t)| \quad (\text{by the strong Markov property}) \leq \\ &\leq \int_0^t \mathbb{E} |V(X(s), i) - W(Y(s), i)| ds \leq \int_0^t \mathbb{E} \left(\sum_{X_j(s) \neq Y_j(s)} G(i, j) \right) ds = \\ &= \int_0^t \sum_{j \in E} G(i, j) \cdot P(X_j(s) \neq Y_j(s)) ds . \end{aligned}$$

As is easy to check, $P(X_i(t) \neq Y_i(t))$, $i \in E$, is dominated by the solution of the integral equation

$$s_i(t) = s_i(0) + \int_0^t \sum_{j \in E} G(i, j) \cdot s_j(s) ds, \quad i \in E,$$

where the initial condition is

$$s_i(0) = 0 \quad \text{for } i \in D, \quad s_i(0) = 1 \quad \text{for } i \in E \setminus D ;$$

that solution has the explicit form

$$s_i(t) = \sum_{j \in E \setminus D} (\exp tG)(i,j) , i \in E ,$$

which is the desired bound for the two measures $Q_t(x, \cdot)$ and $R_t(y, \cdot)$. In view of (+) the lemma is proven.

We remark that the lemma allows one to obtain a lower estimate for a "relaxation time" : if a spin system with the stochastic behaviour governed by the semigroup (P_t) is changed at time 0 arbitrarily outside some finite set D the local behaviour at some fixed $i \in D$ will not be influenced too heavily at a time $t \leq t_0$, where t_0 is the smallest solution of (say)

$$\sum_{j \notin D} (\exp tG)(i,j) = \varepsilon , \text{ and } \varepsilon \text{ is small.}$$

III. In order to consider the semigroup (P_t) in a more concrete situation we will specify our assumptions :

let I be equal to Z^v , the lattice of points with integer valued coordinates in v -dimensional space, $v \geq 1$;

let a finite collection of finite sets K , $K \subset Z^v$ and $0 \in K$, be given and to each K a real number J_K ; put

$$U(x,i) = \sum_K J_K \left(\prod_{j-i \in K} x_j \right) , i \in Z^v ,$$

and as before $V(x,i) = \exp U(x,i)$, $x \in S$, $i \in Z^v$;

let (P_t) be the semigroup constructed in theorem 1.

We will say that a probability measure μ on S is shift invariant, iff for any finite $C \subset Z^v$ and $\alpha_j = \pm 1$, $j \in C$,

$$\mu(X_j = \alpha_j, j \in C) = \mu(X_{j+i} = \alpha_j, j \in C) \text{ for every } i \in Z^v .$$

Then the following theorems hold ([3][4]):

Theorem 2. A shift invariant probability measure μ on S
is Gibbs if and only if it is invariant under (P_t) , i.e. if

$$\mu^{P_t} = \mu \quad \text{for all } t \geq 0 .$$

In that case the process $(X(t))_{t \geq 0}$ is reversible under the
law P^μ , i.e. one has for $t > 0$, A and $B \subset S$

$$P^\mu(X(0) \in A, X(t) \in B) = P^\mu(X(0) \in B, X(t) \in A) .$$

Theorem 3. If there exists a unique shiftinvariant Gibbs measure
 $\bar{\mu}$ and if μ is any shift invariant probability measure on S ,
 μ^{P_t} converges to $\bar{\mu}$ weakly, i.e.

$$\langle \bar{\mu}, f \rangle = \lim_{t \rightarrow \infty} \langle \mu^{P_t}, f \rangle \quad \text{for all } f \in \mathcal{C}(S) .$$

The last theorem says roughly speaking that in our model of a spin system at the end of the relaxation process described by (P_t) the system is found in the equilibrium state $\bar{\mu}$ regardless of what its initial state was.

References

- [1] DOBRUŠIN, R. L.: Description of a random field by means of conditional probabilities and conditions of its regularity. Th. Probability Appl. 13, 197-224 (1968).
- [2] DOBRUŠIN, R. L.: Probl. peredači informacii 7,2 (1971), 70-87.
- [3] DOBRUŠIN, R. L.: Probl. peredači informacii 7,3 (1971), 57-66.
- [4] HOLLEY, R.: Free energy in a Markovian model of a lattice spin system. Commun. math. Phys. 23, 87-99 (1971).
- [5] RUEELLE, D.: Statistical Mechanics. New York, Amsterdam: Benjamin 1969.
- [6] SPITZER, F.: Interaction of Markov Processes. Advances in Math. 5, 246-290 (1970).
- [7] WASSERSTEIN, L. N.: Probl. peredači informacii 5,3 (1969), 64-73.

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