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THE OKA-WEIL THEOREM IN LOCALLY CONVEX SPACES WITH THE APPROXIMATION PROPERTY

by Jorge MUJICA (*)

1. Introduction.

The classical Oka-Weil theorem asserts that every function which is holomorphic on a neighbourhood of a polynomially convex compact set K in $\mathbb{C}^n$ can be approximated uniformly on K by polynomials. In this paper, we show that the Oka-Weil theorem is still true in every quasi-complete locally convex space with the approximation property, thus improving earlier results of NOVERRAZ [5], LIGOCKA [1] and SCHOTTENLOHER [7]. The proof of the theorem is very simple, being no more than a straightforward consequence of a result of LIGOCKA [1]. As an application of the Oka-Weil theorem, we characterize the spectra of certain topological algebras of holomorphic functions.

2. Elementary properties of polynomially convex sets.

Throughout this paper, the letter E denotes a locally convex space, which is always assumed to be complex and Hausdorff. We let $\mathcal{P}(E)$ denote the space of all continuous polynomials on E, and we let $\mathcal{H}(U)$ denote the space of all holomorphic functions on an open subset U of E. We refer to NACHBIN [4] or NOVERRAZ [6] for the basic properties of polynomials and holomorphic functions on infinite dimensional spaces.

2.1 Definition. - Given a compact set $K \subset E$, we define its polynomially convex hull $\hat{K}_E$ by

$$\hat{K}_E = \{x \in E \; ; \; |P(x)| \leq \sup_K |P| \; , \text{ for all } P \in \mathcal{P}(E)\} .$$

We will often write $\hat{K}$ instead of $\hat{K}_E$ when the space E is tacitly understood. The compact set K is said to be polynomially convex if $\hat{K} = K$.

The following easily proved remark is often useful.

2.2 Remark. - Let M be a vector subspace of E, with the induced topology. Then $\hat{K}_M \subset \hat{K}_E \cap M$, for any compact set $K \subset M$, and equality holds, if M is a complemented subspace of E, i. e. if there exists a continuous projection of E onto M.

2.3 PROPOSITION ([6], lemma 2.1.2). - *For each compact subset K of E, the polynomially convex hull $\hat{K}$ of K is contained in the closed, convex hull of K.*

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Proof. - Let L denote the closed, convex hull of K, and let $a \notin L$. By the Hahn-Banach theorem, there exists a continuous linear form φ on E and a real number θ such that

$$\operatorname{Re} \varphi(x) < \theta < \operatorname{Re} \varphi(a) , \text{ for all } x \in L .$$

Since the set $\varphi(L)$ is bounded, we can find a closed disc $D(\zeta ; R)$ containing $\varphi(L)$ and not containing $\varphi(a)$. Then, $P(x) = \varphi(x) - \zeta$ defines a continuous polynomial on E and

$$\sup_K |P| \leqslant \sup_L |P| \leq R < |P(a)| .$$

Hence $a \notin \hat{K}$, and therefore $\hat{K} \subset L$.

2.4 COROLLARY. - For each compact set $K \subset E$, the set $\hat{K}$ is always precompact, and is compact when E is quasi-complete.

2.5 Definition. - Let U be an open set in E.

(a) We say that U is polynomially convex if, for each compact set $K \subset U$, the set $\hat{K} \cap U$ is bounded away from ∂U, i. e. there exists a 0-neighbourhood V such that $\hat{K} \cap U + V \subset U$ (in view of corollary 2.4, when E is quasi-complete, this is equivalent to saying that $\hat{K} \cap U$ is compact).

(b) We say that U is strongly polynomially convex if, for each compact set $K \subset U$, the set $\hat{K}$ is contained in U.

2.6 Remark. - If E is quasicomplete, then clearly every strongly polynomially convex open set in E is polynomially convex. It is not known whether the converse holds in general. See corollary 3.3 below for a partial converse.

2.7 PROPOSITION.

(a) Every convex open set is strongly polynomially convex.

(b) Every open set of the form $U = \{x \in E ; |P(x)| < 1\}$, where $P \in \mathcal{P}(E)$, is strongly polynomially convex.

(c) The intersection of two (strongly) polynomially convex open sets is a (strongly) polynomially convex open set.

Proof. - (b) and (c) are trivial. In view of proposition 2.3, to show (a) it suffices to show that the closed convex hull of each compact set $K \subset U$ is also contained in U. Let V be a convex 0-neighbourhood such that $K + V \subset U$. If L denotes the convex hull of K, then clearly $L + V \subset U$ too. Hence $\bar{L} \subset L + V \subset U$.

3. The Oka-Weil theorem.

The key of the proof of the Oka-Weil theorem is the following result of LIGOCKA ([1], proposition 2.1).

3.1 THEOREM. - Let E be a quasi-complete locally convex space, and let K be a polynomially convex compact subset of E. Then, every open neighbourhood of K contains another open neighbourhood of K which is strongly polynomially convex.

Proof. - Let us write, for each $P \in \mathcal{P}(E)$,

$$A_P = \{x \in E \ ; \ |P(x)| < 1\} , \quad B_P = \{x \in E \ ; \ |P(x)| \leqslant 1\} .$$

Let U be an open neighbourhood of K and let L denote the closed convex hull of K. Then, for each $x \in L \setminus U$, there exists $P \in \mathcal{P}(E)$ such that

$$\sup_K |P| < 1 < |P(x)| .$$

Hence by compactness of $L \setminus U$, we can find $P_1 , \ldots , P_n \in \mathcal{P}(E)$ such that

(1) $$K \subset A_{P_1} \cap \ldots \cap A_{P_n}$$

and

$$L \setminus U \subset \complement B_{P_1} \cup \ldots \cup \complement B_{P_n}$$

and the last written inclusion implies

(2) $$L \cap B_{P_1} \cap \ldots \cap B_{P_n} \subset U .$$

We claim that

(3) $$(L + W) \cap B_{P_1} \cap \ldots \cap B_{P_n} \subset U ,$$

for some convex open 0-neighbourhood W. Let (W_α) be a base of convex open 0-neighbourhoods and let us assume that, for each α, there exists a point

$$x_\alpha \in (L + W_\alpha) \cap B_{P_1} \cap \ldots \cap B_{P_n} \cap \complement U .$$

For each α, we choose $y_\alpha \in L$ such that $x_\alpha - y_\alpha \in W_\alpha$. Since L is compact, the net (y_α) has a subnet (y_β) which converges to a point $y \in L$. But, then the corresponding subnet (x_β) of (x_α) also converges to y, and then, by (2),

$$y \in L \cap B_{P_1} \cap \ldots \cap B_{P_n} \subset U .$$

But, this is impossible, for $x_\beta \longrightarrow y$, and $x_\beta \notin U$, for every β. Thus (3) is proved. We then set

$$V = (L + W) \cap A_{P_1} \cap \ldots \cap A_{P_n} .$$

It follows from (1) and (3) that $K \subset V \subset U$, and in view of proposition 2.7, V is strongly polynomially convex.

Now the proof of the Oka-Weil theorem is straightforward.

3.2 Oka-Weil THEOREM. - Let E be a quasi-complete locally convex space with the approximation property, and let K be a polynomially convex compact subset of E. Then, every function, which is holomorphic on an open neighbourhood of K, can be approximated uniformly on K by continuous polynomials on E.

<u>Proof</u>. - Let $f \in \mathcal{H}(U)$, where U is an open neighbourhood of K. By theorem 3.1, we may assume that U is strongly polynomially convex. Let $\epsilon > 0$ be given. Then, one can easily find an open 0-neighbourhood V such that $K + V \subset U$ and

$$|f(y) - f(x)| < \epsilon \text{, for } x \in K \text{, } y \in x + V \text{.}$$

Since E has the approximation property, there exists a continuous linear operator $T : E \longrightarrow E$, of finite rank, and such that

$$T(x) - x \in V \text{, for every } x \in K \text{.}$$

Then, it follows that

$$|f \circ T(x) - f(x)| < \epsilon \text{, for every } x \in K \text{,}$$

and also that $T(K) \subset K + V \subset U$. Since U is strongly polynomially convex, we get that $\widehat{T(K)}_E \subset U$ and hence

$$\widehat{T(K)}_{T(E)} = \widehat{T(K)}_E \cap T(E) \subset U \cap T(E) \text{.}$$

Set $L = \widehat{T(K)}_{T(E)}$. Then L is a polynomially convex compact subset of $T(E)$ and the restriction of f to $U \cap T(E)$ is holomorphic on a neighbourhood of L in $T(E)$. Then, by the classical Oka-Weil theorem, there exists $P \in \mathcal{P}(T(E))$ such that

$$\sup_L |f - P| \leqslant \epsilon \text{.}$$

Then $P \circ T \in \mathcal{P}(E)$ and

$$\sup_K |f - P \circ T| \leqslant \sup_K |f - f \circ T| + \sup_K |f \circ T - P \circ T| \leqslant 2\epsilon$$

concluding the proof.

3.3 COROLLARY. - <u>Let</u> E <u>be a quasi-complete locally convex space with the approximation property. Then, every polynomially convex open set in</u> E <u>is strongly polynomially convex</u>.

<u>Proof</u>. - The proof is classical. Let U be a polynomially convex open set in E, and let K be a compact subset of U. Then, the compact set $\hat{K}$ may be written as the union of the disjoint compact sets $\hat{K} \cap U$ and $\hat{K} \setminus U$. We define a function f to be equal to zero on a neighbourhood of $\hat{K} \cap U$ and equal to one on a neighbourhood of $\hat{K} \setminus U$. Then, f is holomorphic on a neighbourhood of $\hat{K}$, and, by theorem 3.2, there exists $P \in \mathcal{P}(E)$ such that

$$\sup_{\hat{K}} |f - P| < \frac{1}{2} \text{.}$$

Then, for any point $a \in \hat{K} \setminus U$,

$$\sup_K |P| < \frac{1}{2} < |P(a)| \text{,}$$

a contradiction, unless $\hat{K} \setminus U$ is empty.

4. Elementary properties of topological algebras.

By an algebra, we always mean a commutative algebra over the complex numbers, and having an identity element. By a complex homomorphism of an algebra A, we mean an algebra homomorphism $T : A \longrightarrow \mathbb{C}$ with $T(1) = 1$.

4.1 Definition. - A topological algebra is an algebra and a topological vector space such that ring multiplication is separately continuous. A locally convex algebra is a topological algebra which is a locally convex space. A locally multiplicatively convex algebra is a topological algebra which has a base of convex and idempotent neighbourhoods of zero (V is idempotent if $V^2 \subset V$). A Q-algebra is a topological algebra where the invertible elements form an open set. The spectrum of a topological algebra A is the set of all continuous complex homomorphisms of A.

4.2 PROPOSITION. - *Let (A_α) be a family of subalgebras of an algebra A, directed under inclusion, and satisfying $A = \bigcup A_\alpha$. Let us assume that each A_α is a locally convex algebra and that each inclusion mapping $A_\alpha \hookrightarrow A_\beta$ is continuous. Let A be endowed with the locally convex inductive topology with respect to the inclusion mappings $A_\alpha \hookrightarrow A$. Then :*

(a) *A is a locally convex algebra.*

(b) *If each A_α is a Banach algebra and if each inclusion mapping $A_\alpha \hookrightarrow A_\beta$ has norm not greater than one, then A is a Q-algebra.*

Proof.

(a) It suffices to show that ring multiplication in A is separately continuous. Given $y \in A$, we choose α_0 such that $y \in A_{\alpha_0}$. Then the mapping

$$x \in A_\alpha \longrightarrow xy \in A_\alpha$$

is continuous, for every $\alpha \geq \alpha_0$, and it follows that the mapping $x \in A \longrightarrow xy \in A$ is also continuous.

(b) For each α, let V_α denote the open unit ball of A_α. Since A_α is a Banach algebra, $1 + h$ is invertible in A_α, for every $h \in V_\alpha$. Let $V = \bigcup V_\alpha$. Then V is a convex 0-neighbourhood in A, and $1 + h$ is invertible in A, for every $h \in V$. Let $x \in A$ be invertible. Choose a 0-neighbourhood U in A such that $x^{-1} h \in V$, for every $h \in U$. Then, $x + h = x(1 + x^{-1} h)$ is invertible in A, for every $h \in U$.

4.3 PROPOSITION. - *Let (A_α) be a family of locally convex algebras and, for each α, let $\pi_\alpha : A \longrightarrow A_\alpha$ be an algebra homomorphism of an algebra A into A_α. If A is endowed with the projective topology with respect to the mappings π_α, then A is also a locally convex algebra.*

Proof. - It suffices to show that ring multiplication in A is separately continuous. If $y \in A$, then, for each α, the mapping $\pi_\alpha(x) \in A_\alpha \longrightarrow \pi_\alpha(x)\, \pi_\alpha(y) \in A_\alpha$

is continuous, and it follows that the mapping $x \in E \longrightarrow xy \in A$ is also continuous.

5. Topological algebras of holomorphic functions.

For any open set $V \subset E$, we let $\mathcal{H}^\infty(V)$ denote the Banach space of all bounded holomorphic functions on V with the norm of the supremum. For any compact set $K \subset E$, we define $\mathcal{H}(K)$, the space of all holomorphic germs on K, as the locally convex inductive limit of the Banach spaces $\mathcal{H}^\infty(V)$, where V varies among the open neighbourhoods of K. Then, from proposition 4.2, we get at once the following proposition.

5.1 PROPOSITION. - $\mathcal{H}(K)$ <u>is always a locally convex algebra and a</u> Q-<u>algebra</u>.

Let U be any open set in E. A semi-norm p on $\mathcal{H}(U)$ is ported by a compact set $K \subset U$ if, for each open set V, with $K \subset V \subset U$, there exists a positive constant C such that $p(f) \leq C \sup_K |f|$, for every $f \in \mathcal{H}(U)$. The topology τ_ω, introduced by NACHBIN [4], is the locally convex topology on $\mathcal{H}(U)$ generated by all semi-norms which are ported by compact sets.

5.2 THEOREM. - $(\mathcal{H}(U), \tau_\omega)$ <u>is always a locally convex algebra</u>.

<u>Proof</u>. - [2] (lemma 5.3) tells us that τ_ω is the projective topology with respect a certain family of algebra homomorphisms $\pi_K : \mathcal{H}(U) \longrightarrow \mathcal{H}^K(U)$, where each $\mathcal{H}^K(U)$ is an inductive limit of normed algebras. The conclusion then follows from propositions 4.2 and 4.3.

5.3 <u>Remark</u>. - If E is metrizable, then both $\mathcal{H}(K)$ and $(\mathcal{H}(U), \tau_\omega)$ are locally multiplicatively convex algebras : see [2], theorems 7.1 and 7.2. We do not know whether this remains true for non-metrizable E.

In [3], we characterized the spectra of $\mathcal{H}(K)$ and $(\mathcal{H}(U), \tau_\omega)$ in the case where K and U are polynomially convex and E is a Fréchet space with the approximation property. To obtain those results, we used a version of the Oka-Weil theorem in Fréchet spaces with the approximation property due to SCHOTTENLOHER [7]. If one uses theorem 3.2 instead, then the same proof works in every quasi-complete locally convex space with the approximation property. Thus, we get the following theorem.

5.4 THEOREM. - <u>Let</u> E <u>be a quasi-complete locally convex space with the approximation property, and let</u> K <u>be a polynomially convex compact subset of</u> E. <u>Then, for each complex homomorphism</u> T <u>of</u> $\mathcal{H}(K)$, <u>there exists a unique point</u> $a \in K$ <u>such that</u> $T(f) = f(a)$, <u>for every</u> $f \in \mathcal{H}(K)$.

5.5 THEOREM. - <u>Let</u> E <u>be a quasi-complete locally convex space with the approximation property, and let</u> U <u>be a polynomially convex open subset of</u> E. <u>Then, for each continuous complex homomorphism</u> T <u>of</u> $(\mathcal{H}(U), \tau_\omega)$, <u>there exists a unique point</u> $a \in U$ <u>such that</u> $T(f) = f(a)$, <u>for every</u> $f \in \mathcal{H}(U)$.

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