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# Philosophical Problems of Mathematics in the Light of Evolutionary Epistemology

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# PHILOSOPHICAL PROBLEMS OF MATHEMATICS IN THE LIGHT OF EVOLUTIONARY EPISTEMOLOGY[1]

*Yehuda Rav*

## 1. *Introduction*

When one speaks of the foundations of mathematics or of its foundational problems, it behooves to be reminded that mathematics is not an edifice which risks to collapse unless it is seated on solid and eternal foundations which are supplied by some logical, philosophical, or extra-mathematical construction. Rather, mathematics ought to be viewed as an ever expanding mansion floating in space, with new links constantly growing between previously separated compartments, while other chambers atrophy for lack of interested or interesting inhabitants. The foundations of mathematics also grow, change, and further interconnect with diverse branches of mathematics as well as with other fields of knowledge. Mathematics flourishes on open and thorny problems, and foundational problems are no exception. Such problems arose already in antiquity, but the rapid advance in the second half of the 19th century toward higher levels of abstraction and the recourse to the actual infinite by Dedekind[2] and Cantor, all pressed for an intense concern with foundational questions. The discovery of irrational numbers, the use of negative numbers ('negare' = to deny, to refuse), the introduction of imaginary(!) numbers, the invention of the infinitesimal calculus and the (incoherent) calculations with divergent series, etc., each of these novelties precipitated at their time uncertainties and resulted in methodological reflections. But starting with the creation of non-Euclidean geometries[3] and culminating in Cantor's theory of transfinite numbers, the *rate* at which new foundational problems presented themselves grew to the point of causing in some quarters a sense of crisis – hence the talk of a foundational crisis at the beginning of this century.

The philosophy of mathematics is basically concerned with a systematic reflection about the nature of mathematics, its methodological problems, its relations to reality and its applicability. Certain foundational inquiries, philosophical at the outset, were eventually internalized. Thus, the impetus resulting from philosophically motivated researches produced spectacular developments in the field of logic, with their ultimate absorption within mathematics proper. Today, the various descendants of foundational work, such as proof theory, axiomatic set theory, recursion theory, etc., are part and parcel of the main stream of mathematical research. This does not mean that the philosophy of mathematics has or ought to wither away. On the contrary. Nowadays, many voices hail a renaissance in the philosophy of mathematics and acclaim its new vigor. Note also the current dynamic preoccupation by biologists and philosophers alike with foundational problems of biology.[4] By contrast, the mathematical community is rather insular, and most mathematicians now have a tendency to spurn philosophical reflections. Yet without philosophy we remain just stone heapers:

> "You can certainly reason about the arrangement of the stones of the temple, but you'll never grasp its essence which lies beyond the stones."[5]
>
> (A. de Saint-Exupéry, *Citadel*, LXXXIII)

It is significant to notice that in the current literature on the philosophy of mathematics there is a marked shift towards an analysis of *mathematical practice*.[6] This is most refreshing, for it is high time that the philosophy of mathematics liberates itself from ever enacting the worn-out tetralogy of platonism, logicism, intuitionism, and formalism. As Quine (1948) has pointed out, the "traditional schools of the philosophy of mathematics have their roots in the medieval doctrines of realism, conceptualism and nominalism". Whereas the quarrel about universals and ontology *had* its meaning and significance within the context of medieval Christian culture, it is an intellectual scandal that some philosophers of mathematics can still discuss whether whole numbers exist or not. It *was* an interesting question to compare mathematical 'objects' with physical objects as long as the latter concept was believed to be unambiguous. But, with the advent of quantum mechanics, the very concept of a physical object became more problematic than any mathematical concept.[7] In a nutshell, philosophy too has its paradigms, and a fertile philosophy of mathematics, like any other 'philosophy-of', must be solidly oriented towards the practice of its particular discipline and keep contact with actual currents in the philosophy of science.

The purpose of this essay is to explore one such current in the philosophy of science, namely, *evolutionary epistemology*, with the tacit aim of hopefully obtaining some new insights concerning the nature of mathematical knowledge. This is not a reductionist program. But the search for new insights seems more fruitful than treading forever on the quicksand of neo-scholasticism and its offshoots. We concur with Wittgenstein that "a philosophical work consists essentially of elucidations." (Tractatus 4.112)

## 2. *The main tenets of evolutionary epistemology*

Evolutionary epistemology (EE) was independently conceived by Lorenz, a biologist; Campbell, a psychologist; and Vollmer, a physicist and philosopher. Though its origins can be traced to 19th century evolutionary thinkers, EE received its initial formulation by Lorenz (1941) in a little-noticed paper on Kant. Christened in 1974 by Campbell and systematically developed in bookform by Vollmer in 1975, evolutionary epistemology has quickly become a topic of numerous papers and books.[8] In the opening paragraph of an essay in honor of Sir Karl Popper, where the term 'evolutionary epistemology' appears for the first time, Campbell (1974) states:

> An evolutionary expistemology would be at minimum an epistemology taking cognizance of and compatible with man's status as a product of biological and social evolution. In the present essay it is also argued that evolution - even in its biological aspects - is a knowledge process, and that the natural-selection paradigm for such knowledge increments can be generalized to other epistemic activities, such as learning, thought, and science.

Our aim is to add mathematics to that list.

Riedl (1984, p. 220; 1988, p. 287) characterizes evolutionary epistemology as follows:

> In contrast to the various philosophical epistemologies, evolutionary epistemology attempts to investigate the mechanism of cognition from the point of view of its phylogeny. It is mainly distinguished from the traditional position in that it adopts a point of view outside the subject and examines different cognitive mechanisms comparatively. It is thus

> able to present objectively a series of problems [including the problems of traditional epistemologies] but not soluble on the level of reason alone [but, which are soluble from the phylogenetic point of view].

In an extensive survey article, Bradie (1986) introduced a distinction between two interrelated but distinct programs which go under the name of evolutionary epistemology. On one hand, there is an "attempt to account for the characteristics of cognitive mechanisms in animals and humans by a straightforward extension of the biological theory of evolution to those aspects or traits of animals which are the biological substrates of cognitive activity, e.g., their brains, sensory systems, motor systems, etc." Bradie (p. 403) refers to this as the Evolutionary Epistemology Mechanism program (EEM). On the other hand, the EE Theory program, EET, "attempts to account for the evolution of ideas, scientific theories and culture in general by using models and metaphors drawn form evolutionary biology.[9] Both programs have their roots in 19th century biology and social philosophy, in the work of Darwin, Spencer and others." Popper is generally considered to be the main representative of the EET program, though Popper himself would not call himself an evolutionary epistemologist.[10] The great impetus to the EE Mechanisms program came from the work of Konrad Lorenz and his school of ethology. Through extensive studies of the behavior of animals in their natural habitat, Lorenz has deepened our understanding of the interplay between genetically determined and learned behavioral patterns. To Lorenz, the evolution of the cognitive apparatus is not different in kind from the evolution of organs. The same evolutionary mechanisms account for both. As Lorenz puts it in a famous passage:

> Just as the hoof of the horse, this central nervous apparatus stumbles over unforeseen changes in its task. But just as the hoof of the horse is adapted to the ground of the steppe which it copes with, so our central nervous apparatus for organizing the image of the world is adapted to the real world with which man has to cope. Just like any organ, this apparatus has attained its expedients species-preserving form through this coping of real with the real during its genealogical evolution, lasting many eons.

In the fascinating 1941 paper already mentioned, Lorenz reinterpreted the Kantian categories of cognition in the light of evolutionary biology. By passing from Kant's *prescriptive episte-*

*mology* to an evolutionary *descriptive epistemology*, the category of *a priori cognition* is reinterpreted as the individual's inborn (*a priori*) *cognitive mechanisms* which have evolved on the basis of the species' *a posteriori* confrontation with the environment. In short, the phylogenetically *a posteriori* became the ontogenetically *a priori*. In the words of Lorenz (1977, p. 37; 1985, p. 57):

> the categories and modes of perception of man's cognitive apparatus are the natural products of phylogeny and thus adapted to the parameters of external reality in the same way, and for the same reason, as the horse's hooves are adapted to the prairie, or the fish's fins to the water.

Any epistemology worthy of its name must start form some postulate of realism: that there exists a real world with some organizational regularities. "In a chaotic world not only knowledge, but even organisms would be impossible, hence non-existent." (Vollmer, 1983, p. 29). But the world includes also the reflecting individual. Whereas the idealist, to paraphrase Lorenz, looks only into the mirror and turns his back to reality, the realist looks only outwardly and is not aware that he is a mirror of reality. Each ignores that the mirror also has a non reflecting side which is part and parcel of reality and consists of the physiological apparatus which has evolved in adaptation to the real world. This is the subject of Lorenz's remarkable book *Behind the Mirror*. Yet reality is not given to immediate and direct inspection. *Reality is veiled*, to use the deft expression of d'Espagnat. But the veil can progressively be *transluminated*, so to speak, by conceptual modeling and experimentation. This is the credo of the working scientist. Evolutionary epistemology posits a minimal ontology, known under the name of *hypothetical realism*, following a term coined and defined by Campbell (1959, p. 156) as follows:

> My general orientation I shall call hypothetical realism. An 'external' world is hypothesized in general, and specific entities and processes are hypothesized in particular, and the observable implications of these hypotheses (or hypostatizations, or reifications) are sought out for verification. No part of the hypotheses has any 'justification' or validity prior to, or other than through, the testing of these implications. Both in specific and in general they are always to some degree tentative.[11]

The reader is referred to the treatises by Vollmer (1975 and 1985/86) for a systematic discussion of evolutionary epistemology.[12] Subsequently, I will also draw on the insights furnished by the genetic (= developmental) epistemology of Piaget and his school, (which I consider part of the EE Mechanisms program), as well as on the work of Oeser (1987 and 1988).

### 3. *Some perennial questions in the philosophy of mathematics*

The hungarian mathematician Alfréd Rényi has written a delightful little book entitled *Dialogues on Mathematics*.[13] The first is a Socratic dialogue on the nature of mathematics, touching on some central themes in the philosophy of mathematics. From the following excerps, (quoted with the kind permission of the publishers), we shall extract the topics of our subsequent discussion.

Socrates: What things does a mathematician study? ... would you say that these things exist? ... Then tell me, if there were no mathematicians, would there be prime numbers, and if so, where would they be?

Socrates: Having established that mathematicians are concerned with things that do not exist in reality, but only in their thoughts, let us examine the statement of Theaitetos, which you mentioned, that mathematics gives us more trustworthy knowledge than does any other branch of science.

Hippocrates: ... in reality you never find two things which are exactly the same; ... but one may be sure that the two diagonals of a rectangle are exactly equal... Heraclitus ... said that everything which exists is constantly changing, but that sure knowledge is only possible about things that never change, for instance, the odd and the even, the straight line and the circle.

Socrates: ... we have much more certain knowledge about persons who exist only in our imagination, for example, about characters in a play, than about living persons. ... The situation is exactly the same in mathematics.

Hippocrates: ... But what is the use of knowledge of non-existing things such as that which mathematics offers?

Socrates: ... How to explain that, as often happens, mathematicians living far apart from each other and having no contact, independently discover the same truth?

I never heard of two poets writing the same poem. ... It seems that the objects of [mathematicians'] study has some sort of existence which is independent of their person.

Socrates: But tell me, the mathematician who finds new truths, does he discover it or invent it?

Hippocrates: The main aim of the mathematician is to explore the secrets and riddles of the sea of human thought. *These exist independently of the mathematician, though not from humanity as a whole.* (italics mine).

Socrates: We have not yet answered the question : what is the use of exploring the wonderful sea of human thought?

Socrates: *If you want to be a mathematician, you must realize you will be working mostly for the future.* (italics mine).

Now, let us return to the main question. We saw that knowledge about another world of thought, about things which do not exist in the usual sense of the word, can be used in everyday life to answer questions about the real world. Is this not surprising?

Hippocrates: More than that, it is incomprehensible. It is really a miracle.

Hippocrates: ... but I do not see any similarity between the real world and the imaginary world of mathematics.

Hippocrates: ... Do you want to say that the world of mathematics is a reflected image of the real world in the mirror of our thinking?

Socrates: ... do you think that someone who has never counted real objects can understand the abstract notion of number?... The child arrives at the notion of a sphere through experience with round objects like balls. Mankind developed all fundamental notions of mathematics in a similar way. These notions are crystallized from a knowledge of the real world, and thus it is not surprising but quite natural that they bear the marks of their origin, as children do of their parents. And exactly as children when they grow up become the supporters of their parents, so any branch of mathematics, if it is sufficiently developed, becomes a useful tool in exploring the real world.

Hippocrates: ... Now we have found that the world of mathematics is nothing else but a reflection in our mind of the real world.

Socrates: ... I tell you [that] the answer is not yet complete.

Socrates: *We have kept too close to the simile of the reflected image. A simile is like a bow - if you stretch it too far, it snaps.* (italics mine).

Schematically, the key issues which emerge from the dialogue are the following:

(1) *Ontology*. In what sense can one say that mathematical 'objects' exist?[14] If discovered, what does it mean to say that mathematical propositions are true independently of the knowing subject(s) and *prior* to their discovery?
(2) *Epistemology*. How do we come to know 'mathematical truth' and why is mathematical knowledge considered to be certain and apodictic?
(3) *Applicability*. Why is mathematical knowledge applicable to reality?
(4) *Psychosociology*. If invented, how can different individuals invent the 'same' proposition? What is the role of society and culture?

It has been stressed by Körner (1960) and by Shapiro (1983) that problem (3) is least adequately dealt with by *each* of the traditional philosophies of mathematics. As Shapiro rightly observes: "...many of the reasons for engaging in philosophy at all make an account of the relationship between mathematics and reality a priority... Any world view which does not provide such an account is incomplete at best." (p. 524). To answer this challenge, there are voices which try to revive Mill's long burried empiricist philosophy of mathematics, notwithstanding the obvious fact that mathematical propositions are neither founded on sense impressions nor could any ever be refuted by empirical observations. How are we supposed to derive from experience that every continuous function on a closed interval is Riemann integrable? A more shaded empiricism has been advocated by Kalmár and Lakatos. Their position was sharply criticized by Goodstein (1970) and I fully agree with Goodstein's arguments.[15] In a different direction, Körner (1965) has sought an empiricist *justification* of mathematics via empirically verifiable propositions modulo translation of mathematical propositions into empirical ones. The problematics of translation apart, the knotty question of inductive justification 'poppers' up again, and not much seems to be gained from this move. Though the road to empiricism is paved with good intentions, as with all such roads, the end point is the same. Yet the success of mathematics as a scientific tool is itself an empirical fact. Moreover, empirical elements seem to be present in the more elementary parts of mathematics and they are difficult to account for. To say that some mathematical concepts were formed by 'abstraction' from experience only displaces the problem, for we still don't know how this process of abstraction is supposed to work. Besides, it is not the elementary part of mathematics which plays a fundamental role in the elaboration of scientific theories; rather, it is the totality of mathematics, with its most abstract concepts, which serves as a *pool* from which the scientist draws *conceptual schemes* for the elaboration of scientific theories. In order to account for this process, evolutionary epistemology starts from a minimal physical ontology, known as *hypothetical realism*; it just assumes the existence of an objective reality which is independent of our taking cognizance of it. Living beings, idealistic philosophers included, are of course part of objective reality. It is sufficient to assume that the world is non-chaotic; or put positively, the world is assumed to possess organizational regularities. But I would not attribute to reality 'objective *relations*', 'quantitative relations', 'immutable *laws*', etc. All these are epistemic concepts and can only have a place within the frame of scientific theories. Some philosophers of mathematics have gone far beyond hypothetical realism and thereby skirt the pitfalls of both empiricism and platonism, such as Ruzavin when he writes: "In complete conformity with the assertions of science, dialectical materialism considers mathematical objects as images, photographs, copies of the real quantitative relations and space forms of the world which surrounds us." (p. 193). But we are never told how could, for instance, Urysohn's metrization theorem of topological spaces reflect objective reality. Are we supposed to assume that through its pre-image in objective reality, Urysohn's theorem was already true before anybody ever thought of topological spaces? Such a position is nothing but platonism demystified, and it would further imply that every mathematical problem is decidable independently of any underlying theoretical framework. As Rényi has reminded us: there is some truth in the simile of the reflected image. *But, a "simile is like a bow - if you stretch it too much, it snaps."*

To reiterate: mathematics and objective reality are related, but the relationship is extremely complex and no magic formula can replace patient epistemological analysis. We turn now to the task of indicating a direction for such an analysis form the point of view of evolutionary epistemology.[16]

## 4. *Mathematics and reality*

Many consider it a miracle - as Rényi had Hippocrates say - that mathematics is applicable to questions of the real world. In a famous article, Wigner (1960) expressed himself in a similar way:

> ... the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and ... there is no rational explanation for it.
> ... it is hard to believe that our reasoning power was brought, by Darwin's process of natural selection, to the perfection which it seems to possess.

With due respect to the awe of the great physicist, *there is a rational explanation for the usefulness of mathematics* and it is the task of any epistemology to furnish one. Curiously, we'll find its *empirical basis* in the very evolutionary process which puzzled Wigner. Here is the theory which I propose.

The core element, the depth structure of mathematics, incorporates cognitive mechanisms which have evolved, like other biological mechanisms, by confrontation with reality and which have become genetically fixed in the course of evolution. I shall refer to this core structure as the *logico-operational component* of mathematics. Upon this scaffold grew and continues to grow the *thematic component* of mathematics which consists of the specific content of mathematics. This second level is culturally determined and originated, most likely, from ritual needs. (The ritual origin of mathematics has been discussed and documented by numerous authors. Cf. Seidenberg; Carruccio, p. 10, Michaels; and their respective bibliographies). Notice that ritual needs were practical needs, seen in the context of the prevailing cultures, hence there is no more doubt about the practical origin of mathematics! Marshack (1972) has documented the presence of mathematical notations on bones dating to the Paleolithic of about 30:000 years ago. This puts it 20,000 years prior to the beginnings of agriculture, hence some mathematical knowledge was already available for the needs of land measurements, prediction of tides, etc. Given this remarkable long history, mathematics has been subjected to a lengthy cultural molding process, akin to an environmental selection. Whereas the thematic component of mathematics is culturally transmitted and is in a continuous state of growth, the logico-operational component is based on genetically transmitted cognitive mechanisms and thus is fixed. (This does not mean that the logico-operational level is ready for use at birth; it is still subject to an ontogenetic development.[17] The genetic program is an *open program* (Mayr) which is materialized in the phenotype under the influence of internal and external factors and is realized by stages in the development of the individual).

Let us look closer at the nature of cognitive mechanisms. Cognition is a fundamental physical process; in its simplest form it occurs on the molecular level when certain stereospecific configurations permit the aggregation of molecules into larger complexes.[18] As we move up the ladder of complexity, cognition played a central role in prebiotic chemical evolution, and further so in the formation of self-replicating units. Here, in the evolution of macromolecules, 'survival of the fittest' has a literal meaning: that *which fits sticks* (Chemically so!) That which doesn't fit, well, it just stays out of the game, it is 'eliminated'. These simple considerations should have a sobering effect when looking at more complicated evolutionary processes. The importance of cognition in the process of self-organization of living matter cannot be overemphasized. Thus, Maturna (1980, p. 13) writes: *"Living systems are cognitive systems, and living as a process is a process of cognition".* What I wish to stress here is that there is a continuum of cognitive mechanisms, from molecular cognition to cognitive acts of organisms, and that some of these fittings have become genetically fixed and are transmitted from generation to generation. Cognition is not a passive act on the part of an organism, but a dynamic process realized in and through *action*. Lorenz (1941; 1984, p. 102) has perceptively pointed out that the German word for reality, 'Wirklichkeit', is derived from the verb 'wirken', to act upon. The evolution of cognitive mechanisms is the story of successive fittings of the organism's actions upon its internal and external environment.

It is remarkable how complicated and well-adapted inborn behavioral patterns can be, as numerous studies by ethologists have shown. "Consider, for instance", writes Bonner (1980, p. 40), "a solitary wasp. The female deposits her eggs in small cavities, adds some food, and seals off the chamber. Upon emergence the young wasp has never seen one of its own kind, yet it can walk, fly, eat, find a mate, mate, find prey, and perform a host of other complex behavioral patterns. This is all done without any learning form other individuals. It is awsome to realize that so many (and some of them complex) behavioral patterns can be determined by the genes." Isn't this as remarkable as "that our reasoning power was brought, by Darwin's process of natural selection, to the perfection which it seems to possess (Recall the citation form Wigner). From the rigid *single choice behavior* as in the case of the solitary wasp through the evolution of *multiple choice behavior*[19] and up to our capacity of *planned actions*, all intermediate stages occur and often concur. "As behavior and sense organs become more complex", writes Simpson (1963, p. 84), "perception of sensation from those organs obviously maintained a realistic relationship to the environment. To put it crudely but graphically, the monkey who did not have a realistic perception of the tree branch he jumped for was soon a dead monkey and therefore did not become one of our ancestors. Our perceptions do give true, even though not complete, representations of the outer world because that was and is a biological necessity, built into us by natural selection. If we were not so, we would not be here! We do now reach perceptions for which our ancestors had no need, for example, of X-rays for electrical potentials, but we do so by translating them into modalities that are evolution-tested."

The nervous system is foremost a steering device for internal and external coordination of activities. There is no such thing as an 'illogical' biological coordinating mechanism, else survival would not have been possible. "For survival", writes Oeser (1988, p. 38), "it is not the right images which count but the corresponding (re)actions." The coordinating activities of the nervous system proceed mostly on a subconscious level; we become aware of the hand which reached out to catch a falling glass only at the end of the action. (It is estimated that from an input of $10^9$ bits/sec, only $10^2$ bits/sec reach consciousness.) Yet another crucial mechanism has evolved, known on the human level as *planned action.* It permits a choice of action of hypothetical reasoning: we can imagine, prior to acting, the possible outcome of an action and thereby minimize all risks. The survival value of anticipatory schemes is obvious. *When we form a representation for possible action, the nervous system apparently treats this representation as if it were a sensory input, hence processes it by the same logico-operational schemes as when dealing with an environmental situation.*[20] From a different perspective, Maturana and Varela (1980, p. 131) express it this way:

> ... all states of the nervous system are internal states, and the nervous system cannot make a distinction in its process of transformations between its internally and externally generated changes...

Thus, the logical schemes in hypothetical representations are the same as the logical schemes in coordination of actions, schemes which have been tested through eons of evolution and which by now are genetically fixed.

The preceding considerations have far reaching implications for mathematics. Under *logico-mathematical schemes* Piaget understands the cognitive schemes which concern groupings of physical objects, arranging them in order, comparing groupings, etc. These basic premathematical schemes have a genetic envelope but mature by stages in the intellectual development of the individual. They are based on the equally genetically fixed *logico-operational schemes*, a term which I have introduced, as these schemes operate also on the non-human level. The logico-operational schemes form the basis of our logical thinking. As it is a fundamental property of the nervous system to function through recursive loops, any hypothetical representation which we form is dealt with by the same 'logic' of coordination as in dealing with real life situations. Starting from the elementary logico-mathematical schemes, a hierarchy is established. Under the impetus of socio-cultural factors, new mathematical concepts are progressively introduced, and each new layer fuses with the previous layers.[21] *In structuring new layers, the same cognitive mechanisms operate with respect to the previous layers as they operate with respect to an environmental input.* This may explain perhaps why the working mathematician is so prone to platonistic illusions. The sense of reality which one experiences in dealing with mathematical concepts stems in part from the fact that in all our hypothetical reasonings, the object of our reasoning is treated by the nervous system by means of cognitive mechanisms which have evolved through interactions with external reality. (See also the quotation from Borel in footnote 28).

To summarize: mathematics does not reflect reality. But our cognitive mechanisms have received their imprimatur, so to speak, through dealing with the world. The empirical component in mathematics manifests itself not on the thematic level, which is culturally determined, but through the logico-operational and logico-mathematical schemes. As the patterns and structures which mathematics consists of are molded by the logico-operational neural mechanisms, these abstract patterns and structures acquire the status of *potential* cognitive schemes for forming abstract hypothetical world pictures. Mathematics is a singularly rich *cognition pool* of mankind from which schemes can be drawn for formulating *theories* which deal with phenomena which lie outside the range of daily experience, and hence for which ordinary language is inadequate. Mathematics is structured by cognitive mechanisms which have evolved in confrontation with experience, and in its turn, mathematics is a tool for structuring domains of indirect experience. But mathematics is more than just a tool. *Mathematics is a collective work of art which derives* *its objectivity through social interaction.* "A mathematician, like a painter or a poet, is a maker of patterns", wrote Hardy (1969, p. 84). The metaphor of the weaver has been frequently evoked. But the mathematician is a weaver of a very special sort. For when he arrives at the loom, he finds a fabric already spun by generations of previous weavers and whose beginnings lie beyond the horizons. Yet with the yarn of his creative imagination he extends and sometimes modifies existing patterns. He may only be concerned with adding a beautiful motif, or mend the web as the sees fit, or at times care more about the possible use of the cloth. But the weaving hand, for whatever motive it may reach out for the shuttle, is the very *prehensile* organ which evolved as a grasping and branch clutching organ, and its coordinating actions have stood the test of an adaptive evolution.[22] In the mathematician, the artisan and artist are united into an inseparable whole, an unity which reflects the uniqueness of mankind as *homo artifex*.

## 5. *The trilemma of a finitary logic and infinitary mathematics*

In 1902, *L'enseignement Mathématique* launched an inquiry into the working methods of mathematicians. The questionnaire is reproduced (in English translation) as Appendix I in Hadamard (1945). Of particular interest in Question 30 which, among others, Hadamard addressed to Einstein. (No date of the correspondence is given, but I situate it in the forties when Hadamard was at Columbia University). Question 30 reads as follows:

> It would be very helpful for the purpose of psychological investigation to know what internal or mental images, what kind of 'internal word' mathematicians make use of, whether they are motor, auditory, visual, or mixed, depending on the subject which they are studying.

In his answer to Hadamard (Appendix II), Einstein wrote:

> (A) The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. The physical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be 'voluntarily' reproduced and combined...
> (B) The above mentioned elements are, in my case, *of visual and some of muscular type.* Conventional words or other

signs have to be sought for laboriously only in a secondary stage, when the mentioned associative play is sufficiently established and can be reproduced at will. (italics mine).

In the previous section I have discussed the core structure of mathematics which consists of the logico-operational schemes for the coordination of actions. Throughout the evolution of hominoids, the coordinating mechanisms of the hand and eye played a particularly important role, leading to the feasibility of extensive use of tools, and thereby to further cortical developments. It is therefore not surprising that in dealing with concepts, where the same neural mechanisms are involved, visual and traces of kinesthetic elements manifest themselves in consciousness, as Einstein's testimonial confirms.

The world of our immediate actions is *finite*, and the neural mechanisms for anticipatory representations were forged through dealing with the finite. Formal logic is not the source of our reasoning but only *codifies* parts of the reasoning processes. But whence comes the feeling of safety and confidence in the soundness of the schemes which formal logic incorporates? To an evolutionary epistemologist, logic is not based on conventions; rather, we look for the biological substrata of the fundamental schemes of inference. Consider for instance *modus ponens*:

$$\frac{\begin{array}{c}A \rightarrow B \\ A\end{array}}{\therefore B}$$

If a sheep perceives only the muzzle of a wolf, it flees already for its life. Here, 'muzzle → wolf' is 'wired' into its nervous system. Hence the mere sight of a muzzle - any puzzle of a wolf, not just the muzzle of a particular wolf - results in 'inferring' the presence of a wolf. Needless to day how vital such inborn behavioral patterns are. [23] The necessary character of logic, qua codified logico-operational schemes, thus receives a coherent explanation in view of its phylogenetic origin. It follows furthermore that *as far as logic is concerned, finitism does not need any further philosophical justification.* It is biologically imposed. The situation is different with respect to the *thematic component* of mathematics. Once the cultural step was taken in inventing number words and symbols which can indefinitely be extended, mathematics proper, as "*the science of the infinite*" (Zermelo) came into being. The story of the early philosophical groping with mathematical and possible physical infinity is well known.[24] When at last full citizenship was conferred on the *actual infinite* - de facto by Kummer and Dedekind; *de jure by* Cantor and

Zermelo - an intense preoccupation with foundational problems was set in motion.[25] The first school to emerge was logicism à la Frege and Russell. "The logicistic thesis is", writes Church (1962), "that logic and mathematics are related, not as the different subjects, but as earlier and later parts of the same subject, and indeed in such a way that mathematics can be obtained from pure logic without the introduction of additional primitives or additional assumptions." *Had the logicist programme succeeded, then infinitary mathematics, a cultural product, would have received a finitary foundation in finitary, biologically based logic.* But as early as 1902, Keyser already showed that mathematical induction required an axiom of infinity, and finally, Russell had to concede that such an axiom (plus the axiom of reducibility) had to be added to his system. Thus, *the actual infinite is the rock upon which logicism foundered.* Still, the efforts of the logicist school were not in vain, as Church (1962, p. 186) has pointed out: "... it does not follow that logicism is barren of fruit. Two important things remain. One of these is the reduction of mathematical vocabulary to a surprisingly brief list of primitives, all belonging to the vocabulary of pure logic. The other is the basing of all existing mathematics on one comparatively simple unified system of axioms and rules of inference."[26]

The second attempt of finitist foundations for mathematics was undertaken by Hilbert in his famous programme. It may not be inopportune to stress that Hilbert never maintained seriously that mathematics is devoid of content, and his oft-cited *mot d'esprit* that "mathematics is a game played according to certain simple rules with meaningless marks on paper" has regrettably resulted in unwarranted philosophical extrapolations. Hilbert's formalist programme is a *technique*, a device, for proving the consistency of infinitary mathematics by finitistic means. In the very article in which he outlines his programme, Hilbert said the following concerning Cantor's theory of transfinite numbers: "This appears to me the most admirable flower of the mathematical intellect and in general one of the highest achievements of purely rational human activity" (p. 373).[27] A meaningless game? Hardly!

Through formalization of thematic mathematics, Hilbert proposed that "contentual inference (be) replaced by manipulation of signs according to rules" (p. 381). This *manipulation* ('manus'= hand), this *handling* of inscriptions in the manner one handles physical objects would be founded, from the perspective of evolutionary epistemology, on the safe *logico-operational schemes for dealing with the finite*. It was a magnificent programme, and though in view of Gödel's incompleteness theorem could not be carried out as originally conceived, its offshoot, proof theory, is a major flourishing branch of mathematical logic. Thus, the contributions of logicism and Hilbert's programme are of lasting values. As to the original intent, we just have to accept that *one cannot catch an infinite fish with a finite net!* Thus there remain three alternatives:

(1) Use an infinite net, say of size $\varepsilon_0$. (Gentzen)
(2) Eat only synthetic fish. (Brouwer)
(3) Be undernourished and settle for small fish. (Strict finitism)

A chacun son goût!

## 6. *Invention versus discovery*

"But tell me", asked Socrates in Rényi's Dialogue, "the mathematician who finds new truth, does he discover it or invent it?" We all know that a time-honored way to animate an after-dinner philosophical discussion is to ask such a question. People agree that following common usage of language, Columbus did not invent America, nor did Beethoven discover this 9th symphony. But when a new drug has been synthesized we commonly speak of a discovery, though the molecule never existed anywhere prior to the creative act of its synthesizers. Hadamard, in the introduction to his book *The Psychology of Invention in the Mathematical Field* observes that "there are plenty of examples of scientific results which are as much discoveries as inventions", and thus he prefers not to insist on the distinction between invention and discovery. Yet there are philosophies of mathematics which are committed to an essential distinction between discovery and invention. To the intuitionist, mathematical propositions are mental constructions, and as such, could not result form a discovery. The platonist, on the other hand, believes "that mathematical reality lies outside us, that our function is to discover or *observe* it", as Hardy (1969, p. 123) put it. The conventionalist, thought for different reasons, would side with the intuitionist and consider mathematics to be invented. Apparently, neither logicism nor formalism is committed to a discovery/invention dichotomy. Is the debate about invention versus discovery an idle issue or can one use the common sense distinction between the two terms in order to elucidate the distinct components in the growth of mathematical knowledge? Let us examine the issue through a standard example. I propose to argue that: (a) the *concept* of a prime number is an invention; (b) the *theorem* that there are infinitely many prime numbers is a discovery. (N.B. Euclid's formulation, Book IX, Prop. 20, reads: "Prime numbers are more than any assigned multitude of prime numbers".)

Why should the concept of prime number be considered an invention, a purely creative step which need not have been taken, while contrariwise it appears that an examination of the factorization properties of the natural numbers leads immediately to the 'discovery' that some numbers are composite and others are not, and this looks like a simple 'matter of fact'. Weren't the prime numbers already there, tucked away in the sequence of natural numbers prior to anyone noticing them? Now things are not that simple. First of all, the counting numbers, like other classificatory schemes, did not make a sudden appearance as an indefinitely extendable sequence. Some cultures never went beyond coining words for the first few whole numbers. There are even languages destitute of pure numberal words. But even in cultures with a highly developed arithmetic, like the ones of ancient Babylonia, Egypt or China, the concept of prime number was absent. Mow (1982) has shown how mathematicians in ancient China, though lacking the concept of prime number, solved problems such as reduction of fractions to lowest terms, addition of fractions, and finding Pythagorian triplets. Could it reasonably be said that the Chinese just missed 'discovering' the prime numbers, and that so did the Babylonians and the Egyptians, in spite of their highly developed mathematical culture extending over thousands of years? I don't think so. In retrospect it seems to us that there was some sort of necessity that the concept of prime number be stumbled upon. But this is a misleading impression. *Evolution, be it biological or cultural, is opportunistic.* Much of our modern mathematics would still stay intact if the concept of prime number were lacking, though number theory and hence portions of abstract algebra would be different. There are $2^{\aleph_0}$ subsets of $\mathbb{N}$ of which only $\aleph_0$ can be defined by any linguistic means. We neither discover nor invent any one of these subsets separately. But when the inventive step was taken in formulating he *concept* of prime numbers, one of the subsets of $\mathbb{N}$ was singled out, (i.e. to serve as a *model*, in modern terminology). Some historians of mathematics attribute to the Pythagorians certain theorems involving primes, but it is more likely that the concept of prime number is of a later date. It is conceivable that there is a connection between cosmological reflections about the ultimate constituents of matter by the Greek atomists and the thought about numerical atoms, i.e., prime numbers. Whatever the tie may be, one thing is certain: the invention of mathematical concepts is tied to culture. As White (1956) affirmed contra platonistic doctrines, the "locus of mathematical reality is cultural tradition". The evolution of mathematical concepts can only be understood in the appropriate sociocultural context.[26]

Let us note that concepts can be defined explicitly, as in the case of prime numbers, or implicitly, by a system of axioms, like the concept of a group. In either case it is an inventive act. Theorems, on the other hand, have more the character of a discovery, in the sense that one *discovers* a road linking different localities. Once certain concepts have been introduced, and so to speak are already there, it is a matter of discovering their connection, and this is the function of proofs. To come back to the theorem that no finite set of primes can contain all the prime numbers, it has the character of a discovery when one establishes a *road map* (Goodstein) linking 'set of primes', 'number of elements', etc., to yield a path to the conclusion. A proposed path may or may not be valid, beautiful, or interesting. But to say that a proof renders a proposition 'true' is as metaphorical as when one says to have found a 'true' path. It *seems best to dispense altogether with the notion of mathematical truth.*[29] Gone is then too the outdated Aristotelian conception of 'true axioms'. (Think of euclidean and non-euclidean geometries). Such a 'no truth' view also resolves the infinite regress involved in the apparent flow of truth form axioms to theorems which Lakatos (1962) endeavored to eliminate by an untenable return to empiricism. The creative work of the mathematician consists of inventing concepts and developing methods permitting to chart paths between concepts.[30] This is how mathematics grows in response to internal and external *problems* and results in an edifice which is beautiful and useful at the same time.

## 7. *Recapitulation and concluding remarks*

The evolutionary point of view dominated this essay, both in its metaphorical as well as in its strict biological sense. We started with the view of mathematics as an evolving mansion, foundations included. In harmony with the current emphasis in the philosophy of mathematics on actual mathematical practice, one of our chief concerns was to elucidate the relationship between mathematics and external reality. Though we rejected empiricism as an inadequate philosophy of mathematics, we endeavored to account for the empirical components in mathematics whose presence is clearly felt but which are difficult to locate. Mathematics is a science of structures, of abstract patterns. (cf. Resnik). It is a human creation, hence it is natural to look for biological as well as socio-cultural factors which govern the genesis of mathematical knowledge. The success of mathematics as a cognitive tool leaves no doubt that some basic biological mechanisms are involved. The acquisition of knowledge by organisms, already in its simplest form, presuppose mechanisms which could only have evolved under environmental pressure. Evolutionary epistemology starts from the empirical fact that our cognitive apparatus is the result of evolution and holds that our world picture must be appropriate for dealing with the world because otherwise survival would not have been possible. (Cf. Vollmer, 1975, p. 102). Indeed, it is from the coordination of *actions* in dealing with the world that anticipatory schemes of action have evolved, which in turn are at the root of our logical thinking. Thus, the *phylogenetically* but not individually empirical element manifests itself in our logico-operational schemes of actions, which lie at the root of the elementary logico-mathematical operations as studied by Piaget. On the other hand, the *content* of mathematical theories is culturally determined, but the overall mathematical formation sits on the logico-operational scaffold. Mathematics is thus seen as a two-tiered web: a *logico-operational level* based on cognitive mechanisms which have become fixed in adaptation to the world, and a *thematic level* determined by culture and social needs and hence in a continuous process of growth. This special double-tiered structure endows mathematics in addition to its artistic value with the function of a *cognition pool* which is singularly suitable beyond ordinary language for formulating scientific concepts and theories.

In the course of our discussion we also reassessed the rationale of logicism and Hilbert's programme. Of the traditional philosophies of mathematics, only platonism is completely incompatible with an evolutionary epistemology. "How is it that the Platonistic conception of mathematical objects can be so convincing, so fruitful and yet so clearly false?", writes Paul Ernest in a review. (Math. Reviews 83k:00010).[31] I only disagree with Ernest on one point: I do not think that platonism is fruitful. As a matter of fact, platonism has *negative* effects on research by blocking a dynamical and dialectic outlook. Just think of set-theorists who keep looking for '*the* true axioms' of set theory, and the working mathematicians who will not explore on equal footing the consequences of the negation of the continuum hypothesis as well as the consequences of the affirmation of the continuum hypothesis. For the same reason too many logicians still ignore paraconsistent and other 'deviant' (!) logics. Like the biological theory of preformation - which is just another side of the same coin - platonism has deep sociological and ideological roots. What Dobzhansky (1955, p. 223) had to say about the preformist way of thinking applies *mutatis mutandis* to platonism: "The idea that things are preformed, predestined, just waiting around the corner for their turn to appear, is pleasing and comforting to many people. Everything is destiny, fate. But to other people predestination is a denial of freedom and novelty. They prefer to think that the flow of events in the world may be changed creatively, and that new things do arise. The influence of these two types of thinking is very clear in the development of biological theories." And so it is in the philosophy of mathematics.

Starting with the misleading metaphor of mathematical truth, platonists graft upon it the further misleading metaphor of mathematical object in the manner of physical objects and to which 'truth' is supposed to apply. Metaphors are illuminating, but when metaphors are stacked one upon the other without end, the result is obscurity, and finally, obscurantism. I frankly confess that I am absolutely incapable of understanding what is meant by "ontological commitment" and the issue of the "existence of abstract objects", and I begin to suspect that the king wears no clothes. No, there are no preordained, predetermined mathematical 'truths' which lie just out or up there. Evolutionary thinking teaches us otherwise.

| | |
|---|---|
| Caminante, son tus huellas | Walker, just<br>your footsteps |
| el camino y nada más; | are the path and<br>nothing more; |
| caminante, no hay camino, | walker, no path<br>was there before, |
| se hace camino al andar. | the path is<br>made by act of walking. |

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NOTES

1. This is an expanded version of talks presented at the International Congress "Communication and Cognition. Ap-

plied Epistemology", Ghent, December 6–10, 1987; at the Logic of Seminar of the Kurt-Gödel-Gesellschaft, Technical University, Vienna, May 30, 1988; and at the Séminaire de Philosophie et Mathématique, Ecole Normale Supérieure, Paris, Novembre 7, 1988. May I thank the various organizers and participants for the numerous stimulating conversations and discussions.

2. Concerning the role of Dedekind, often neglected in foundational discussion, see Edwards (1983). In his review of Edwards' paper, Dieudonné makes the following significant observation: "*Dedekind broke entirely new ground in his free use of 'completed' infinite sets as single objects on which one could compute* as *with numbers, long before Cantor began his work on set theory.*" (Math. Reviews 84d:01028).
3. "With non-Euclidean geometry came into being a new state of mind which impressed its spirit of freedom on the whole development of modern mathematics." (Toth, 1986, p. 90; this fascinating essay deals at considerable depth with the epistemological problems of non-Euclidean geometries).
4. A special journal *Biology & Philosophy*, was created in 1986 to serve as a common forum.
5. "Tu peux certes raisonner sur l'arrangement des pierres du temple, tu ne toucheras point l'essentiel qui échappe aux pierres."
6. Cf. Feferman, Hersh, Kitcher, Kreisel, Resnik, Resnik & Kusher, Shapiro, Steiner, and Van Bendegem.
7. From a current point of view, physical objects are considered as events or states which rest unaltered for a non-negligible time interval. Though 'event' and 'state' *refer* to reality, in order to *speak* of them one needs the mathematical apparatus incorporated in physical theories. Thus one ends up again with mathematical concepts. Hence it is futile to look at mathematical concepts as *objects* in the manner of physical objects and then, to crown it all, relegate them to a platonic abode. For a further discussion of ontological questions concerning physical objects, see the joint and separate papers by Dalla Chiara and Toraldo di Francia, as well as Quine (1976).
8. See the 30-pages bibliography by Campbell, Heyes, and Callebaut (1987).
9. Vollmer (1987, in Riedl et al., 1987) also stresses the difference between EE à la Lorenz as a *biological theory of the evolution of cognitive systems* and EE à la Popper as a theory of the *evolution of scientific ideas*. In particular, see

Vollmer's Table 3 on p. 145.

10. Cf. Popper's disclamer in Riedl et al., (1987), p. 24. In Popper's philosophy, factual knowledge cannot serve as a basis for an epistemology, whereas evolutionary epistemology is committed to an "irresolvable nexus between empirical knowledge and metatheoretical reflections", following Vollmer. Moreover, the great strides of science in the last fifty years are due to ever-refined experimental techniques and technologies coupled with piecemeal modelling, rather than to the elaboration of grand theories. When one peeks into a modern research institute one scarsely finds scientists in a grandiose search for bold hypotheses and a frantic pursuit of refutations, but rather humbly approaching "nature with the view, indeed, of receiving information form it, not, however, in the character of a pupil, who listens to all that his master chooses to tell him, but in that of a judge, who compels the witness to reply to those questions which he himself thinks fit to propose." (*Critique of Pure Reason*, B XIII, Introd. second ed., transl. by Meiklejohn).
11. For a further discussion, cf. Vollmer, 1975, pp. 34-40.
12. For a short introduction to evolutionary epistemology, with references to the philosophical pro and con debate, see the article by Ursúa. See also Vollmer's survey article (1983).
13. See Rényi (1967). The booklet contains three dialogues: (1) "A Socratic Dialogue on Mathematics", whose protagonists are Socrates and Hippocrates. (2) "A Dialogue on the Applications of Mathematics", featuring Archimedes and Hieron. (3) "A Dialogue on the Language of the Book of Nature", whose chief character is Galileo.
14. Aristotle already discussed the difficulties with the platonistic notion of mathematical *objects* and their existence. See Metaphysics, Book XIII, chapts. 1-3, $1076^{a}33$-$1078^{b}6$.
15. Cf. also the discussion by Lolli (1982).
16. Cf. also Vollmer (1983).
17. We owe much of our understanding of the ontogenetic development of the various logico-mathematical schemes to the work of Piaget and his school. Cf. Müller (1987), pp. 102-106, for a succint summary of Piaget's theory. Note that much thought not all of Piaget's (onto)genetic epistemology is compatible with evolutionary epistemology. See the discussion by Apostel (1987) and Oeser (1988, p. 40 and p. 165).
18. There is no more anthropomorphism here in speaking of *molecular cognition* as in using the term *force* in physics.
19. These terms are due to Bonner (1980). Of particular importance is Bonner's extension of the concept of *culture* which

he defines (p. 10) as follows: "By culture I mean the transfer of information by behavioral means, most particularly by the process of teaching and learning. It is used in a sense that contrasts with the transmission of genetic information passed by direct inheritance of genes. The information passed in a cultural fashion accumulates in the form of knowledge and tradition, but the stress of this definition is on the mode of transmission of information, rather than its result. *In this simple definition I have taken care not to limit it to man...*" (italics mine).

20. Cf. Shepard & Cooper (1981) for some fascinating data.
21. It is a "fundamental principle of neuro-epistemology", writes Oeser (1988, p. 158), "that each new cognitive function results from an integration with previously formed and already existing functions."
22. The evolution of the hand as a prehensible organ enabled not only to *grasp* physical objects but led concomitantly to neural mechanisms enabling to *grasp relationships* between objects. This is the path form *prehension to comprehension*, or in German, as Lorenz has pointed out, from '*greifen*' (to grasp), via '*begreifen*' (to understand), to '*Begriff*'(concept). See (Lorenz, 1973, pp. 192-194; Vollmer, 1975, pp. 104-105; Oeser & Seitelberger, 1988, p. 159). From a neurophysiological point of view , notice the large area of the cortical maps of the hands. (See Granit, 1977, pp. 64-65).
23. For related examples, cf. Lorenz (1973) and Riedl (1979).
24. For a collection of most of the relevant passages in Aristotle, see Apostel (1952). Aurelius Augustinus (354-430) had no qualms about the *actual infinite* in mathematics, to wit: "Every number is defined by its own unique character, so that no number is equal to any other. They are all unequal to one another and different, and the individual numbers are finite but *as a class they are infinite.*" (*De Civitate Dei*, Book XII, chapt. 19; italics mine; English translation from Penguin Classics, 1984).
25. The 'paradoxes' played only a minor role in this process, and none in the case of Frege. For a discussion, cf. Garciadiega (1986) and the review by Corcoran in Math. Reviews 1988 (88a: 01026).
26. However there is no *unique* set theory with a unique underlying logic from which all presently known mathematics can be derived. (Just recall the numerous independence results and the needs of category theory). Moreover, when one examines actual mathematical practice, the deficiencies of 'standard logic' are apparent, as Corcoran (1973) has perspi-

caciously discussed. Furthermore, cognitive psychologists and workers in artificial intelligence are keenly aware of the fact that our current schemes of formal logic are inapplicable for analyzing actual reasoning processes. Cf. Gardner, 1985, pp. 368-370 and the references cited therein. Much work needs to be done in developing a logic of actual reasoning.

27. Hilbert (1926). This and the subsequent citation and page indications refer to the English translation in van Heijenoort.
28. For a further discussion, see White (1956) and Wilder (1981). And Borel (1983, p. 13) adds the following perceptive observation: "...we tend to posit existence on all those things which belong to civilization or culture in that we share them with other people and can exchange thoughts about them. Something becomes objective (as opposed to 'subjective') as soon as we are convinced that it exists in the minds of others in the same form as it does in ours, and that we can think about it and discuss together. Because the language of mathematics is so precise, it is ideally suited to defining concepts for which such a consensus exists. In my opinion, that is sufficient to provide us with a *feeling* of an objective existence, of a reality of mathematics..."
29. This has no bearing on the *technical* metamathematical notion of 'truth' in the sense of Tarski.
30. A radioscopy of mathematical proofs reveals their logical structure, and this aspect has traditionally been over-emphasized at the expense of seeing the meat and flesh of proofs. The path between concepts not only has a logical part which serves *to convince*, but establishes interconnections which *modify and illuminate* complexes of mathematical ideas, and this is how proofs differ from derivations.
31. Similarly, Machover (1983) writes concerning platonism: "The most remarkable thing about this utterly incredible philosophy is its success." (p. 4) And further down (p. 5): "The clearest condemnation of Platonism is not so much its belief in the occult but its total inability to account for constructive mathematics."

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