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RELATIVISTIC HYDRODYNAMICS OF ROTATING FLUID MASSES

by David BOHM and Jean-Pierre VIGIER

1. Abstract.

With the aid of the new notion of center of matter density, we give a relativistic treatment of the behaviour of finite size masses of rotating fluid. This treatment is based on an analysis of the relative motion of this center of matter density and the more familiar center of mass. In this way, we obtain a clear physical interpretation of the equations studied by MATHISSON, WEYSENHOFF and MØLLER. We also show that more general types of motions are possible related to additional degrees of freedom of the relativistic fluid droplet. These degrees of freedom provide a framework for a theory of the quantum numbers of the elementary particles (isotopic spin, strangeness, etc.) which will be developed in detail in a subsequent paper.

In a series of very interesting papers, MATHISSON [5], MØLLER [6], WEYSENHOFF [11] and PRYCE [8] have developed a relativistic theory of the motions of rotating masses of matter. Their equations are deduced from the conservation of energy-momentum and angular-momentum tensors. From these conservation assumptions they demonstrate the possibility of qualitatively new types of motion resulting from the coupling of a mean velocity with the total angular-momentum of the system. However, they do not make it clear to what this mean velocity refers. In fact, MOLLER suggested that these new motions are purely formal, or in other words, that the mean velocity defined in these theories refers only to the behaviour of fictitious and purely mathematical "center of gravity" points.

Moreover, in all these papers, the deduction of the equations of motion is based in a very essential way on the assumption that the time-like components of a certain angular momentum vanish in the mean rest frame of the body, or in other words, that :

$$\mathfrak{M}_{\alpha\beta}\, u^{\beta} = 0$$

where $\mathfrak{M}_{\alpha\beta}$ is the anti-symetric tensor for the total internal angular momentum, and u_{α} is the four velocity with $u_{\alpha} u_{\alpha} = 1$. This assumption however has not been justified by any specific physical arguments ; so that it constitutes a further somewhat arbitrary mathematical restriction on the theory.

In the present paper, we shall give a relativistic treatment of the general problem of the behaviour of a mass of conserved fluid, that is in some kind of rotational motion. We shall begin by giving a clear physical interpretation of the meaning of the time-components of the angular-momentum. Then with the aid of the new concepts of center of matter density, defined along with the already well-known concept of center of mass, we shall see that the ambiguity of the meaning of the motions described in the above quoted papers can be removed. Indeed, it will become clear that in our interpretation of the theory, the equations of Weysenhoff, Moller, etc. refer to the relative motion of center of mass and center of matter density. Furthermore, we shall see that motions are possible which are more general than these treated by WEYSENHOFF and MOLLER, with the result that equation [5] need no longer be satisfied. We then obtain a set of equations for these more general cases, and we show that the motion cannot be fully determined without further physical assumptions replacing equation [5].

In another paper we develop an example of one of these more general theories, leading to a classical motions equation of the same form as the Dirac relativistic equation in quantum mechanics. When this classical equation is quantized in the usual way, one obtains a set of quantum numbers similar to those which have been proposed recently (1) for the elementary particles.

2. Relativistic fluid masses.

The difficulty of treating rotating masses in the theory of relativity is connected with the impossibility of defining a relativistic rigid body in a consistent way (2). We may however overcome theses difficulties of formulation by considering instead relativistic fluid masses, which are held together by appropriate internal tensions that tend to hold these masses in some stable forms (3). Relative to such stable forms, the body of fluid may be subjected to all kinds of internal movements, such as rotations, vibrations, creation and destruction of inner closed

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(1) GELL MANN, VISHIJIMA, TIOMONO and others.

(2) Many authors have discussed this problem and have proposed various solutions, but these proposals are in any case very complicated and it is not yet clear wether they are completely free of contradictions. See for example J.L. SYNGE [9].

(3) POINCARÉ [7] has shown that rotating fluid masses with internal tensions tend to go into stable equilibrium forms, one of which is a rotating torus.

vortex structures, etc., each corresponding to different possible physical motions.

At first sight a general treatment of the problem of describing the behaviour of such masses raises insuperable difficulties. If we attempted to treat,of all the details of these possible complex motions, we would find ourselves blocked not only by mathematical difficulties, but also by the fact that we do not even know in general what the fluid equations are . Fortunately another point of view is possible if we are willing to restrict ourselves to an overall average description. In this case, if we suppose that however complicated the motion may be, there is a conserved energy momentum tensor density $T_{\mu\nu}$ (which of course contains the tensions that hold the body of fluid together), it is possible to define certain average properties of the motion, independently of the complex details that we ignore. These average properties can be treated mathematically and lead to a description of the general features of the motions of relativistic rotating fluid masses.

The assumption of a conserved energy momentum tensor density takes the form

(2) $$\partial^{\nu} T_{\mu\nu} = 0$$

We assume further that the energy momentum tensor is symmetric (as has been the case for all fluids treated so far). This means that

(3) $$T_{\mu\nu} = T_{\nu\mu}$$

As a result, the angular momentum tensor

(4) $$L_{\mu\nu,\lambda} = u_{\mu} T_{\nu\lambda} - x_{\nu} T_{\mu\lambda}$$

satisfies the conservation equation

(5) $$\partial^{\lambda} L_{\mu\nu,\lambda} = 0$$

On the basis of equation [6] one can easily show [4] that if the fluid body is localized (so that $T_{\mu\nu}$ vanishes outside of a space-like three dimensional limited region) the total energy and momentum integrated over all space in any specified Lorentz frame are constants. In other words,

(6) $$G_{\mu} = \int T_{\mu 0}\, dv = \text{constant}$$

and

(7) $$\frac{d\, G_{\mu}}{dt} = 0$$

where dv represents the element of volume.

Moreover, it also follows from equation (2) that G_{μ} transforms as a 4 vector under Lorentz transformations. On physical grounds we suppose that G_{μ} is a time-like vector ; otherwise there would have to be a Lorentz frame in which the fluid had momentum but no energy.

We shall assume further that we can define at each point of the fluid mass a 4 vector density j_μ satisfying the conservation equation

(8) $$\partial^\mu j_\mu = 0$$

This 4 vector density can be written

$$j_\mu = D\, u_\mu$$

where u_μ $(u_\mu u^\mu = 1)$ represents the components of the local unitary 4 velocity and $D = (j_\mu j^\mu)^{\frac{1}{2}}$ the invariant matter density.

The "matter density" j_o is proportional to the quantity of matter in a given region, which could for example be the number of molecules while the vector j_i represents the rate of flow of this matter across a unit area in the direction of the coordinate vector i . If the fluid consists of charged particules with a constant ratio of e/m , then the density of charge will be $\rho = e\, j_o$. But wether the fluid is charged or not, there will be a set of quantities j_μ satisfying equation (8).

The quantities j_μ will clearly be important for the determination of the way in which the body of fluid will change its shape, size, position and orientation in space (5). Indeed, one can in principle deduce all these properties on the basis of the fundamental hydrodynamic equations satisfied by the local flow velocity, which is just

$$u_i(\vec{x}\,,\, t) = j_i(\vec{x}\,,\, t)/j_o(\vec{x}\,,\, t)$$

Hence, if we wish to treat any of these properties of the motion of the fluid, we shall evidently have to study the behaviour of the j_μ .

In the non relativistic limit, the mass density T_{oo}/c^2 and the matter density j_o are proportional. But in the relativistic domain these two quantities may be different. For if the fluid is in motion, the kinetic energy E contributes a term E/c^2 to the mass, or in any case the internal tensions which are contained in the tensor $T_{\mu\nu}$ may make a similar contribution.

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(5) One can even regard a "rigid" solid as a limiting case of a fluid in which neighbouring points are restricted to remain at constant distances from each other.

The essential features of the distinction between mass density and matter density arising in the theory of relativity may be brought out most clearly in terms of the conceptions of center of mass and center of matter density. We shall discuss the center of matter density in the next section ; and here we shall consider only the center of mass X_μ which is defined as

(9) $$G_o X_i = \int T_{oo} x_i \, d$$

where $G_o = \int T_{oo} \, dv$ is the total energy of the body (the Latin subscript i refers to space like indices).

The most interesting property of the center of mass is that it moves at a constant velocity proportional to the total momentum. To prove this, we write

(10) $$G_o \frac{d X_i}{dt} = \int \frac{\partial T_{oo}}{\partial t} x_i \, dv$$

But by the conservation equation (2) we have

(11) $$G_o \frac{d X_i}{dt} = - \int \frac{\partial T_{oj}}{\partial x_i} x_i = \int T_{oj} \frac{\partial x_i}{\partial x_j} dv = G_i$$

where we have integrated $\partial^i T_{oj}$ by parts and used the vanishing of T_{oj} outside the fluid body. This gives

(12) $$\frac{d X_i}{dt} = \frac{G_i}{G_o}$$

which is the usual relativistic relation between the velocity of a particle and its momentum.

If we integrate on the total volume of the liquid droplet the time components of the angular momentum density $L_{[\mu\nu]\lambda} = x_\mu T_{\nu\lambda} - x_\nu T_{\mu\lambda}$ we form the total angular momentum

(13) $$\boxed{L_{[\mu\nu]} = \iiint L_{[\mu\nu]o} \, dv}$$

In the same way, we can show that $L_{\mu\nu}$ is a conserved skew tensor of second rank, for

$$\frac{d}{dt} L_{\mu\nu} = \iiint \frac{\partial}{\partial t} L_{\mu\nu,o} \, dv = - \iiint \frac{\partial}{\partial x_i} L_{\mu\nu,i} \, dv = 0$$

since $L_{[\mu\nu]\lambda} = 0$ on the surface of the droplet.

From the conservation of $L_{[\mu\nu]\lambda}$ it follows that $L_{[\mu\nu]}$ is a tensor. Moreover, $L_{\mu\nu}$ is evidently a constant, for

$$\frac{d}{dt} L_{\mu\nu} = \iiint \frac{\partial}{\partial t} L_{\mu\nu,o}\, dv = -\iiint \frac{\partial}{\partial x_i} L_{\mu\nu,i}\, dv = 0$$

since $L_{\mu\nu,i} = 0$ on the surface of the droplet.

The center of mass has a close relation-ship to the time like components of the angular momentum. To see this, we obtain from equation (13) and (4)

$$(14) \qquad L_{io} = \int L_{io,o}\, dv = \int (x_i T_{oo} - x_o T_{oi})dv$$

If we integrate at a constant value of the time coordinate x_o we obtain (using equation 12) and (6) :

$$(15) \qquad L_{io} = G_o X_i - G_i X_o$$

If we choose $X_o = 0$ then the time component of the angular momentum is proportionate to the center of mass coordinate. More generally, we have, by integrating equation (12),

$$G_o X_i = G_i X_{o+\alpha}$$

where α is a constant. We then obtain :

$$L_{io} = \alpha$$

Thus we verify the constancy of the time component of the angular momentum which we have already derived from the conservation law directly.

The space components of the angular momentum are, of course, just the usual moments of the momenta

$$(16) \qquad L_{ij} = \int (x_i T_{jo} - x_j T_{io})dv$$

It can be seen from equation (15) that the center of mass varies from one Lorentz frame to another (6). For example, let us choose the origin of our space time coordinate system such that $X_o = 0$ and $X_i = 0$ (so that the origin is at the center of mass and L_{oi} is zero). If $X_{oi}\ X_i$ were a four vector, then evidently under Lorentz transformation we would obtain for the new coordinates $X'_o = X'_i = 0$ and L'_{oi} would also have to be zero. To show that this cannot be true consider the infinitesimal Lorentz transformation (which consists only of a change of velocity with no rotation) :

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(6) cf. [5], [6] PAPATROU, etc.

$$X'_o = X_o - \varepsilon_{oi} X_i$$
$$X'_i = X_i - \varepsilon_{io} X_o$$

with

$$\varepsilon_{io} = - \varepsilon_{oi}$$

We obtain immediately

$$L'_{io} = L_{io} + \varepsilon_{o\ell} L_{i\ell} = \varepsilon_{o\ell} L_{i\ell}$$

since $L_{io} = 0$; now $L_{i\ell}$ is just the space like part of the angular momentum ; hence, if the fluid body is spinning (so that $L_{i\ell} \neq 0$), then L'_{io} will not be zero.

We conclude from the above that the center of mass coordinates do not transform as a 4 vector, and that in different Lorentz frames, the center of mass correspond to physically different points.

3. Center of matter density.

We shall now define the center of matter density. In analogy with what was done with the center off mass, one would be led to assume that this center, Y_μ is given by

(17) $$Y_\mu J_o = \int j_o x_i \, dv$$

where $J_o = \int j_o \, dv$ is the total amount of matter in the droplet (which is a scalar constant because of the conservation equation). The time derivative $\frac{dv_\mu}{dt}$ would then be given by

(18 a) $$J_o \frac{dY_o}{dt} = \int \frac{\partial j_o}{\partial t} x_i \, dv = \int \frac{\partial j_k}{\partial x_k} x_i \, dv = \int j_k \, dv$$

Thus, the velocity ω_k of the center of matter density is

(18b) $$\omega_k = \frac{dY_k}{dt} = \frac{\int j_k \, dv}{J_o} = \frac{J_k}{J_o}$$

where

$$J_k = \int j_k \, dv$$

The difficulty with this definition is that the quantities J_k do not form a four vector, because they depends on the volume element corresponding to the chosen Lorentz frame. (On the other hand, we recall that the velocity of the center of mass is proportional to a four vector). As a result, we cannot obtain for example an unambiguous definition of the frame in which the center of matter density is at rest.

We can remove this ambiguity by defining the total current J_k and the center of matter density Y_i according to equation (18), but in the special Lorentz frame π in which the center of mass is at rest (so that $G_i = 0$). This frame does have a unique meaning because the total momentum is a four vector. Then if we wish to know J_i and Y_μ in another frame π' we simply take their values in the frame π_o and transform them according to the Lorentz transformation laws of a four vector. In other words J_i and Y_μ are defined by (18) only by integration over the volume element associated with the frame π_o ; and we must be careful not to use this equation in any other frame.

To express this definition of (Y_μ) in more detail, we first denote by the superscript zero all quantities which refer to the frame π_o in which the total momentum G_i is zero. The velocity of the center of matter-density in the frame is then

(19a) $$\omega_k^o(t_o) = \frac{J_k^o(t^o)}{J_o^{\,o}}$$

From this we can define a four velocity (where we choose units such that $c = 1$)

(19b) $$v_\mu^o(t^o) = \frac{J_\mu^{\,o}(t^o)}{D^o(t^o)}$$

with

$$D^o(t^o) = \sqrt{-\,J_\nu^o(t_o)\;J_\nu^{\,o}(t_o)}$$

(where $J_4 = i\,J_o$) . The four velocity v_μ is then evidently unitary ; that is $v_\mu^o\;v_\mu^o = 1$.

On the other hand, the four velocity of the center of mass will be

(20) $$u_\mu = \frac{G_\mu}{M_o}$$

with

$$M_o^{\,2} = -\,G_\mu\;G^\mu$$

To go to an arbitrary frame, for example the laboratory frame Σ one simply makes a Lorentz transformation with the velocity u_μ . This transformation is defined by

(21a) $$x_\mu = \sum_\nu a_{\mu\nu}\;x_o^{\,o}$$

then

$$(21b)\qquad \begin{cases} a_{ij} = \delta_{ij} + \dfrac{u_i\, u_j}{1 + u_o} \\ a_{ko} = - a_{uk} = i\, u_k \\ a_{ko} = u_o \end{cases}$$

Then in the frame Σ the center of matter density has the coordinates

$$(22a)\qquad Y_k = Y_k^o(t^o) - \frac{u_j\, u_j\, Y_\ell^o(t^o)}{i+u_o} - u_k\, t^o$$

$$(22b)\qquad Y_o = u_o\, t_o - u_k\, Y^k(t_o)$$

where Y_o represents the time coordinate of the center of matter density in the frame Σ. The above equations provide a parametric representation of the trajectory of the center of matter density in Σ the parameter being t_o.

The velocity of matter density in Σ can be obtained by Lorentz transformation of (19b). This gives

$$(23a)\qquad v_i(Y_o) = \frac{i}{D_o} \left\{ J_i{}^o + \frac{G_i (G^k J_\mu^o - M_o\, J_o{}^o)}{M_o\, (M_o + G_o)} \right\}$$

$$(23b)\qquad V_o(Y_o) = - \frac{G^\nu J_\nu^o}{M_o\, D_o}$$

Notice that the time parameter appearing in the above equations is still t^o. We could transform to the parameter $t = Y_o$ with the aid of the equation (22b). However, it will be more convenient to use the proper time τ of the center of matter density as a parameter. τ is then defined by the relation

$$\frac{d\tau}{dY_o} = \frac{1}{V_o} = - \frac{M_o\, D_o(t^o)}{G_o^\nu\, J_\nu^o(t^o)}$$

4. <u>Internal angular momentum</u>.

Thus far we have defined the angular momentum relative to points fixed in space. We wish now to define an internal angular momentum analogous to the non relativistic angular momentum relative to the center of mass. In the relativistic theory the center of mass varies from one Lorentz frame to another. Moreover, the center of mass and the center of matter density are not in general the same. Thus there is an ambiguity with regard to the point relative to which the inner angular momentum of the fluid droplet ought to be defined. This ambiguity could be removed for example by choosing as the point relative to which the inner angular momentum is to be taken the center of mass (X_μ). This is in fact one of the possibilities that MOLLER [6] considered. However, as we shall see in section (5) such a choice leads to results having little physical significance with regard to the motion of the droplet as a whole. We shall choose instead for this purpose the center of matter density (Y_μ) for this points reflects in a better way the average velocity of the droplet. If the angular momentum relative to this point is taken, then the resulting equations will, as we shall see, describe the fluctuating motion of the droplet as a whole, relative to that of the center of mass which moves at a constant velocity. Thus, a general description of the overall motion of the fluid droplet is obtained.

In accordance with these considerations, we define the inner angular momentum of the fluid droplet as

(24) $$\mathfrak{M}_{\mu\nu} = \int [(x_\mu - Y_\mu)J_{\nu o} - (x_o - Y_o)T_{\mu o}]dv$$

We can express $\mathfrak{M}_{\mu\nu}$ in terms of $L_{\mu\nu}$ with the aid of equation (13). We have

$$L_{\mu\nu} = \int (x_\mu T_{\nu o} - x_\nu T_{\mu o})dv =$$
$$\int [(x_\mu - Y_\mu)T_{\nu o} - (x_\nu - Y_\nu)T_{\mu o}]dv + \int (Y_\mu T_{\nu o} - Y_\nu T_{\mu o})dv$$

since Y_ν is a constant in the integration, we then obtain

(25) $$L_{\mu\nu} = \mathfrak{M}_{\mu\nu} + Y_\mu G_\nu - Y_\nu G_\mu$$

As $L_{\mu\nu}$ is a tensor, it follows from (25) that $\mathfrak{M}_{\mu\nu}$ is also a tensor. By differentiating (25) with respect to the proper time τ and by noting that $d G_\mu/d\tau = 0$ we obtain

(26) $$\frac{d\mathfrak{M}_{\mu\nu}}{d\tau} = G_\mu u_\nu - G_\nu u_\mu$$

This is one of the basic equations postulated by WEYSENHOFF.

It is particularly instructive to consider equation (26) in the special frame Σ_o in which $v_i = 0$. This we shall call the rest frame of the particle, because it is the frame in which the center of matter density is at rest. In that frame we choose a set of axes such that $Y_i = 0$ and $Y_o = 0$ for the moment of interest. Then we have

$$\frac{d Y_o}{dt} = 1$$

We then obtain for the time component of the angular momentum

$$L_{io} = \mathfrak{M}_{io} + Y_i G_o - Y_o G_i = \mathfrak{M}_{io} - G_i t .$$

since

$$L_{io} = G_o X_i - G_i t$$

We obtain

$$G_o X_i - G_i t = \mathfrak{M}_{io} - G_i t$$

and thus

(27) $$\mathfrak{M}_{io} = X_i G_o$$

The above relation shows that in the frame Σ_o the time component of inner angular momentum is proportional to the vector joining the center of mass and the center of matter density. This interpretation of $\mathfrak{M}_{io}$ will be seen to play an important role in the further development of the theory.

5. Brief review of Weysenhoff's theory.

We now proceed to give a brief review of the Weysenhoff theory, in order to lay the foundations for a discussion if its physical significance in terms of our fluid model.

The basic starting point is equation (25). Now equation (25) consists of six equations determining the anti-symmetric tensor $d\mathfrak{M}_{\mu\nu}/d\tau$ in terms of g_μ and v_μ. Moreover, there are four more equations coming from the conservation of the total momentum, viz. :

(7) $$\boxed{\frac{dg_\mu}{d\tau} = 0}$$

There are still however no equations to determine the time variation of the v_μ (of which only three are independant, since $v_\mu v_\mu = 1$). In order to determine those equations, some further hypothesis is needed. Such a hypothesis is essentially a supplementary assumption connecting the center of mass and the center of matter density.

In the Weysenhoff theory, the supplementary assumption is

(1) $$\mathfrak{M}_{\mu\nu} v^\nu = 0$$

By going to the rest frame of the particle Σ_o (where $v^i = 0$, $v^o = 1$), we see that the above reduces to the three conditions

(28) $$\mathfrak{M}_{oi} = 0$$

Thus, equation (1) which at first sight seems to contain four conditions, is seen actually to contain only three. And by equation (27) it follows that in this frame

$$X_i = 0$$

Thus, Weysenhoff's assumption implies, in our theory, that in the rest frame Σ_o the center of mass and the center of matter density concide. It is clear from this that Weysenhoff's assumption serves to complete the definition of the equations of motion of the particle.

To obtain the equations of motion in detail, we first multiply (25) by v^ν. This gives (with $v_\mu v_\mu = 1$) :

(29) $$G_\mu = (G_\nu v^\nu) v_\mu + v^\nu \frac{d \mathfrak{M}_{\mu\nu}}{d\tau}$$

Now because $\mathfrak{M}_{\mu\nu} v^\nu = 0$ we have $v^\nu \frac{d \mathfrak{M}_{\mu\nu}}{d\tau} + \mathfrak{M}_{\mu\nu} \frac{dv^\nu}{d\tau} = 0$. Equation (29) then becomes

$$G_\mu = m_o v_\mu - \mathfrak{M}_{\mu\nu} \frac{dv^\nu}{d\tau}$$

which gives, when multiplied by $dv^\mu / d\tau$

$$G_\mu \frac{dv^\mu}{d\tau} = m v_\mu \frac{dv^\mu}{d\tau} - \mathfrak{M}_{\mu\nu} \frac{dv^\mu}{d\tau} \frac{dv^\nu}{d\tau}$$

But because $v^\mu v_\mu = 1$ the first term on the right hand side vanishes ; while because of the antisymmetry of $\mathfrak{M}_{\mu\nu}$ the second term also vanishes. Thus we obtain $\frac{d}{d\tau} (G^\mu u_\mu) = 0$ and $G^\mu u_\mu$ = constant. In fact $G^\mu u_\mu$ plays just the role of a rest mass, which we denote by m. Thus we have from (29)

(30) $$\boxed{G_\mu = m v_\mu - \mathfrak{M}_{\mu\nu} \frac{dv^\nu}{d\tau}}$$

This is one of Weysenhoff's set of equations.

To obtain the other set of Weysenhoff's equations, we differentiate equation (30) with regard to noting that

$$\frac{dG_\mu}{d\tau} = 0$$

This yields

$$\frac{m\, dv_\mu}{d\tau} - \frac{d \mathfrak{M}_{\mu\nu}}{d\tau} \frac{dv^\nu}{d\tau} - \mathfrak{M}_{\mu\nu} \frac{d^2 v^\nu}{d\tau^2} = 0$$

~~By~~ applying (28) and $G_\mu \frac{dv^\mu}{d\tau} = 0$ we obtain $\frac{d\mathfrak{M}_{\mu\nu}}{d\tau} \frac{dv^\nu}{d\tau} = 0$ and are left with :

(31) $$\boxed{m \frac{dv^\mu}{d\tau} = \mathfrak{M}_{\mu\nu} \frac{d^2 v^\nu}{d\tau^2}}$$

The physical meaning of these relations can be further clarified by the introduction of a spin vector (7) s_μ defined by the relation

$$s_\mu = \widetilde{\mathfrak{M}}_{\mu\nu} v_\nu = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} v^\nu \mathfrak{M}^{\alpha\beta}$$

s_μ is a space like vector, for we have evidently

$$s_\mu v_\mu = 0$$

In the rest frame we get :

$$s_i = \frac{1}{2} \varepsilon_{ijk} \mathfrak{M}^{jk} \qquad (\varepsilon_{ijk} \doteq \varepsilon_{ijko})$$

which implies that the spin is the space dual of the angular momentum in the rest frame.

Reciprocally we can write $\mathfrak{M}_{\alpha\beta}$ in terms of s_μ and v_μ. The preceeding relation gives evidently :

$$\mathfrak{M}_{jk} = \varepsilon_{ijk} s^i$$

which can be written in the covariant form :

(32) $$\mathfrak{M}_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} s^\alpha v^\beta$$

The above implies the identity :

(33) $$\varepsilon_{\alpha\beta\mu\nu} \mathfrak{M}^{\mu\nu} = 2(s_\alpha v^\beta - s_\beta v^\alpha)$$

From equation (19), it is then possible to calculate the derivatives of s_μ. We find immediately

$$\frac{d s_\mu}{d\tau} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \left(\mathfrak{M}^{\alpha\beta} \frac{dv^\nu}{d\tau} + \frac{d\mathfrak{M}^{\alpha\beta}}{d\tau} v^\nu\right)$$

$$= \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \left(\mathfrak{M}^{\alpha\beta} \frac{dv^\nu}{d\tau} + G^\alpha u^\beta v^\nu - G^\beta u^\alpha v^\nu\right)$$

$$= \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \mathfrak{M}^{\alpha\beta} \frac{dv^\nu}{d\tau}$$

(7) The need for the introduction of a spin 4 vector density, instead of tensor density $\mathfrak{M}_{\mu\nu}$ has been stressed by de BROGLIE ([1], p. 54). It has also been introduced independently of us by HALBWACHS [3] and [4].

By utilising the decomposition (32) of $\mathfrak{M}^{\alpha\beta}$ we obtain :

(34) $$\frac{ds^\mu}{d\tau} = \frac{dv^\nu}{d\tau}(s_\mu v_\nu - s_\nu v_\mu) = -(s_\nu \frac{dv^\nu}{d\tau})v_\mu$$

where we have also used

$$v^\nu \frac{dv^\nu}{d\tau} = 0$$

In terms of the spin components s_μ equation (29) then becomes :

$$G_\mu = m\, v_\mu - \varepsilon_{\mu\nu\alpha\beta}\, s^\alpha\, v^\beta \frac{dv^\nu}{d\tau}$$

which can be written as

(35) $$G_\mu = m\, v_\mu - p_\mu$$

if we introduce the 4 vector

(36) $$\boxed{p_\mu = \varepsilon_{\mu\nu\alpha\beta} \frac{dv^\nu}{d\tau}\, s^\alpha\, v^\beta}$$

orthogonal to s_μ and v_μ . p_μ evidently represents usual energy momentum of rotation in the rest frame.

The Weysenhoff equations (31) can then be expressed in a simplified form. We obtain after a simple calculation

(37) $$\frac{dp^\mu}{dt} = m\,\frac{dv^\mu}{dt} = \varepsilon_{\mu\nu\alpha\beta} \frac{d^2 v^\nu}{d\tau^2}\, s^\alpha\, v^\beta + \varepsilon_{\mu\nu\alpha\beta} \frac{dv^\nu}{d\tau} \frac{ds^\alpha}{d\tau}\, v^\beta$$

Multiplying (37) by s_μ we then obtain

(38) $$m\,\frac{dv^\mu}{d\tau}\, s^\mu = 0$$

which, when compared with (34) implies finally the relation :

(39) $$\frac{ds^\mu}{d\tau} = 0$$

This shows that the spin s_μ of the droplet is a constant, according to the Moller-Weysenhoff theory.

From (37) and (38) we then obtain :

(40) $$\boxed{m\,\frac{dv^\mu}{d\tau} = \varepsilon_{\mu\nu\alpha\beta} \frac{d^2 v^\nu}{d\tau^2}\, s^\alpha\, v^\beta}$$

Equation (39) constitues a set of second order differential equations for the velocity. These equations imply that, unlike what happens with Newton's laws of motion, not only are the initial values of Y_i and $\frac{dY_i}{d\tau}$ arbitrary, but so also are these of $\frac{dv_i}{d\tau} = \frac{d^2 Y_i}{d\tau^2}$. As a result, new motions are possible not contained within the framework of Newton's laws. These new motions must however be consistent with (30) and (35) from which they were derived by differentiation.

To investigate the solutions to (40), it will be adequate to consider what happens in a special frame, namely that in which the space components G_i of the momentum are zero. For because of the Lorentz invariance of the theory, another solution of these equations corresponding to a non zero value of G_i can always be obtained by Lorentz, transforming the solution corresponding to $G_i = 0$.

As the velocity of the center of mass corresponding to each frame Σ is given by the relation $\frac{d X_i}{d\tau} = \frac{G_i}{G_o}$ we see that in the frame π_o where $G_i = 0$ (inertial rest frame of Weysenhoff) the center of mass is at rest. In π_o the space components v_i are then the components of a space vector $\vec{v}$ which represents the velocity of the center of matter relative to the center of mass.

The velocity v_μ also has a time component $v_o = \frac{J_o}{D}$ since $\frac{d J_o}{d\tau} = \frac{d D}{d\tau} = 0$ we obtain $\frac{dv_o}{d\tau} = 0$.

We see also that in the special frame π_o the acceleration of the center of matter is a space like $\frac{d\vec{v}}{d\tau}$ and the general relation $v_\mu \frac{dv^\mu}{d\tau} = 0$ can be written $v_i \frac{dv_i}{d\tau} = 0$ which implies that the two vectors $\vec{v}$ and $\frac{d\vec{v}}{d\tau}$ are orthogonal.

In π_o the components of the 4 vector p_μ are $p_i = u\, v_i$, $p_o = m\, v_o - G_o$.

The relation $p_\mu s_\mu = 0$ which follows from (36) can be written as $m\, v_i s_i + p_o s_o = 0$. From $v_\mu s^\mu = 0$ we then deduce $v_i s_i = - v_o s_o$ so that we find $- m\, v_o s^o + p_o s_o = 0$ or equivalently : $s_o(p_o - m\, v_o) = - s_o G_o = 0$. As $G_o \neq 0$ we see finally that $s_o = 0$ in π_o which implies that s_μ is s space like vector in that frame. Then $v_\mu s_\mu = 0$ becomes $v_i s^i = 0$ in π_o and we get $s^\mu \frac{dv_\mu}{d\tau} = s_i \frac{dv_i}{d\tau} = 0$ (because of $\frac{dv_o}{d\tau} = 0$). This shows that the three vector $\vec{v}$ $\frac{d\vec{v}}{d\tau}$ and $\vec{s}$ form an orthogonal instantaneous system of axis which generalizes to our case the Darboux-Freinet moving system of axis. The integration of the laws of motion results immediately from these considerations. The space like components of p_μ in the frame π_o can be written as

$$p_i = m\, v_i = \varepsilon_{iojk} \frac{dv_o}{d\tau} s^j v^k + \varepsilon_{ijok} \frac{dv^i}{d\tau} s^o v^k + \varepsilon_{ijko} \frac{dv^i}{d\tau} s^k v^o = \varepsilon_{ijko} \frac{dv^i}{d\tau} s^k v_o$$

since $\frac{dv_o}{d\tau}$ and s^o are zero. In ordinary vector notation, this relation becomes

(41) $$n \vec{v} = v_o (\frac{d\vec{v}}{d\tau} \times \vec{s})$$

The above equation implies that the motion of the center of matter remains in a plane orthogonale to s_μ . As $\vec{s}$ and v_o are constants in π_o (since $\frac{ds_\mu}{d\tau} = \frac{dv_o}{d\tau} = 0$) this motion reduces to a circular uniform motion with an angular velocity $\omega = \frac{n}{|s|}$. If we multiply (41) vectorially by $\vec{s}$ we obtain

$$m \vec{v} \times \vec{s} = v_o (\frac{d\vec{v}}{d} \times \vec{s}) \times \vec{s}$$

$$= (\vec{s} \frac{d\vec{v}}{d\tau})\vec{s} - s^2 \frac{d\vec{v}}{d\tau}$$

$$= - s^2 \frac{d\vec{v}}{d\tau}$$

since by equation (37) $\vec{s}$ is orthogonal to $\frac{d\vec{v}}{d\tau}$. This shows that $\frac{d\vec{v}}{d\tau} = \frac{n}{s^2} \vec{s} \times \vec{v}$ Weysenhoff's equations therefore completely determine the time rate of change of $\vec{v}$.

We conclude that in π_o Weysenhoff's equations imply that the center of matter density executes the above-mentioned uniform circular motion around a fixed center of mass. This gives a clear physical meaning to the motions described by the Weysenhoff equations.

Finally we remark that if the external forces are acting on the fluid droplet, their effect can be taken into account by adding them to the energy momentum conservation equation (2), so that we have $\partial^\nu T_{\mu\nu} = \bar{p}_\mu$ where F_μ is the applied force. External torques can be taken into account in a similar way. This, in fact, has already been done by MOLIER.

6. On the physical meaning of the Weysenhoff motions.

Thus far we have seen in section 4 that Weysenhoff's assumption $\mathfrak{M}_{\mu\nu} V^\nu = 0$ leads to interesting new kinds circular motion.

We can obtain a better understanding of this problem by going to the Lorentz frame Σ_o in which the space like parts v_i of the velocity are zero (while v_o is unity). Equation (29) then takes the form

$$G_i = - \mathfrak{M}_{i\nu} \frac{dv^\nu}{d\tau}$$

Thus a non vanishing acceleration will imply that the momentum and velocity are not colinear, so that even when the mean matter current is zero, there is still some momentum.

To show how this situation could come about, consider a mass of fluid in its rest frame (so that the total current is zero). Now suppose that this fluid is, to begin with, in a symmetrical and uniform rotational motion about its center of matter density. Then, as is evident from the symmetrical distribution of the energy, the center of mass will coincide with the center of matter density and Weysenhoff's condition will be satisfied. However, there will be no acceleration since the mean momentum is zero because of the symmetry. Thus the fluid will simply continue to rotate about a fixed point.

If this fluid is viewed from another Lorentz frame in which the body of the fluid moves with a velocity $\vec{v}$ the part of the body in which rotational and translational velocities add, will then be moving faster than the part in which they substract. Thus the energy density will be higher on the former side than on the latter ; and as a result, the center of mass will move away from the center of the matter density.

In this way, we see qualitatively, the origin of the relation $\frac{d\vec{v}}{d\tau} = \frac{m}{s^2} \vec{s} \times \vec{v}$ since $\frac{d\vec{v}}{d\tau}$ is just proportional to the difference between these two centers.

However, it is clear that as long as the distribution of velocity in the rest frame is symmetrical, there will be no net acceleration. We may however suppose a further disturbance in the rotating mass of fluid ; for example, localized vortices which do not contribute to the net current, but which do contribute to the energy density. Such a vortex would have two effects on the net motion.

First, it would displace the center of mass away from the center of matter density. Secondly, it would contribute to the mean momentum, since it would place a high mass density in a region of high velocity. Thus the mean momentum could fail to be zero even when the mean current was zero.

Of course, to satisfy the Weysenhoff condition without reducing to the trivial case of rectilinear motion, it is necessary to bring the center of mass back to the center of matter density, without bringing the momentum back to zero. This could be done by supposing further vortices on the opposite side of the body which lead to an energy density that cancels the moments of the original vortices in the determination of the center of mass, without cancelling their momentum completely. To show that this is possible, consider two vortices on opposite sides of a diameter of the body. Let r_1 be the distance of the first vortex from the center of matter density, r_2 that of the second. Let ω_1 be the energy of the first vortex, ω_2 that of the second. Then we choose $\omega_1 r_1 = \omega_2 r_2$ in order to satisfy to the Weysenhoff condition. Now the momentum of the first vortex will be $\vec{p_1} = \omega_1 \vec{u_1}$ where is the local mean stream velocity around the vortex, while that of the second vortex will be $\vec{p_2} = \omega_2 \vec{u_2}$. If the angular velocity ω were a constant throughout the body, (so that it was rotating as if it were rigid), then we would have $\vec{u_1} = \omega \vec{r_1}$ and $\vec{u_2} = \omega \vec{r_2}$ so that $\vec{p_1} + \vec{p_2}$ would be $\omega(\omega_1 \vec{r_1} - \omega_2 \vec{r_2}) = 0$. But suppose ω were a function of r. This would imply, of course, a non rigid rotation (of a type which is evidently quite common in fluids). Then we would have :

$$\vec{p_1} + \vec{p_2} = \omega_{(r_1)} \omega_1 \vec{r_1} - \omega_{(r_2)} \omega_2 \vec{r_2} = (\omega_{(r_1)} - \omega_{(r_2)}) \omega_1 \vec{r_1}$$

which is in general not zero. We see then that the Weysenhoff condition would be satisfied by suitable distribution of motions in the fluid. Of course, it could also be satisfied with much more complex distributions, but the principle is essentially the same.

We can now easily see qualitatively, the reason for the Weysenhoff motion, for the velocity of the center of mass is proportional to the total momentum. Thus, if the frame where $v_i = 0$, G_i is not zero, the center of mass will move and separate from the center of the matter density. Since the fluid body tends to maintain in a certain shape, this process cannot continue indefinitely without some change in the pattern of the fluid motion and internal tensions, which leads to acceleration of the center of matter density. Indeed, if the fluid satisfies the Weysenhoff condition (1) for all times, then the distribution of motion will be such as to lead an acceleration of the center of matter density which satisfies the Weysenhoff equations. Thus, the Weysenhoff assumption implies certain restrictions on the general features of the internal motions of the fluid body.

7. Extension to more general motions not satisfying the Weysenhoff condition.

We have seen that the Weysenhoff condition (1) represents a certain state of the internal motion of the fluid. The most general state of motion evidently need not satisfy this condition. We shall now formulate the problem of how to treat this more general type of motion.

First of all, we no longer require that the time component of the angular momentum vanish in the rest frame. Is then becomes convenient to split the angular momentum into two parts, one of which is purely space like, and the other which is purely time like. To do this, we first define the 4 vector :

$$t_\mu = \mathfrak{M}_{\mu\nu} v^\nu$$

(42)

$$s_\mu = \widetilde{\mathfrak{M}}_{\mu\nu} v^\nu = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} v^\nu \mathfrak{M}^{\alpha\beta}$$

where $\widetilde{\mathfrak{M}}_{\mu\nu}$ is the dual of $\mathfrak{M}_{\mu\nu}$

We note that because of the anti-symmetry of $\mathfrak{M}_{\mu\nu}$ we have :

(43a) $$v_\mu t_\mu = 0$$

and similarly

(43b) $$v_\mu s_\mu = 0$$

Thus, t_μ and s_μ are both vectors whose time component vanish in the rest frame. Hence they have a total of six independant components. Since $\mathfrak{M}_{\mu\nu}$ also has six independant components, this suggests that it should be possible to express $\mathfrak{M}_{\mu\nu}$ completely in terms of s_μ and t_μ. This is indeed possible, and the expression is :

(44) $$\mathfrak{M}_{\mu\nu} = t_\mu v_\nu - t_\nu v_\mu + \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}(s_\alpha v_\beta - s_\beta v_\alpha)$$

To verify this, one need merely to go to the rest frame (where $v_i = 0$ and $v_o = 1$ while t_o and s_o are zero). For since (44) is a tensor relation it will be true in every frame if it is true in any one frame. But in the rest frame we have from (42) :

$$\mathfrak{M}_{\nu o} = t_i$$

Since $\frac{1}{2}\varepsilon_{o\nu\alpha\beta}(s_\alpha v_\beta - s_\beta v_\alpha) = 0$ in the rest frame, the time like components of (44) are identical.

To deal with the space like components, we take the dual of both sides of (29). This gives

$$\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\mathfrak{M}^{\alpha\beta} = \varepsilon_{\mu\nu\alpha\beta} t^\alpha v^\beta + s_\mu v_\nu - s_\nu v_\mu$$

In the rest frame, the time components of this relation reduce to :

$$\frac{1}{2}\varepsilon_{oijk}\mathfrak{M}^{jk} = - s_i$$

which is also an identity. Thus equation (44) is proved.

The Weysenhoff theory then corresponds to the choice

$$t_\mu = 0$$

If we wish to make a more general choice, we are then faced with the problem of determining the equation of motion of the t_μ . For as we saw the original equation (26) on which the Weysenhoff theory is based are just sufficient to determine the equations of motion when t_μ is chosen equal to zero. To proceed further we shall therefore need some additional physical hypothesis.

One can obtain some restrictions on t_μ by returning to a more detailed consideration of our model. Now, as POINCARÉ [7] has shown, there exist stable rotating configurations of fluid that are symmetrical around their axis of rotation (for example ring-shaped masses). If some inhomogeneities are introduced which destroy the symmetry of the velocity distribution around the axis of symmetry, then with the passage of time they will tend to be redistributed uniformly by oscillation and rotation ; and eventually, the configuration will approach its stable symmetrical form again.

Let us then suppose that our rotating fluid mass takes one of these stable symmetrical forms. Now in the rest frame (where $v_i = 0$) equation (25) shows that $\frac{d\mathfrak{M}_{ik}}{d\tau}$ vanishes. Thus the spatial part of the angular momentum is a constant : and it is clear that the stable axis of symmetry around which the fluid rotates must coincide with the direction of the angular momentum vector.

Let us now suppose that some asymmetric distribution of vortices were introduced into the rotating fluid mass which results in a non zero value of the time component of the angular momentum in the rest frame (and therefore a non zero value of t_μ). We then consider two cases :

a. t_i is perpendicular to the angular momentum

b. t_i is parallel to the angular momentum.

It is clear that in case a. the inhomogeneities will be whirled around along with the rotating fluid mass, so that their constribution to t_i will tend to cancel out in a time that is not long compared with the period of rotation of the fluid mass. Moreover, according to Poincaré's result quoted in the previous paragraph, the fluid mass will finally approach a symmetrical form in which the center of mass will clearly coincide with the center of matter density.

In case b. where t_i is parallel to the angular momentum vector, there will be in general, however, no such a tendency for the inhomogeneities to become uniformly distributed and to produce a zero average time component of the angular momentum. For in this case, the vortices are distributed asymmetrically long the direction of the angular momentum $\vec{s}$. Thus, the rotational motion will not act to produce a rapid equilibrium, since the distribution will already be symmetric about the axis of rotation.

Of course, tensions may be set in the fluid which eventually will bring the center of matter density back to the center of mass. But these tensions may in general be expected to act much more slowly than the rapid rotational motion. Thus there will be a wide range of motions in which it will be a good approximation to assume that in the rest frame, the components of t_i that are perpendicular to $\vec{s}$ are zero, while the component parallel to $\vec{s}$ will still have to be taken into account.

Thus in the rest frame we have

$$t_i = \lambda s_i$$

where λ is a proportionality factor. Since $t_o = s_o = 0$ the above can also be written as a 4 vector relation

$$t_\mu = \lambda s_\mu$$

which must hold in every frame if it holds in the rest frame. (λ is evidently a pseudo scalar).

We have thus reduced the number of new variable (relative to those considered in the Weysenhoff theory) to one, namely λ. To determine λ still further, some physical hypothesis is needed, with regard to the component of t_μ that is directed along the angular momentum vector. We shall consider examples of such a hypothesis in a later paper. For the present we merely point out that the fluid model indicates a direction in which the Weysenhoff theory can be generalized.

8. Comparison with treatments of other authors.

We shall now compare our treatment of this problem with that used by others.

First of all, an essential new step proposed here was the introduction of the concept of center of matter density. We recall that because this center is not a 4 vector, it was necessary to evaluate it in a definite Lorentz frame ; namely the one in which the quantities v_i vanished.

As we have already pointed out in section 4, however, there exists another natural frame, namely the one π_o in which the space components G_i of the momentum vanish. Previous work on this problem has been based on defining the point Y_μ of equations (25) and (26) as the "center of gravity" which is the center of mass evaluated in this rest frame. In this way the fact that the center of mass is not a 4 vector is circumvented.

However, as we have seen from equation (12) the velocity of the center of mass is proportional to the momentum. Equation (26) then reduces to $\frac{d \mathfrak{M}_{\mu\nu}}{d\tau} = 0$. Thus, all components of the angular momentum remain constant ; the "center of gravity" moves at a constant rate ; and Weysenhoff's equations reduce to the trivial case of uniform rectilinear motion. Hence, the whole treatment loses its interest, and nothing qualitatively new is learned about the motion from the Weysenhoff equations.

In order to give more physical relevance to the Weysenhoff equations, two types of proposals have been made.

a. MOLLER [6] has shown that these equations could describe the motion of a certain set of purely mathematically defined "pseudo centers of gravity", whose connection with any aspect of the motion of the body has not been defined,

b. It has been shown that under suitable conditions points fixed in the body will undergo the Weysenhoff notions.

Possibility a. is evidently not satisfactory since it describes no real physical properties of the motion. Possibility b. also is not entirely satisfactory, especially for a fluid where elements undergo complex motions an where it is arbitrary to choose a particular element to describe the behaviour of the mass of fluid.

Our proposal of a center of matter density provides a natural refenrence point to describe the behaviour of the fluid ; for the matter current determines the general motion of the fluid, while the center of mass is a point which moves at a constant rate. By studying the motion of the center of matter density relative to that of the center of mass, we obtain a general idea of how the fluid motion differs from uniform rectilinear motion, without the need for going into a detailed treatment of all the complex motions inside the fluid body (8).

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(8) These details could, for example, be treated by considering moments of the first energy-momentum and current densities higher than the first, but since such moments are not in general conserved, their effects tend to be lost by random mixing processes analogous to collisions in the Boltzman equation in statistical mechanics. Thus, the conserved moments not only satisfy equations that are independant of the higher moments, but also describe the major part of the "bulk" properties of the fluid averaged over some period of time.

We also give a clear interpretation of the Weysenhoff condition $s_{\mu\nu} v_{\nu} = 0$ for it means that in the Lorentz frame in which the spacial components of the mean velocity are zero, there is no time component to the internal angular momentum so that in this frame the center of mass and the center of matter density are the same. As we have seen in the previous sections, this is a possible state of motion of the fluid droplet, and one which could readily be set up. As we have also seen, however, more general states of motion are possible. Thus our physical interpretation of these equations opens up the possibility for studying a broader range of motions. Indeed, as we shall show in a later paper, some of these new possibilities correspond to a set of classical equations which, when quantized lead to a generalization of the Dirac equation. In these generalized equations, there is a new set of quantum numbers very similar to those (such as isotopic spin and strangeness) which have been used recently for classifying the various types of elementary particles. Thus, the new quantum numbers can be interpreted as representing states of rotation and internal excitation of a relativistic liquid droplet.

This general theory of rotating relativistic droplets is also interesting from another point of view. It provides a clear physical model for the "molecules" which could constitute relativistic fluids with spin, that is, fluids characterized at the macroscopic level by a continuous distribution of internal angular momentum. By adding to G_{μ} and $\mathfrak{M}_{\mu\nu}$ suitable tensions $\Theta_{\mu\nu}$ representing intersections between the rotating droplets which constitutes such fluids (so that the total energy momentum tensor of such a fluid can be represented by D , $G_{\mu} v_{\nu} + \Theta_{\mu\nu}$ where the total current is $D v_{\mu}$ and the internal angular momentum $D \mathfrak{M}_{\mu\nu}$), we can formulate new types of relativistic hydrodynamics. The theory of these types is now being developed. Indeed, it has already been shown that they provide a model for the hydrodynamical representation of the Dirac and Kemmer wave equations (see [2] and [10]), thus furnishing a physical basis for the causal interpretation of relativistic wave equations.

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