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A simple proof of Doob’s convergence theorem

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<http://www.numdam.org/item?id=SPS_1971__5__76_0>
Doob's version of the fundamental convergence theorem of potential theory asserts that if \((f_n)\) is a decreasing sequence of excessive function and \(f\) is the supermedian function \(\inf f_n\), then the set where \(f\) differs from \(\hat{f}\) (its regularized function) is semi-polar. Many beautiful proofs of this result are available in the literature. Here is a trivial one.

By truncation we can assume \(f\) is bounded. Set \(A = \{f \geq \hat{f} + \varepsilon\}, \varepsilon > 0\). Since \(f\) is fine u.s.c., \(\hat{f}\) fine continuous, \(A\) is finely closed. Calling \(P_A\) the usual réduit operator, we have

\[
P_t f \geq P_A(\hat{f} + \varepsilon) = P_A \hat{f} + P_A \varepsilon
\]

( the first \(\geq\) follows from the second \(\geq\) from the fact that the measures \(P_A(x, \cdot)\) are carried by \(A\) ). Applying the semi-group

\[
P_t f \geq P_t P_A \hat{f} + P_t P_A \varepsilon
\]

As \(t \to 0\), we get

\[
\hat{f} \geq P_A \hat{f} + P_A \varepsilon
\]

At a point \(x\) regular for \(A\), this implies the absurd inequality \(\hat{f}(x) \geq \hat{f}(x) + \varepsilon\). Thus \(A\) has no regular point, and \(|f - \hat{f}| = \cup |f \geq \hat{f} + \frac{1}{n}|\) is semi-polar.

The same proof shows that if \(f\) is nearly Borel, positive and such that \(f \geq \hat{f}\) in every compact set \(K\), then \(|f - \hat{f}|\) is semi-polar. One must just take care to apply the above reasoning, not to \(A\) (which isn't known to be finely closed) but to \(K \subseteq A\), compact. If \(x\) is regular for \(A\) one chooses an increasing sequence \(K_n\) of compact subsets of \(A\) such that \(T_n \xrightarrow{\text{P}} 0\) \(P_K\)-a.s. and gets the same contradiction as above.

As a matter of fact the latter version, given very recently by Getoor and Rao, is the one that I stumbled into. On going over the above proof with Sieveking he pointed out the even shorter cut to the original Doob version.