MURALI RAO

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DOOB DECOMPOSITION AND BURKHOLDER INEQUALITIES*

Murali Rao

Let $X_0,\ldots,X_N$ be a martingale relative to σ-fields $F_0,\ldots,F_N$. Let $x_0 = X_0$, $x_i = X_i - X_{i-1}$ for $i=1,\ldots,N$ so that $X_n = \sum_0^n x_i$. Let $|v_i| \leq 1$, $0 \leq i \leq N$ and v_{i+1} F_i-measurable for $i=0,\ldots,N-1$. Put $g_n = \sum_0^n v_i x_i$, $n=0,\ldots,N$, $g_N^* = \max_n |g_n|$ and $S_N = S_N(X) = (\sum x_i^2)^{\frac{1}{2}}$. In [1] Burkholder proved the following remarkable inequalities:

For $a > 0$,

(1) $$a\, P[g_N^* > a] \leq 52 \; E[|X_N|]$$

(2) $$a\, P[S_N > a] \leq 52 \; E[|X_N|] \; .$$

In [2] Gundy, making use of his decomposition for L'-bounded martingales obtaines inequalities for "class B mappings" which include (1) and (2) above. In this note we exploit Doob decomposition to give completely elementary proofs of (1) and (2) thus answering a question raised by Luis Baez-Duarte [3]. Let us add in passing that our method also gives inequalities for class B mappings of Gundy. For terminology not defined here we refer to [4].

For random variables $f_0,\ldots,f_N$, f_N^* will denote $\max_{0 \leq i \leq N} |f_i|$.

We shall show that if the martingale $X_0,\ldots,X_N$ is non-negative (1) and (2) can be replaced by

(3) $$a\, P[g_N^* > a] \leq 13 \; E[|X_0|]$$

(4) $$a\, P[S_N > a] \leq 13 \; E[|X_0|] \; .$$

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* **Prof. Neveu pointed out to P.A.Meyer that he has given in his Cours de 3e Cycle on martingale theory, Paris 1969/70, a proof of the Burkholder maximal lemma which is very closely related to that of Prof. Rao.**

__Lemma__. Let $Z_o,\dots,Z_N$ be a square integrable super martingale and $Z_i = M_i - A_i$ be its Doob-decomposition. Then

(5) $$E[M_N^2] \leq E[Z_N^2] + 2E[\sum_{o}^{N-1} Z_i (A_{i+1} - A_i)] .$$

__Proof__. Noting $E[(Z_{i+1} - Z_i) \mid F_i] = A_i - A_{i+1}$

$$E[M_{i+1}^2 - M_i^2] = E[(M_{i+1} - M_i)^2]$$

$$= E[(Z_{i+1} - Z_i)^2 + 2(A_{i+1} - A_i)(Z_{i+1} - Z_i) + (A_{i+1} - A_i)^2]$$

$$= E[(Z_{i+1} - Z_i)^2] - E[(A_{i+1} - A_i)^2]$$

$$\leq E\,|(Z_{i+1} - Z_i)^2|$$

$$= E[Z_{i+1}^2 - Z_i^2] + 2E[Z_i(Z_i - Z_{i+1})]$$

$$= E[Z_{i+1}^2 - Z_i^2] + 2E[Z_i(A_{i+1} - A_i)] .$$

And since $M_o = Z_o$ we get

$$E[M_N^2] = E[M_o^2] + \sum_{o}^{N-1} E[M_{i+1}^2 - M_i^2]$$

$$\leq E[Z_o^2] + \sum_{o}^{N-1} E[Z_{i+1}^2 - Z_i^2] + 2\sum_{o}^{N-1} E[Z_i(A_{i+1} - A_i)]$$

$$= E[Z_N^2] + 2\sum_{o}^{N-1} E[Z_i(A_{i+1} - A_i)] .$$

That proves the Lemma.

If $Z_i \geq 0$, $E[A_N] \leq E[M_N] = E[Z_o]$ and we have

__Corollary__. If $0 \leq Z_i \leq a$ is a super martingale and $Z_i = M_i - A_i$ its Doob decomposition then

(6) $$E[M_N^2] \leq 3a\, E[Z_o] .$$

Now let us prove (3). Let $a > 0$ and $Z_i = X_i \wedge a$. Let $Z_i = M_i - A_i$ be its Doob decomposition and

$$U_n = Z_o v_o + \sum_1^n (Z_i - Z_{i-1}) v_i$$
$$V_n = v_o M_o + \sum_1^n (M_i - M_{i-1}) v_i \ .$$

By martingale inequality $aP(X_N^* > a) \leq E(X_o)$. On the set $(X_N^* \leq a)$, $g_n = U_n$ for all n. Thus

(7) $$aP(g_N^* > a) \leq aP[X_N^* > a] + aP[g_N^* > a, X_N^* \leq a]$$
$$\leq E[X_o] + aP[U_N^* > a] \ .$$

Clearly $|U_n| \leq |V_n| + A_n$ (note that $|v_i| \leq 1$ and $A_n \geq 0$) and $|V_n| + A_n$ is a submartingale. Submartingale inequality gives

$$P[U_N^* > a] \leq P[(|V|+A)_N^* > a]$$
$$\leq \frac{1}{a^2} E[(|V_N|+A_N)^2]$$
$$\leq \frac{2}{a^2} E[V_N^2 + A_N^2]$$
$$\leq \frac{4}{a^2} E[M_N^2]$$

since $E[V_N^2] \leq E[M_N^2]$ and $A_N \leq M_N$. Using (6) and that $Z_o \leq X_o$

$$P[U_N^* > a] \leq \frac{12}{a} E[X_o] \ .$$

Together with (7) this gives (3).

As another example of application of the Lemma let us derive (4). Put again $Z_i = X_i \wedge a$, and let $Z_i = M_i - A_i$ be its Doob decomposition.

(8) $$P[S_N(X) > a] \leq P[X_N^* > a] + P[S_N(X) > a, X_N^* \leq a] \leq \frac{1}{a} E[X_o] + P[S_N(Z)^2 > a^2] \ .$$

Clearly $S_N(Z)^2 \leq 2S_N(M)^2 + 2S_N(A)^2 \leq 2S_N(M)^2 + 2A_N^2$.

$$P[S_N(Z)^2 > a^2] \leq P[S_N(M)^2 + A_N^2 > \frac{a^2}{2}]$$

$$\leq \frac{2}{a^2} E[S_N(M)^2 + A_N^2]$$

$$= \frac{2}{a^2} E[M_N^2 + A_N^2]$$

$$\leq \frac{4}{a^2} E[M_N^2] \leq \frac{13}{a} E[X_o] .$$

This together with (8) gives (4). Similar argument applies to any class B mapping. We remark that (3) implies (4) but with 54 instead of 13.

References

[1] D. L. Burkholder: Martingale Transforms, Ann. Math. Stat. 37 (1966), 1494-1504.

[2] R. F. Gundy: A decomposition for L'-bounded martingales, Ann. Math. Stat. 39 (1968), 134-138.

[3] Luis Baez-Duarte: On the convergence of Martingale transforms, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 19 (1971), 319-322.

[4] P. A. Meyer: Probability and Potentials, Blaisdell (1966).

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