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APPENDIX: from the JOURNAL OF SYMBOLIC LOGIC (to appear)

REVIEW OF:

Solomon Feferman: " Weyl vindicated: **Das Kontinuum** 70 years later "

The review of this interesting paper is divided into two parts, as the paper itself is split into a philosopical or historical part and a technical one. I will first hint the technical aspects, which are very relevant for predicativity and set on firm grounds an established area of investigation, and later discuss the informal preliminaries (and the title of the paper!). These present a rather partial view of Weyl's contribution to the foundation of Mathematics, a view which I do not share, because of both the historical and the foundational perspective proposed by the author.

The core of the approach presented beginning with section 4, is the notion of "definability". In the "definitionist" approach, which is essentially the Feferman's view, given (possibly) for granted the totality of the collection of the natural numbers, the other mathematical notions must be defined in some sort of "stratified" way. Stratification may mean entirely typed definitions following Russell, or, more weakly, "predicative" definitions, in the sense made clear by Poincare' and Weyl. In short, << ... we draw a distinction between two types of classifications...: the predicative classification which cannot be disordered by the introduction of new elements; the non-predicative classifications in which the introduction of new elements necessitates constant modification>> (Poincaré [1913, p.47]). As the author is interested, in the end, in the foundation of Analysis, impredicativity is understood as a second order notion as it typically applies in the definition of sets. And a set is impredicatively given when <<...quantified variables may range on a set which includes the definiendum...>>, Weyl[1918, I.6]. That is, a set b is defined in an impredicative way if it is given by

(1) $b = \{ x \mid \text{for all } y \text{ in } A, P(x,y) \}$

where b itself may be an element of A.

As a matter of fact, the author refers to Weyl's ideas in his 1910 paper and in the 1918 book with a remarkable philological attention. The core of the technical part reconstructs with great care Weyl's hint towards a predicative Analysis. Indeed, Weyl's book is extremely incomplete and vague in this proposal, while rich of mathematically deep observations. The author takes up Weyl's largely informal remarks step by step and turns them into a clear and complete approach to predicative mathematics. In particular, section 5 carefully describes iteration and induction, and clarifies the connections between the various different forms of Weyl's "principle of iteration", by a careful formalization of Weyl's informal hints. Section 6 faithfully rewrites in modern formalism Weyl's axiom system: the structure and role of explicit definitions and comprehension axioms is clearly presented and this contributes, in an essential way, to the understanding of a few very obscure pages of Weyl's book. Section 7 compares Weyl's approach, as formalized in the previous section, to extant approaches to predicative Analysis. This sections, in particular, refers to the author's (and others', such as Kreisel's) work in predicative systems, since the early 60-ties (see references). Weyl's approach turns out to be "equivalent" to sufficiently large fractions of (formally) rather expressive systems. By this, one may say that Weyl's aim of setting on clear grounds large part of Analysis is achieved. Indeed, by the extension to "flexible types" in section 8, the author updates even further the previous approach to the current debate on higher order type systems. An interesting result of "conservative extension" of the system w.r.t. to PA is also stated. In section 9, a few arguments are hinted towards the actual expressiveness of the predicative systems proposed, for the purposes of applicable Analysis, to Physics, tipically. Recent and unpublished work by the author and Simpson reinforces the thesis just sketched in this section, by a technically deep insight into various area of "predicative" mathematics. However, it is known that Lebesgue measures, least upper and largest lower bounds escape predicativity. Indeed, "the continuum" goes beyond it.

Let's now go back to a "critique" of the historical and epistemological perspective presented by the author in the first part. For this purpose, I will also borrow a few ideas from the first part of Longo[1989]. First of all, in my view, the main relevance of Weyl's "The Continuum" is not due to the informal hints towards predicative Analysis, but to the critique of the formalist approach to Analysis and to a deep and original discussion on the relation between our intuition of the physical continuum and its mathematical formalizations. In my opinion, instead, the paper "Weyl vindicated: ..." (and its title), suggests a rather partial view of this broad, deep, wonderful thinker and mathematician. Of course, the author's work towards the clarification of Weyl's specific proposal is an interesting and a relevant contribution to predicative Analysis, as I tried to say. The danger is that the reader may classify and reduce by this Weyl's contribution to the foundation of mathematics as one of the many (attempted) "reductionist" formalizations. As a matter of fact, Weyl's very broad scientific experience led him to explore and appreciate, over the years, several approaches to the foundation of Mathematics, sometimes wavering between different viewpoints. The actual unity of Weyl's epistemological view may be found in his overall philosophy of Mathematics and of scientific knowledge, a matter he treated in several writings from 1910 to 1952, the time of his retirement from the Institute for Advanced Studies, in Princeton. As an important aspect of this view, one should stress a crucial critique to Hilbert's approach: Weyl keeps stressing, in several writings, that what really matters in (meta)mathematics is the relevance of axiom systems, in their broad connections to the structures under investigation and the physical reality; consistency is a necessary, but far less relevant condition. A further critique of Hilbert's program is clearly expressed in Weyl's book. If we could <<...decide the truth or falsity of every geometric assertion (either specific or general) by methodically applying a deductive technique (in a finite number of steps), then Mathematics would be trivialized, at least in principle...>> (Weyl[1918; I.3]). Weyl's awareness of the limitations of formalism is so strong (and his mathematical intuition so deep) that, at the reader's surprise, a few lines below, he conjectures that there may be number theoretic assertions independent of the axioms of Arithmetic (in 1918!). (Indeed, he suggests, as an example, the assertion that, for reals r and s, $r < s$ iff there exists a rational q such that $r < q < s$. There may be cases where <<... neither the existence nor the non-existence of such a rational is a consequence of the axioms of Arithmetic>>. Can we say anything more specific about this, now that we also know of mathematical independence results such as Paris-Harrington's ? This would really vindicate Weyl, whose views on this matter greatly annoyed his former professor and major academic authority, David Hilbert). Two sections later, Weyl conjectures that <<...there is no reason to believe that any infinite set must contain a countable set>>. This is a very early hint in the right direction for the independence of the axiom of choice (!).

This insight of Weyl's into mathematical structures seems scarcely influenced, either positively or negatively, by the predicativist approach he is proposing. It is more related to an "objective" understanding of mathematical definitions and to his practical work. What "objective" or "independent from specific formalizations" means here is a delicate issue, for which one may consult Weyl[1918] (or Longo[1989;1991]).

Indeed, what really interests Weyl is the understanding of mathematics as part of our human endeavour towards knowledge, in particular of the physical world. Weyl stresses the inadequacies of the mathematical formalization with respect to a crucial aspect of our physical experience (see chapter II, §.6): our intuition of the continuity of space and time (Weyl greatly contributed to the mathematics of Einstein's relativity). In his view, the phenomenal experience of time, as past, present and future, is unrelated to the mathematical treatment of the real numbers. Time cannot be decomposed in points. Present lasts continuously, it is <<something ever new which endures and changes in consciousness>>. In our perception of time, <<an individual point is non-independent... it exists only as a point of transition.. it cannot be exhibited in any way ... only an *approximate,* never an exact determination of it is possible>> (again, chapter II, §.6). Even the use of limit points or ideal constructions, the essence of mathematics according to Weyl, do not help us sufficiently in grasping the <<irreducible>> perception of the continuum (references are made for this to Husserl and Bergson).

The depth and philosophical difficulties of Weyl's chapter II do not allow us to go any further into this here. I believe though that there is a strong need to revisit these aspects of Weyl's reflexion. In particular today, in view of the increasing interests in "theories of knowledge" as part of broadly construed attempts to reconsider our understanding and (possibly mathematical) description of the world.

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