SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

NORIIHIKO KAZAMAKI

Krickeberg's decomposition for local martingales

Séminaire de probabilités (Strasbourg), tome 6 (1972), p. 101-104

[<http://www.numdam.org/item?id=SPS_1972__6__101_0>](http://www.numdam.org/item?id=SPS_1972__6__101_0)

© Springer-Verlag, Berlin Heidelberg New York, 1972, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (<http://portail.mathdoc.fr/SemProba/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

KRICKEBERG'S DECOMPOSITION FOR LOCAL MARTINGALES

by N. KAZAMAKI

Any L^1 -bounded martingale can be uniquely decomposed into two positive martingales possessing some additional property : this is the well known Krickeberg decomposition, which will be recalled below. In the present note we extend this fact to the local martingale case, following the same idea.

1. - Let Ω be a set, $\underline{\mathbb{F}}$ a Borel field of subsets of Ω , P a probability measure defined on $(\Omega, \underline{\mathbb{F}})$. We are given a family $(\underline{\mathbb{F}}_t)$ of Borel subfields of $\underline{\mathbb{F}}$, increasing and right continuous. We may, and do, assume that $\underline{\mathbb{F}}$ has been completed with respect to P , and that each $\underline{\mathbb{F}}_t$ contains all sets of measure zero. We assume that the reader knows the usual definitions, for example : stopping times, changes of times, martingales, etc (see [2]). We don't distinguish two processes X and Y such that for a.e. ω $X_t(\omega) = Y_t(\omega) \forall t \geq 0$; this is important for the understanding of uniqueness statements below.

2. - All martingales considered below are assumed to be right continuous.

Proposition 1.- A martingale $X = (X_t, \underline{\mathbb{F}}_t)$ is L^1 -bounded if and only if it can be written as the difference of two positive martingales. These martingales $X^{(1)}$ and $X^{(2)}$ then can be chosen so as to realize the equality

$$(1) \quad \sup_t E[|X_t|] = E[X_0^{(1)}] + E[X_0^{(2)}]$$

This decomposition then is unique.

Proof. The "if" part is clear. To prove the "only if" part, set

$$X_t^{(1)} = \lim_n E[X_n^+ | \underline{\mathbb{F}}_t] \quad , \quad X_t^{(2)} = \lim_n E[X_n^- | \underline{\mathbb{F}}_t]$$

The monotone convergence theorem shows that if $s < t$, we have $E[X_t^{(i)} | \underline{\mathbb{F}}_s] = X_s^{(i)}$, $i=1,2$. This is the martingale equality, and since the family $(\underline{\mathbb{F}}_t)$ is right continuous we may assume that right continuous modifications of the above processes have been chosen. Then it is easy to

see that $X = X^{(1)} - X^{(2)}$, and that the equality (1) holds.

If we have another decomposition $X = Y - Z$ of X into two positive martingales, then $Y_t \geq X_t^{(1)}$ and $Z_t \geq X_t^{(2)}$. If this decomposition satisfies (1), we must have $E[Y_0] = E[X_0^{(1)}]$ and $E[Z_0] = E[X_0^{(2)}]$ and the uniqueness statement follows from it. It is interesting for the sequel to remark that the conclusion $Y_t \geq X_t^{(1)}$, $Z_t \geq X_t^{(2)}$ is true also if Y, Z are just assumed to be supermartingales ≥ 0 .

3.- Definition 2. A process $X = (X_t, \mathbb{F}_t)$ is said to be a local martingale if there exists an increasing sequence (T_n) of stopping times of $(\mathbb{F}_t)$ such that $\lim_n T_n = \infty$ and for each n the process $(X_{t \wedge T_n} I_{\{T_n > 0\}}, \mathbb{F}_t)$ is a martingale which belongs to the class (D).

To be short, we shall say that a stopping time T reduces the process X if $(X_{t \wedge T} I_{\{T > 0\}})$ belongs to the class (D) - one may then show that it is a martingale - and we shall call a sequence T_n as above a fundamental sequence for the local martingale X .

Now we set $\|X\|_1 = \sup E[|X_T|]$, T ranging over the set of all a.s. finite stopping times. If $\|X\|_1 < \infty$, the local martingale is said to be bounded in L^1 .

Theorem 3 . Let X be a local martingale. Then $\|X\|_1 = \sup_n E[|X_{T_n}|]$ for any fundamental sequence (T_n) consisting of a.s. finite stopping times. If X is L^1 -bounded, then X can be written as the difference $X^{(1)} - X^{(2)}$ of two positive ^{local} martingales, which can be chosen so as to realize the equality

$$(2) \quad \|X\|_1 = E[X_0^{(1)}] + E[X_0^{(2)}]$$

This decomposition ^{then} is unique.

Proof. We have $E[|X_{T_n}|] \leq \|X\|_1$ for all n . Let T be any finite stopping time. A well known submartingale inequality gives us $E[|X_{T \wedge T_n} I_{\{T_n > 0\}}|] \leq E[|X_{T_n} I_{\{T_n > 0\}}|]$, and $E[|X_T|] \leq \sup_n E[|X_{T_n}|]$ now comes from Fatou's lemma. This proves the first statement.

Assume $\|X\|_1 < \infty$. Then X_0 is integrable. The process $(X_{t \wedge T_n})$ is a local martingale (stopping preserves the local martingale property) and belongs to the class (D), hence is a martingale of the class (D), and we have no need to insert $I_{\{T_n > 0\}}$. For each n , denote by $X_t^{(1,n)}$ and $X_t^{(2,n)}$ the martingales appearing in the Krickeberg decomposition of $X_{t \wedge T_n}$.

The processes $X_{t \wedge T_{n-1}}^{(1,n)}$, $X_{t \wedge T_{n-1}}^{(2,n)}$ are positive martingales, and their difference is the martingale $X_{t \wedge T_{n-1}}$. Therefore we have

$$X_{t \wedge T_{n-1}}^{(1,n)} \geq X_t^{(1,n-1)}, \quad X_{t \wedge T_{n-1}}^{(2,n)} \geq X_t^{(2,n-1)}$$

and $E[X_0^{(1,n)} + X_0^{(2,n)}] = \sup_T E[|X_{T \wedge T_n}|]$. The processes $Y_t^{(i,n)} =$

$X_{t \wedge T_n}^{(i,n)} I_{\{t \leq T_n\}}$ ($i=1,2$) are supermartingales and increase with n , therefore their limit still is a right continuous process (see [1], chapter VI, theorem 16). Denote this limit by $X_t^{(i)}$. We also have

$$X_t^{(i)} = \lim_n X_t^{(i,n)}$$

The processes $X_t^{(i)}$ are positive supermartingales, their difference is X_t , and we have $E[X_0^{(1)} + X_0^{(2)}] \leq \|X\|_1$ from Fatou's lemma - in fact, this must be an equality, since the reverse inequality is obvious. On the other hand, $X_{t \wedge T_k}^{(i)}$ is the limit of the increasing sequence of martingales $X_{t \wedge T_k}^{(i,n+k)}$ as $n \rightarrow \infty$. We now remark that $X_{t \wedge T_k}^{(i,n+k)} = E[X_{T_k}^{(i,n+k)} | \mathcal{F}_t]$

and, using monotone convergence, that $X_{t \wedge T_k}^{(i)} = E[X_{T_k}^{(i)} | \mathcal{F}_t]$. Hence this process is a class (D) martingale, T_k reduces $X^{(i)}$, which therefore is a local martingale. The existence part is proved.

To prove the uniqueness, consider another decomposition $X = Y^{(1)} - Y^{(2)}$ where $Y^{(1)}, Y^{(2)}$ are positive local martingales, and $E[Y_0^{(1)} + Y_0^{(2)}] = \|X\|_1$. Note that $Y^{(i)}$ is a supermartingale, $i=1,2$. Stopping at time T_k , and using our remark at the end of the proof of proposition 1, we get that $Y_{t \wedge T_k}^{(i)} \geq X_t^{(i,k)}$. Letting $k \rightarrow \infty$, we have $Y^{(i)} \geq X^{(i)}$, and the condition on expectations implies $E[Y_0^{(i)}] = E[X_0^{(i)}]$. The positive supermartingale $Y^{(i)} - X^{(i)}$ being equal to 0 for $t=0$ must be identically 0, and the theorem is proved.

Corollary 1. For any local martingale X

$$(\forall \lambda > 0), \lambda P\left\{ \sup_t |X_t| > \lambda \right\} \leq \|X\|_1$$

Corollary 2. If $\|X\|$ is finite, then X_t converges a.s. to an integrable random variable as $t \rightarrow \infty$.

Proof. X is the difference of two positive supermartingales.

Remark that for any normal change of time $\Theta = (\mathcal{F}_t, \Theta_t)$ we have $(\Theta X)_t^{(i)} = X_{\Theta_t}^{(i)}$, $i=1,2$.

REFERENCES

- [1]. P.A.Meyer, Probabilités et Potentiels, Hermann, Paris, 1966
- [2]. P.A.Meyer. Non square integrable martingales, etc. Martingale theory (Oberwolfach meeting) Lecture Notes in Mathematics, vol.190, Springer-Verlag 1970.

Norihiko KAZAMAKI
Mathematical Institute
Tôhoku University
Sendai, Japon