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NOTE ON A STOCHASTIC INTEGRAL EQUATION

by N.KAZAMAKI

In the present paper we shall consider the stochastic integral equation

$$(1) \quad Z_t = x + \int_0^t f(Z_u) dM_u + \int_0^t g(Z_u) dU_u, \quad x \in R^1$$

where $M=(M_t), M_0=0$, is a locally square integrable martingale and $U=(U_t), U_0=0$, is a continuous increasing process.

Let $(\Omega, \underline{F}, P)$ be a complete probability space, given an increasing right continuous family (\underline{F}_t) of sub σ -fields of \underline{F} . We suppose as usual that \underline{F}_0 contains all the negligible sets. By a normal change of time $A=(\underline{F}_t, a_t)$ we mean a family of stopping times of the family (\underline{F}_t) , finite valued, such that for $\omega \in \Omega$ the sample function $a_\cdot(\omega)$ is strictly increasing, $a_0(\omega)=0$, $a_\infty(\omega)=\lim_{t \rightarrow \infty} a_t(\omega) = \infty$ and continuous. We don't distinguish two processes X and Y such that for a.e $\omega \in \Omega$ $X_\cdot(\omega)=Y_\cdot(\omega)$. We assume that the reader knows the usual definitions.

THEOREM.- Assume that the family (\underline{F}_t) is quasi-left continuous. Then for coefficients f and g belonging to $C^1(R^1)$ and of bounded slope the equation (1) has one and only one solution.

PROOF.- From the quasi-left continuity of (\underline{F}_t) , it follows that there exists a unique continuous increasing process $\langle M \rangle$ such that $M^2 - \langle M \rangle$ is a local martingale.

Define

$$(2) \quad b_t = t + \langle M \rangle_t + U_t, \quad a_t = \inf(u; b_u > t) .$$

Then an easy computation shows that $A=(\underline{F}_t, a_t)$ and $B=(\underline{F}_{a_t}, b_t)$ are normal changes of time.

For every t , put

$$(3) \quad Y_t = Z_{a_t}, \quad N_t = M_{a_t}, \quad V_t = U_{a_t}.$$

The process N is a square integrable martingale and V is the natural increasing process associated to N ; clearly V is, in fact, continuous. It is shown in [1] that we have

$$(4) \quad \int_0^{a_t} f(Z_u) dM_u = \int_0^t f(Y_u) dN_u.$$

Thus, in order to show the existence of the unique solution of (1), it suffices to consider the following stochastic integral equation

$$(1^*) \quad Y_t = x + \int_0^t f(Y_u) dN_u + \int_0^t g(Y_u) dV_u.$$

For simplicity, the proof is spelled out for $0 \leq t \leq 1$ only. Without loss of generality, we may assume that $\max(\|f'\|_{\infty}, \|g'\|_{\infty}) \leq 1/2$.

Define in succession

$$(5) \quad \begin{aligned} Y_t^0 &= x \\ Y_t^n &= x + \int_0^t f(Y_u^{n-1}) dN_u + \int_0^t g(Y_u^{n-1}) dV_u, \quad n=1,2,\dots \end{aligned}$$

Put now

$$c_t^n = f(Y_t^n) - f(Y_t^{n-1}), \quad d_t^n = g(Y_t^n) - g(Y_t^{n-1}).$$

As $t = b_{a_t} = a_t + \langle N \rangle_t + V_t$ by the definition of a_t , we have

$$\begin{aligned} D_n(t) &= E[(Y_t^{n+1} - Y_t^n)^2] \\ &\leq 2E[(\int_0^t c_u^n dN_u)^2 + (\int_0^t d_u^n dV_u)^2] \\ &\leq 2 \left\{ E[\int_0^t (c_u^n)^2 d\langle N \rangle_u] + E[V_t \int_0^t (d_u^n)^2 dV_u] \right\} \\ &\leq 2 \left\{ E[\int_0^t (c_u^n)^2 du] + E[\int_0^t (d_u^n)^2 du] \right\} \\ &\leq 2(\|f'\|_{\infty}^2 + \|g'\|_{\infty}^2) \int_0^t E[(Y_u^n - Y_u^{n-1})^2] du \\ &\leq \int_0^t D_{n-1}(u) du \leq \text{Const.} \cdot t^n/n! \end{aligned}$$

Since the process $(\int_0^t c_u^n dN_u)$ is a martingale, the extension of Kolmogorov's inequality^λ shows that for any $\varepsilon > 0$ to martingales

$$\begin{aligned} \varepsilon^2 P\left(\sup_{0 \leq t \leq 1} \left| \int_0^t c_u^n dN_u \right| \geq \varepsilon\right) &\leq E\left[\left(\int_0^1 c_u^n dN_u\right)^2\right] \\ &\leq E\left[\int_0^1 (c_u^n)^2 du\right] \\ &\leq \|f\|_{\omega}^2 \int_0^1 D_{n-1}(u) du \\ &\leq \text{Const.} \cdot 1/n! \quad . \end{aligned}$$

Similarly, we get by using the Schwarz inequality

$$\begin{aligned} P\left(\sup_{0 \leq t \leq 1} \left| \int_0^t d_u^n dV_u \right| \geq \varepsilon\right) &= P\left(\sup_{0 \leq t \leq 1} \left[\int_0^t d_u^n dV_u \right]^2 \geq \varepsilon^2\right) \\ &\leq P\left(\sup_{0 \leq t \leq 1} V_t \cdot \int_0^t (d_u^n)^2 dV_u \geq \varepsilon^2\right) \\ &\leq P\left(\int_0^1 (d_u^n)^2 du \geq \varepsilon^2\right) \\ &\leq \varepsilon^{-2} E\left[\int_0^1 (d_u^n)^2 du\right] \\ &\leq \text{Const.} \cdot \varepsilon^{-2}/n! \quad . \end{aligned}$$

Thus $P\left(\sup_{0 \leq t \leq 1} |Y_t^{n+1} - Y_t^n| \geq 2\varepsilon\right) \leq \text{Const.} \cdot \varepsilon^{-2}/n!$. Pick $\varepsilon^{-2} = (n-2)!$. Then $\varepsilon^{-2}/n!$ is the general term of a convergent sum, and so the Borel-Cantelli lemma shows that Y_t^n converges uniformly a.s for $0 \leq t \leq 1$ to some random variable Y_t^* ; clearly Y_t^* is \mathbb{F}_{a_t} -measurable and for a.s ω the sample function $Y_t^*(\omega)$ is right continuous.

Because of this, $f(Y_t^n)$ (resp. $g(Y_t^n)$) converges uniformly a.s to $f(Y_t^*)$ (resp. $g(Y_t^*)$).

According to THEOREM 10 of [1], $\int_0^t f(Y_u^n) dN_u$ converges uniformly in probability to

$$\int_0^t f(Y_u^*) dN_u, \text{ i.e. for each } \varepsilon > 0, \lim_n P\left(\sup_{0 \leq t \leq 1} \left| \int_0^t f(Y_u^n) dN_u - \int_0^t f(Y_u^*) dN_u \right| > \varepsilon\right) = 0.$$

Thus, for some subsequence (n_k) , we get

$$(6) \quad \lim_k \sup_{0 \leq t \leq 1} \left| \int_0^t f(Y_u^{n_k}) dN_u - \int_0^t f(Y_u^*) dN_u \right| = 0 \quad \text{a.s}$$

As $\int_0^t g(Y_u^n) dV_u$ converges uniformly a.s to $\int_0^t g(Y_u^*) dV_u$, we have

$$(7) \quad Y_t^* = x + \int_0^t f(Y_u^*) dN_u + \int_0^t g(Y_u^*) dV_u .$$

This completes the proof of existence. We are now going to show its uniqueness.

Let (Y_t^1) and (Y_t^2) be solutions of (1*). Then the random variable r defined by

$$r = \inf (t; \max_1 |Y_t^i| \geq n) .$$

is a stopping time of the family (\underline{F}_t) . We denote $Y_t^i I_{[t < r]}$ by \hat{Y}_t^i . Then for $t < r$ we have

$$\hat{Y}_t^2 - \hat{Y}_t^1 = \int_0^t [f(\hat{Y}_u^2) - f(\hat{Y}_u^1)] dN_u + \int_0^t [g(\hat{Y}_u^2) - g(\hat{Y}_u^1)] dV_u .$$

From the definition of r , $\hat{D}(t) = E[(\hat{Y}_t^2 - \hat{Y}_t^1)^2] \leq 4n^2 < \infty$. On the other hand,

$\hat{D}(t) \leq \int_0^t \hat{D}(u) du$ as in the proof of existence. Thus $\hat{D}(t) \equiv 0$, and making $n \rightarrow \infty$

we obtain the uniqueness statement. Consequently $BY^* = (Y_{b_t}^*, \underline{F}_t)$ is the unique solution of the equation (1). Hence the theorem is established.

REFERENCE

- [1] N.KAZAMAKI ; Some properties of martingale integrals , Ann.Inst.Henri Poincaré, vol.VII, n°1, 1971.

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