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TWO FOOTNOTES TO A THEOREM OF RAY

John B. Walsh

Let E be a locally compact metric space and $\underline{\underline{E}}$ its Borel field. Let $(R_p)_{p>0}$ be a Markov resolvent family on $(E,\underline{\underline{E}})$, that is, for $p > 0$ and $x \in E$, $R_p(x,.)$ is a measure on E of mass $1/p$, for each $A \in \underline{\underline{E}}$ $R_p(.,A)$ is Borel measurable, and the resolvent equation is satisfied:

$$R_p - R_q = (q - p)R_pR_q \ .$$

Recall that a positive universally measurable function f is said to be q-supermedian if for each $p > 0$, $f \geq pR_{p+q}$. Let $C(E)$ be the class of functions continuous with finite limits at infinity and $C_0(E)$ the class of continuous functions with limit zero at infinity, and let $S(E)$ be the class of continuous 1-supermedian functions. We say (R_p) is a Ray resolvent if $S(E)$ separates points of E and for each p $R_p:C_0(E) \to C_0(E)$. A very important theorem of Ray states:

<u>THEOREM</u> (Ray) Let (R_p) be a Ray resolvent. Then there exists a unique semigroup $(P_t)_{t \geq 0}$ satisfying

(i) $t \to P_t(x,.)$ is vaguely right continuous;

(ii) (P_t) has resolvent (R_p).

Further, for each probability measure ν on $\underline{\underline{E}}$ there exists a right continuous strong Markov process whose paths have left limits in E, and which has transition semigroup (P_t) and absolute distributions $(\nu P_t)_{t \geq 0}$.(1),(2)

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(1) In general P_t is not the only semigroup corresponding to the resolvent (R_p). In fact, let (Q_t) be defined by

$$Q_t(x,\cdot) = \begin{cases} \delta_x(\cdot) & \text{if } t = 0 ; \\ \text{vague} \lim_{s \uparrow t} P_s(x,.) & \text{if } t > 0. \end{cases}$$

Then (Q_t) is also a semigroup, and is the transition function of a left continuous moderately Markov process. Recall that a left continuous process is said to be moderately Markov relative to an increasing family $(\underline{\underline{F}}_t)_{t \geq 0}$ of fields and transition function $P_t(x,A)$ if it is adapted to $\underline{\underline{F}}_t$ and if for each predictable stopping time T, $t \geq 0$, and bounded Borel f

$$E\{ f(X_{t+T}) \mid \underline{\underline{F}}_{T-}\} = P_t f(X_T) .$$

This is a natural form of the strong Markov property for a left continuous process; moderately Markov processes have recently been shown to be interesting in their own right. The left handed version of Ray's theorem is:

THEOREM 1 Let (R_p) be a Ray resolvent. Then there exists a unique semigroup $(Q_t)_{t \geq 0}$ satisfying

(i) $t \to Q_t(x,.)$ is vaguely left continuous and $Q_0(x,.) = \delta_x(\cdot)$ for each $x \in E$;

(ii) (Q_t) has resolvent (R_p).

Further, for each probability measure ν on $\underline{\underline{E}}$ there exists a left continuous moderately Markov process whose paths have right limits in E and which has transition semigroup (Q_t) and absolute distributions $(\nu Q_t)_{t \geq 0}$.

Some of the standard proofs of Ray's theorem can be modified to prove both right and left-handed versions at the same time, and this is doubtless the most satisfying method of proof, but Theorem One can be derived from the other without great difficulty. It is interesting, tho, that the semigroup property of Q_t doesn't seem to follow easily unless one uses some properties of the sample paths. A corollary of this theorem is that if X is a right continuous strong Markov process with the above semigroup (P_t), then the left continuous process (X_{t-}) is moderately Markov with semigroup (Q_t).

<u>PROOF</u> Let (P_t) be the vaguely right continuous semigroup corresponding to (R_p) and define (Q_t) as above. Fix a probability measure ν on $\underline{\underline{E}}$, and let X be a right continuous strong Markov process with semigroup (P_t) and absolute distributions (νP_t). It is enough to prove the theorem in the case $\nu = \delta_{x_0}$ for some fixed x_0. Define a process Y by

$$Y_t = \begin{cases} \lim\limits_{s \uparrow t} X_s & \text{if } t > 0, \\ x_0 & \text{if } t = 0. \end{cases}$$

Now $Q_t(x,.) = P_t(x,.)$ except for possibly countably many points, so certainly $\int_0^\infty e^{-pt} Q_t(x,A)dt = \int_0^\infty e^{-pt} P_t(x,A)dt$ for each $p > 0$, $x \in E$ and $A \in \underline{\underline{E}}$, so that (Q_t) also has (R_p) for resolvent.

Suppose $f \in C_0(E)$. Then $E^x\{f(Y_t)\} = E^x\{f(X_{t-})\} = Q_t f(x)$. Further, $R_p f \in C_0(E)$ so that $t \to R_p f(Y_t)$ is left continuous. If $T > 0$ is a predictable stopping time and (T_n) is a sequence of stopping times announcing T, that is $T_n < T$ and $T_n \uparrow T$, then in the limit the equation

$$E^{x_0}\left\{ p \int_0^\infty e^{-pt} f(X_{T_n+t})dt \,\middle|\, \underline{\underline{F}}_{T_n} \right\} = pR_p f(X_{T_n})$$

gives

$$E^{x_0}\left\{ p \int_0^\infty e^{-pt} f(X_{T+t})dt \,\middle|\, \underline{\underline{F}}_{T-} \right\} = pR_p f(Y_T) .$$

Therefore if $p \to \infty$ we have

$$(1.1) \qquad E^{x_0}\left\{ f(X_T) \,\middle|\, \underline{\underline{F}}_{T-} \right\} = P_0 f(Y_T) .$$

For each $t > 0$ let us calculate

$$E^{x_0}\left\{ f(Y_{Y+t}) \,\middle|\, \underline{\underline{F}}_{T-} \right\} = E^{x_0}\left\{ E^{x_0}\{f(Y_{T+t}) \mid \underline{\underline{F}}_T\} \,\middle|\, \underline{\underline{F}}_{T-} \right\}.$$
$$= E\left\{ Q_t f(X_T) \,\middle|\, \underline{\underline{F}}_{T-} \right\} = P_0 Q_t f(Y_T) .$$

But if $t > 0$ we claim $P_0 Q_t = Q_t$. Since for $f \in C_0(E)$ both $P_0 Q_t f$

and $Q_t f$ are left continuous functions of t, it is enough to prove equality of the Laplace transforms. But that is immediate:

$$\int_0^\infty e^{-pt} P_0 Q_t f(x)dt = P_0 R_p f(x) = R_p f(x) = \int_0^\infty e^{-pt} Q_t f(x)dt .$$

Therefore $E^{x_o}\{f(Y_{T+t}) \mid \underline{\underline{F}}_{T-}\} = Q_t f(Y_T)$. Hence Y is moderately Markov. To show that (Q_t) is actually a semigroup, Set $T \equiv s > 0$ and take expectations of both sides. We get

$$Q_{s+t} f(x_o) = Q_s Q_t f(x_o) \quad \text{if } t > 0 , s > 0 .$$

But Q_0 is the identity, so clearly $Q_{s+t} = Q_s Q_t$ if either s or t is zero, and we are done.

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(2) It is tempting to say that for each ν, "there is a process with initial measure ν." Generally, however, this is false; there exist points, called branching points, at which $P_0(x,.) \neq \delta_x$. A process "starting from x" in such a case has initial measure $P_0(x,.)$, that is to say it starts from some point entirely different from x! The possibility of having branching points is one of the most interesting aspects of these processes. It is well known, for instance, that a right continuous strongly Markov process with a Ray resolvent (Ray process for short) is quasi left continuous except when it approaches a branching point; at such a time it always jumps. This is in fact evident from (*). Another question involving the branching points is the question of discontinuities of the fields $\underline{\underline{F}}_t$. Recall that an increasing family $(\underline{\underline{F}}_t)$ of Borel fields is said to be free of discontinuities if for each stopping time T and sequence $T_n \uparrow T$ of stopping times, $\underline{\underline{F}}_T = \bigvee \underline{\underline{F}}_{T_n}$. It is well known that $(\underline{\underline{F}}_t)$ is free of discontinuities iff for each predictable T, $\underline{\underline{F}}_T = \underline{\underline{F}}_{T-}$, where $\underline{\underline{F}}_{T-}$ is given by $\underline{\underline{F}}_{T_n}$, and (T_n) is any sequence of stopping times announcing T.

For concreteness, let Ω be the space of right continuous functions from $[0,\infty)$ to E and let X_t be the coordinate random variable. $\underline{\underline{F}}_t^0$ denotes the Borel field generated by $\{X_s, s \le t\}$ and $\underline{\underline{F}}^0 = \bigvee \underline{\underline{F}}_t^0$. If P is a measure on $\underline{\underline{F}}^0$, then $\underline{\underline{F}}^P$ is the completion of $\underline{\underline{F}}^0$ with respect to P, and $\underline{\underline{F}}_t^P$ is the Borel field generated by $\underline{\underline{F}}_t^0$ plus all P-null sets of $\underline{\underline{F}}^P$. The following theorem is implicit in the field continuity theorems of Meyer; we provide a proof for the sake of completeness.

<u>THEOREM 2</u> Suppose that X is strongly Markov (and necessarily right continuous but not necessarily a Ray process) on $(\Omega, \underline{\underline{F}}^P, P)$. Let (T_n) be a sequence of stopping times increasing to a stopping time T. Then a necessary and sufficient condition that $\underline{\underline{F}}_T^P = \bigvee \underline{\underline{F}}_{T_n}^P$ is that X_T be measurable with respect to $\bigvee \underline{\underline{F}}_{T_n}^P$.

<u>PROOF</u> We will write $\underline{\underline{F}}$ and $\underline{\underline{F}}_t$ instead of $\underline{\underline{F}}^P$ and $\underline{\underline{F}}_t^P$ respectively. The condition is surely necessary, for X_T is $\underline{\underline{F}}_T$ measurable. On the other hand the martingale convergence theorem assures us that if $\bigvee \underline{\underline{F}}_{T_n} = \underline{\underline{F}}_T$, then for each integral random variable Y

(2.1) $$E\{Y \mid \underline{\underline{F}}_T\} = \lim_{n\to\infty} E\{Y \mid \underline{\underline{F}}_{T_n}\} \quad \text{w.p.1.}$$

Following Meyer, it is enough to show (2.1) for Y of the form $\prod_1^n Y_j$, where $Y_j = \int_0^\infty f_j(X_t)dt$ and f_j is continuous of compact support, for these span a space dense in $L_1(\Omega, \underline{\underline{F}}^P, P)$.

For each $n = 1,\dots,\infty$, (with $T_\infty = T$) write

$$Y_j = Y_j^-(n) + Y_j^o(n) + Y_j^+ = \int_0^{T_n} f_j(X_s)ds + \int_{T_n}^{T} f_j(X_s)ds + \int_T^\infty f_j(X_s)ds .$$

Then $E\{Y \mid \underline{\underline{F}}_{T_n}\}$ is a sum of products of the form

$$Y_a^-(n)\dots Y_b^-(n) E\{Y_c^o(n)\dots Y_d^o(n) Y_e^+ \dots Y_k^+ \mid \underline{\underline{F}}_{T_n}\},$$

where $a,\dots,b,\dots,k$ is a permutation of $1,\dots,n$.

As $n \to \infty$, $Y_j^o(n)$ tends to 0, hence so does any term containing $Y_j^o(n)$. An integral is a continuous function of its upper limit, so $Y_j^-(n)$ tends to $Y_j^-(\infty)$ for each j. Let $Z = Y_e^+, \dots, Y_k^+$. It remains to show $E\{Z \mid \underline{\underline{F}}_{T_n}\} \to E\{Z \mid \underline{\underline{F}}_T\}$. But

$$E\{Z \mid \underline{\underline{F}}_{T_n}\} = E\left\{E\{Z \mid \underline{\underline{F}}_T\} \middle| \underline{\underline{F}}_{T_n}\right\} = E\left\{E\{Z \mid X_T\} \middle| \underline{\underline{F}}_{T_n}\right\}$$

by the strong Markov property. As n tends to infinity, this tends to

$$E\left\{E\{Z \mid X_T\} \middle| \bigvee \underline{\underline{F}}_{T_n}\right\} = E\{Z \mid X_T\}$$

since X_T is $\bigvee \underline{\underline{F}}_{T_n}$ measurable, and we are done. qed

Now let us return to the case where X is a Ray process, with resolvent (R_p) and (right continuous) semigroup (P_t). For each probability measure ν on $\underline{\underline{E}}$ there is a measure P^ν on $\underline{\underline{F}}^0$ corresponding to the process X "with initial measure ν." We define the fields $\underline{\underline{F}} = \bigcap_\nu \underline{\underline{F}}^{P^\nu}$ and $\underline{\underline{F}}_t = \bigcap_\nu \underline{\underline{F}}_t^{P^\nu}$; in short, the usual situation. We say x is a branching point if $P_0(x,.) \neq \delta_x$. The set K of branching points is a Borel set. If T is a predictable time, then $X_T = X_{T-}$ a.s. on $\{X_{T-} \in K^c\}$. This is well known, but is also a trivial consequence of (1.1): for f bounded and Borel measurable

$$E\left\{[f(X_T) - f(X_{T-})]^2 \middle| \underline{\underline{F}}_{T-}\right\} = P_0 f^2(X_{T-}) - 2f(X_{T-})P_0 f(X_{T-}) + f^2(X_{T-})$$

which is zero on $\{X_{T-} \in K^c\}$.

<u>THEOREM 3</u> Let X be a Ray process, and suppose the resolvant separates points. Let T be a predictable stopping time. Then T is a time of discontinuity for the fields $(\underline{\underline{F}}_t)$ if and only if for some probability measure ν on $\underline{\underline{E}}$,

$$P^\nu\{X_{T-} \in K\} > 0 .$$

<u>REMARK</u> The hypothesis that the resolvent separates points is less for truth than for beauty. If it doesn't, one can always identify points with the same resolvent, or simply replace K by $L = \{x: P_0(x,.) \text{ is not a point mass}\}$.

PROOF According to Theorem 2, $\underline{\underline{F}}_{T-} = \underline{\underline{F}}_T$ iff for all bounded Borel f and probability measures ν on $\underline{\underline{E}}$, $E\{ f(X_T) \mid \underline{\underline{F}}_{T-}\} = f(X_T)$, or equivalently, iff

(2.2) for all $f \in C(E)$ and probability measures ν on $\underline{\underline{E}}$,

$$P_0 f(X_{T-}) = f(X_T) \quad \text{a.s. } (P^\nu) \; .$$

Suppose $P^\nu\{X_{T-} \in K\} = 0$ for all ν. Then for all ν: $X_{T-} = X_T$ a.s. (P^ν) so

$$P_0 f(X_{T-}) = f(X_{T-}) = f(X_T) \quad \text{a.s. } (P^\nu) \; .$$

Conversely, suppose that for some ν, $P^\nu\{ X_{T-} \in K\} > 0$. We remark that a necessary and sufficient condition for a measure N on $\underline{\underline{E}}$ to be a point mass is that for any two functions f and g in C(E) $N(f \wedge g) = N(f) \wedge N(g)$. Now since $x \in K$, $P_0(x,.)$ cannot be a point mass, for if $P_0(x,\{y\}) = 1$ for some $y \neq x$, say, then $R_p(x,.) = P_0 R_p(X,.) = R_p(y,.)$, contradicting the hypothesis that the resolvent separates points. Thus as C(E) is separable , we can find a pair f and g in C(E) such that with positive P^ν-probability

$$P_0(f \wedge g)(X_{T-}) > P_0 f(X_{T-}) \wedge P_0 g(X_{T-}) \; .$$

But this clearly implies that we can't have (2.2) for all three functions f, g, and $f \wedge g$. qed