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EXAMPLES ON LOCAL MARTINGALES

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In this note we shall give two remarks relative to changes of time for local martingales. Example 2 shows that the local martingale property is not invariant through changes of time.

We assume that the reader knows the usual definitions; for example, local martingales, stopping times, etc. By a change of time $A=(\underline{\underline{F}}_t, a_t)$ we means a family of stopping times of the $\underline{\underline{F}}_t$ family, finite valued, such that for $\omega \in \Omega$ the sample function $a_\cdot(\omega)$ is right continuous and increasing. All martingales below are assumed to be right continuous.

EXAMPLE 1.- Let $\Omega = R_+$, $\underline{\underline{F}}^o$ the class of all linear Borel sets in Ω and we designate by S the identity function of Ω into R_+. Let $\underline{\underline{F}}_t^o$ be the Borel field generated by $S \wedge t$. We define the probability measure P on Ω by $P(S>t)=e^{-t}$. Let $\underline{\underline{F}}_t$ be the P-completed Borel field of $\underline{\underline{F}}_t^o$. Note that the family $(\underline{\underline{F}}_t)$ is right continuous and quasi-left continuous.

PROPOSITION 1.- Let $A=(\underline{\underline{F}}_t, a_t)$ be a change of time such that $P(S > a_t) > 0$ for each t. Then for any martingale $M=(M_t, \underline{\underline{F}}_t)$, the process $AM=(M_{a_t}, \underline{\underline{F}}_{a_t})$ is also a martingale.

PROOF.- According to THEOREM 1 of [1], it follows that for each t there exists some $s_t \in \bar{R}$ such that

$$(1) \quad \begin{cases} \text{(i)} & a_t \geqq S \quad \text{if } S \leqq s_t \\ \text{(ii)} & a_t = s_t \quad \text{if } S > s_t \, . \end{cases}$$

Obviously s_t is right continuous. As $P(S > a_t) > 0$ for each t from the assumption, each s_t is finite. On the other hand, there exists a constant process $(c_t, \underline{\underline{F}}_t)$ such that

$$(2) \qquad M_t = M_S I_{[S \leqq t]} + c_t I_{[S>t]} \quad .$$

It follows from (1) that we have

$$M_{a_t} = M_S I_{[S \leqq a_t]} + c_{a_t} I_{[S>a_t]} \qquad = M_S I_{[S \leqq s_t]} + c_{s_t} I_{[S>s_t]} = M_{s_t} \, .$$

Furthermore it is easy to show that for each t we have $\underline{\underline{F}}_{a_t} = \underline{\underline{F}}_{a_t \wedge S}$ and $\underline{\underline{F}}_{s_t} = \underline{\underline{F}}_{s_t \wedge S}$. This implies that $\underline{\underline{F}}_{s_t} = \underline{\underline{F}}_{a_t}$ for each t. Consequently TM is also a martingale. This completes the proof.

PROPOSITION 2.- If $M=(M_t, \underline{\underline{F}}_t)$ is a continuous martingale, then $M_t \equiv C$, where C is a constant.

PROOF.- From formule(2) the continuity of c_t can be deduced by noting that M is continuous. Clearly we then have $M_S(\omega) = c_\omega$ a.s. On the other hand, it follows from formule(2) that the martingale equality implies the following:

$$(3) \qquad \int_s^t M_S \, dP = c_s e^{-s} - c_t e^{-t} \quad (s<t).$$

Then an easy computation shows

$$(4) \qquad \frac{d}{dt} c_t = 0 \qquad \text{i.e} \quad c_t = C.$$

Consequently we have $M_t \equiv C$.

The above proposition implies that any non-constant martingale on this probability space is quasi-left continuous but not continuous.

EXAMPLE 2.- Let $(\Omega, \underline{\underline{F}}, P)$ be a complete probability space,given an increasing right continuous family$(\underline{\underline{F}}_t)$ of Borel subfields of $\underline{\underline{F}}$ as usal. Note that if M is a weak martingale, for any change of time $A=(\underline{\underline{F}}_t, a_t)$ the process AM is also a weak martingale.(see[2]).

We suppose now that there exists a continuous martingale $M = (M_t, \underline{\underline{F}}_t)$, $M_0 = 0$, with the property $P(\limsup_{t \to \infty} M_t = \infty) = 1$; for example one dimensinal Brownian motion. Then the random variable a_t defined by

$$(5) \qquad a_t = \inf(u: M_u > t)$$

is a finite stopping time of the family$(\underline{\underline{F}}_t)$. Clearly $a_0=0$ and $a_\infty = \infty$ a.s. It is easy to see that the change of time A satisfies $M_{a_t} = t$ from the continuity of M. The process $AM = (t, \underline{\underline{F}}_{a_t})$ is not a local martingale. Thus the local martingale property is not invariant through changes of time. This fact should be noted.

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