Séminaire sur les équations non linéaires
École Polytechnique

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Séminaire sur les équations non linéaires (Polytechnique) (1977-1978), exp. n° 2, p. 1-25
<http://www.numdam.org/item?id=SENL_1977-1978____A3_0>

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LES EQUATIONS NON-LINEAIRES

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POLE INTERPRETATION OF ONE-DIMENSIONAL COMPLETELY INTEGRABLE SYSTEMS OF KORTEWEG-de VRIES AND BURGERS-HOPF TYPE.

D.V. CHOODNOVSKY

§ 0. INTRODUCTION

This paper is devoted to detailed analysis of behaviour of poles of meromorphic solutions for two types of non-linear partial differential equations : Korteweg-de Vries (KdV) and similar ones and Burgers-Hopf (BH) and its analogues.

This paper consists of results of D.V. Chudnovsky contained in a preprint [21] of two authors which remains unpublished because of political persecution of the authors (though it was distributed).

Most of the results presented here have been already published in [1] and text in this volume contains also references for new publications in same field.

The first part (§ 1) give definitions of completely integrable systems with which the motion of poles is connected. Then we concentrate our attention on many-particle interpretation of the motion of poles. Always in the problem of pole motion for classical completely integrable systems, there arise some funny functional equations.

In § 2 we investigate these functional equations which are naturally equivalent to different forms of law of additions for Abelian varieties and give information on pole motion for KdV, BH, mKdV and higher types of KdV equations. E.g. the poles of meromorphic solutions of KdV behave like particles with potential X^{-4}. Such ideology was used since [1] for different kinds of equations (see e.g. [31]), as well as for equations in two dimensions (see papers in this volume) and e.g. for Benjamin-Ono equations (paper No 7 in this volume).

§ 3 is devoted to consideration of properties of solutions of higher order BH equations. As it was realized since [21] and [1] that these equations play an important role for multisoliton solutions in two-dimensions. Two papers in this volume (namely papers No 8 and 9) deal with such applications.

§1. MANY-PARTICLE COMPLETELY INTEGRABLE SYSTEMS.

Lax' procedure gives the first examples of the many particle completely integrable systems. The most known example is the Toda lattice [13]:

(1) $$\ddot{x}_n = e^{x_{n+1}-x_n} - e^{x_n-x_{n-1}} .$$

This system is equivalent to the Lax representation $\frac{dL}{dt} = [A,L]$, where the matrices L and A are the following

$$L_{mn} = i\sqrt{C_n}\,\delta_{n,m+1} - i\sqrt{C_m}\,\delta_{n+1,m} + v_n\delta_{mn},$$

$$A_{mn} = \frac{i}{2}\,(\sqrt{C_n}\,\delta_{n,m+1} + \sqrt{C_m}\,\delta_{n+1,m}),$$

where $C_n = e^{x_n-x_{n-1}}$ and $v_n = \dot{x}_n$.

In fact, the infinite Toda lattice (1) can be considered just in the same way as the KdV equation and in fact, from the topological point of view, it is the same kind of equation.

The system (1) as well as the KdV or non-linear Schrödinger equations arises from the moduli of hyperelliptic curves and there exists a natural analogue of finite band solutions and these solutions can be represented using the same θ-functions on the Jacobian varieties of hyperelliptic curves. Naturally, with the system (1) are also associated higher analogues of the Toda lattice.

In fact these systems arise in some physical problems and, for example, are related to the well-known Fermi-Pasta-Ulam chains [21]. These chains are obtained from (1) by taking only 3 terms in the expansion for the exponent.

There is however mechanical systems of complete different character having no topological relations to the Burchnall-Chaundy-Lax [23], [24], [25] procedure. We mean the system of finitely (or infinitely) many particles interacting via the potential $G\wp(x)$ (where $\wp(x)$ is a Weierstrass elliptic function [21]). In the degenerate case we obtain a system of particles $x_i = x_i(t)$ interacting via the Jacobi potential x^{-2}. Thus there occurs an Hamiltonian of the form

$$(2) \qquad H_{\wp} = \frac{1}{2}\sum_{i\in I} \dot{x}_i^2 + G\sum_{i\neq j} \wp(x_i - x_j)$$

or

$$(3) \qquad H = \frac{1}{2}\sum_{j=1}^{n} y_j^2 + \sum_{1\leq j<k\leq n} (x_j - x_k)^{-2}.$$

It was a surprising result obtained by Moser [3] for finite n in (3) and by Calogero [5] for finite I in (2) that the corresponding systems possess Lax representation

$$\frac{dL}{dt} = [A,L]$$

for finite matrices A and L and, so (as n eigenvalues of L are conserved) possess n first integrals. The form of A and L is very easy for (3):

$$L_{ij} = (1 - \delta_{ij})\sqrt{-1}(x_i - x_j)^{-1} + \delta_{ij} y_i;$$

$$A_{ij} = (1 - \delta_{ij})\sqrt{-1}(x_i - x_j)^{-2} - \delta_{ij} \sum_{j \neq k} (x_k - x_j)^{-2}\sqrt{-1}.$$

For the case of (2) the matrix $L = (L_{ij})$, $i,j \in I$ have the form

$$L_{ij} = (1 - \delta_{ij})\sqrt{-G}\,\alpha(x_i - x_j) + \delta_{ij}\dot{x}_i,$$

where $\alpha^2(x) = \mathcal{P}(x)$.

Then, of course, the quantities $J_n = \frac{1}{n}\operatorname{tr}(L^n)$, $n \geq 1$ are the first integrals of $H_{\mathcal{P}}$. Moreover it is proved that the J_n are involutive and that they are sums of polynomials in $\dot{x}_i$, $\mathcal{P}(x_i - x_j)$, G with rational coefficients. The form of the first terms of J_n is the following

$$(4) \qquad J_n = \frac{1}{n}\sum_{i \in I} \dot{x}_i^n + G\sum_{i \neq j} (\dot{x}_i^{n-2} + \dot{x}_i^{n-3}\dot{x}_j + \ldots + \dot{x}_j^{n-2})\mathcal{P}(x_i - x_j) + \ldots$$

For the Hamiltonians (2) and (3) the exact formulae for solutions can be given for finite $|I| = n$. For $\mathcal{P}(x) = x^{-2}$ all the solutions $x_i = x_i(t)$ for the Hamiltonian H and for the Hamiltonian J_n are algebraic functions. In fact the x_i are the roots of some polynomial $P(x,t)$ having degree n on x. These solutions can be easily obtained using the matrix L for $\wp(x) = x^{-2}$. For the matrix [2], [8]:

$$M(t) = \mathrm{diag}(x_1(t_0),\ldots,x_n(t_0)) + L(t_0)(t - t_0)$$

the eigenvalues $x_i(t)$ are solutions with given initial values $x_i(t_0)$, $\dot{x}_i(t_0)$, corresponding to the Hamiltonian $H = 1/2 \operatorname{tr} L^2$. For the Hamiltonian $J_n = 1/n \operatorname{tr} L^n$, in $M(t)$, L must be replaced by L^{n-1}.

Because of this rational character it is unclear how the system $H_\wp$ is connected with the usual one. But the connection with the elliptic curve remains: in fact the existence of Lax representation for (2) is simply equivalent to the functional equation defining the $\wp(x)$, namely to the law of addition for $\wp(x)$ [12]. Nevertheless it was very interesting to show that in fact there is a close relation between many particle systems (2)-(3) and the solutions of known completely integrable equations. This connection lies in the so-called pole interpretation. The idea of such pole interpretations were first proposed by Kruskal [7], but at that time they were not taken into consideration.

POLE INTERPRETATION OF ONE-DIMENSIONAL COMPLETELY INTEGRABLE SYSTEMS OF KORTEWEG-DE VRIES AND BURGERS-HOPF TYPE

The most intriguing feature of exact solutions of non-linear differential equations, that were found during these years (1922-1931 [23], [25] and 1967-1977 [18], [20]) is the meromorphic character of the solutions. Moreover the meromorphic functions that can appear as solutions are of a particular nature, because almost all exact formulae are obtained from multidimensional (or even infinite-dimensional) θ-functions.

So it is natural to start the investigation of solutions from meromorphic ones. While studying meromorphic solutions $u(x,t)$ or $u(x,y,t)$ as functions of x we suppose that at least for some non-trivial interval $[0,t_0]$ or $[0,y_0] \times [0,t_0]$, the function $u(x,t)$ or $u(x,y,t)$ is meromorphic as function of x for $t \in [0,t_0]$ or $(y,t) \in [0,y_0] \times [0,t_0]$.

It is necessary to mention that the meromorphy or even rationality of the initial condition $u(x,0)$ or $u(x,y,0)$ does not imply the meromorphy of $u(x,t)$ $(u(x,y,t))$ for $t \gtrless 0$. Evenfor the KdV equation if the solution $u(x,t)$ is meromorphic for $t \in [0,t_0]$, $t_0 > 0$, then all the poles $x_i = x_i(t_1)$ of $u(x,t)$ for any given $t_1 \in [0,t_0]$ are of second order and of particular type [i.e. $\Sigma_{j\neq i}(x_i - x_j)^{-3} = 0$ for all $x_i = x_i(t_1)$ and any $t_1 \in [0,t_0]$].

In particular this shows why the "algebraic inverse scattering method" cannot give complete solution of completely integrable equations: it is necessary to consider also non-meromorphic $u(x,t)$. But for meromorphic solutions the problem of finding exact analytical formula of meromorphic solutions is solvable by so-called method of pole interpretation (pole expansion).

The ideas of this method briefly are the following: to consider solutions $u(x,t)$ of non-linear partial differential equations as meromorphic functions in the complex x-plane and to investigate the motion of the poles $x_i = x_i(t)$ as particles with self-consistent potential. In fact, for all the classical non-linear completely integrable systems the pole interpretation (or zero-interpretation in the sense that we consider entire functions, entire solutions) leads either to systems (2), (3) or to Hamiltonians J_n. In this situation there can arise systems of differential equations with some constraints.

The procedure of pole (zero) interpretation is of particular interest as the process of finding all the rational solutions [8], [ ].

§2. POLE INTERPRETATION OF KORTEWEG-DE VRIES EQUATION AND SOME FUNCTIONAL EQUATIONS.

In this paragraph we display the connection between Korteweg-de Vries and other equations and many-particle systems with potentials x^{-2}, x^{-4} introducing pole expansions of certaion solutions of various partial differential equations, and showing that the time evolution of the position of the poles corresponds to the motion of classical particles.

In this paragraph we concentrate mainly on the following two types of systems of differential equations:

$$\dot{x}_i = k \sum_{j \neq i} (x_i - x_j)^{-1}, \qquad i \in I, \tag{1}$$

$$\dot{x}_i = G \sum_{j \neq i} (x_i - x_j)^{-2}, \qquad i \in I, \tag{2}$$

satisfied by the system of functions $x_i(t)$, $i \in I$. The analogy between (1) and (2) and the equations of hydrodynamics describing the motion of systems of vortices or dipoles should be noted. For this, however, one should consider complex $x_i(t)$. This analogy (in relation with (2) and the Korteweg-de Vries (KdV) equation) was mentioned by W. Thikstun [6] but this analogy isincomplete, because in real hydrodynamical equations, $\dot{x}_i^*$ should appear in place of $\dot{x}_i$ in (1) and (2) (this is a mistake in [6]).

Recently (independent from [1], [21]) rational solutions of KdV equations and Boussinesq equations were investigated by Airault, McKean, Moser [8], Airault and Moser [26], [29], Satsuma, Ablowitz [27] and [28]. Methods of [27], [28], [ ], are different: they consider rational solutions as a limit case of multi-soliton ones. On the other hand detailed investigations of rational and elliptic solutions of KdV were made in [1], [8]; the paper [8] contains especially some interesting exact formulae for solutions of KdV.

We show below that any solution of (1) is a trajectory of the one-dimensional many-body system with two-body potential x^{-2} [21], [3]. On the other hand, solving (1) for $x_i = x_i(t)$ is equivalent to solving the Burgers-Hopf (BH) equation $2u_t = -2kuu_x - ku_{xx}$ for the function $u(x,t) = \Sigma_{i\in I}[x - x_i(t)]^{-1}$. The BH equation, as is well known, by putting $u = p(\partial/\partial x)\ln \varphi$ is transformed into the heat equation and can be explicitly solved. The systems of type (2) with additional algebraic restrictions describe the evolution of poles of solutions of the KdV equation or of the modified KdV equation (mKdV).

In the pole interpretation, to the KdV equation there corresponds a system (2) with the invariant algebraic restrictions $\Sigma_{j\neq i}\, \wp'(x_i - x_j) = 0,\ i \in I$. The set of such

$(x_i : i \in I)$ is the manifold M characterized by grad $H_{\mathcal{P}} = 0$ (where the variables $\dot{x}_i$ are independent of the conjugate variables x_i). Then system (2) restricted to the manifold M coincides with the action of J_3 also restricted to M. Analogously the second KdV equation in the pole interpretation leads to J_5 with the restriction to the same manifold M, etc.

We also give some systems of equations, connected with other nonlinear partial differential equations, that do not appear to be related to $H_{\mathcal{P}}$ or J_5.

2.1: We consider several classes of functional equations analogous with the Calogero equation [5], [9] and the Sutherland equation [10], [11]. All equations of this kind are variants of the law of addition for Abelian varieties of genus greater than 0.

Proposition 2.1: All (analytical) solutions of the system of functional equations

$$(3) \qquad \varphi(x)\varphi'(y) - \varphi'(x)\varphi(y) = \varphi(x+y)[\varphi'(y) - \varphi'(x)] + \Gamma(x) - \Gamma$$

$$(4) \qquad \varphi(-x) = -\varphi(x), \quad \Gamma(-x) = -\Gamma(x)$$

with given asymptotic in zero

$$(5) \qquad \varphi(\varepsilon) = \varepsilon^{-1} + \gamma\varepsilon + \ldots, \qquad \varepsilon \to 0,$$

have the form

$$(6) \qquad \varphi(x) = \zeta(x) + \gamma x,$$

where $\zeta(x)$ is an arbitrary Weierstrass zeta-function (including degenerate cases). If $\varphi(x)$ is a solution of (3)-(5), then $\Gamma(x)$ has the form

$$(7) \qquad \Gamma(x) = \frac{1}{2}\{[\varphi^2(x)]' + \varphi''(x)\} = \varphi(x)\varphi'(x) + \frac{1}{2}\varphi''(x).$$

The law of addition for the ζ-function [12] shows that, for $\varphi(x) = \zeta(x) + \gamma x$ and $\Gamma(x)$ having the form (7), conditions (3)-(5) are satisfied. On the other hand, for any solution $\varphi(x)$ of the system (3)-(5) we put in (3) $y = -z$, $x = z + \epsilon$ and, taking into account (4)-(5), we expand right- and left-hand sides of (3) in powers of ϵ. Then we immediately obtain

$$2\Gamma(z) - \varphi''(z) = 2\varphi(z)\varphi'(z)$$

and

$$-\frac{1}{6}\varphi \quad (z) - \gamma\varphi''(z) + \frac{1}{2}\Gamma''(z) = \frac{1}{2}[\varphi''(z)\varphi'(z) + \varphi''' \ (z)\varphi(z)].$$

We put $\Psi(z) = -\varphi'(z)$. Then

$$[\Psi'(z)]^2 = 4[\Psi(z)]^3 + 12\gamma[\Psi(z)]^2 + A_1\Psi(z) + A_2.$$

This implies that $\Psi(z) = \wp(z) - \gamma$, where $\wp(z)$ is the Weierstrass function. Thus $\varphi(z) = \zeta(z) + \gamma z + \rho$ and from (5) we have $\rho = 0$.

Corollary 2.2: For the functions $\varphi(x) = x^{-1}$, $\varphi(x) = \operatorname{actg}(ax)$, $\varphi(x) = \operatorname{actgh}(ax)$, we have

$$\varphi(x)\varphi'(y) - \varphi'(x)\varphi(y) = \varphi(x+y)[\varphi'(y) - \varphi'(x)]. \tag{8}$$

2.2: Let us link (1) with the BH equation. We consider a solution of the BH equation of the form

$$u(x,t) = \sum_{v\in I} (x - a_v)^{-1},$$

$a_v = a_v(t)$, $v \in I$. For finite I, $u(x,t)$ is a rational function of x. For infinite I, it is necessary to investigate the convergence of the series $\Sigma_{v\in I}\ (x - a_v)^{-1}$. We work with $\Sigma_{v\in I}$ as with a formally convergent series.

Proposition 2.3: For the function

$$u(x,t) = \sum_{v\in I} (x - a_v)^{-1},$$

the BH equation $u_t = 2cuu_x + cu_{xx}$ is satisfied if and only if the following system of differential equations is satisfied:

$$\dot{a}_{v_0} = -2c \sum_{v\neq v_0,\, v\in I} (a_{v_0} - a_v)^{-1}, \qquad v_0 \in I.$$

We use (3) or (8) for $\varphi(x) = x^{-1}$. Then

$$2cuu_x = -2c \sum_{v\in I} (x - a_v)^{-3} - 2c \sum_{v\in I} \{ \sum_{v_1\in I,\, v_1\neq v} (a_v - a_{v_1})^{-1} \} (x - a_v)^{-3}$$

2.3: We will show that the systems of types (1)-(2), that are not Hamiltonian by themselves, may be imbedded into Hamiltonian systems. The similarity of this situation with previously known cases should be mentioned. For instance, the Kac-Moerbeke lattice [13], [14]:

$$\dot{x}_k = \frac{1}{2}(-\exp[x_k - x_{k+1}] - \exp[x_{k-1} - x_k]), \tag{9}$$

that is not itself Hamiltonian, may be imbedded into the system corresponding to the Toda lattice (see supra and [3]):

$$\ddot{\xi}_j = \exp[\xi_{j-1} - \xi_j] - \exp[\xi_j - \xi_{j+1}], \quad \xi_j = x_{2j}, \quad \tau = \frac{1}{2}t.$$

We put this problem in the wider context of the following problem: for what $\Psi(z)$ and $\Phi(z)$ the system

$$\ddot{x}_i = \sum_{j\in I, j\neq i} \Phi(x_i - x_j), \qquad i \in I, \tag{10}$$

is the corollary of

$$\dot{x}_i = \sum_{j\in I, j\neq i} \Psi(x_i - x_j), \qquad i \in I. \tag{11}$$

Lemma 2.4: If the odd function $\Psi(z)$ satisfies the functional equation

$$-\Psi'(x)\Psi(z) - \Psi'(x)\Psi(y) + \Psi'(z)\Psi(x) + \Psi'(z)\Psi(y) = 0 \tag{12}$$

for $x + y + z = 0$, then from (11) it follows (10) for $\Phi(z) = 2\Psi(z)\Psi'(z)$.

Lemma 2.5: If $\Psi(z)$ is an odd function and

$$(13) \quad -\Psi'(x)\Psi(z) - \Psi'(x)\Psi(y) + \Psi'(z)\Psi(x) + \Psi'(z)\Psi(y) = F(x) - F(z)$$

is satisfied whenever $x + y + z = 0$ for odd $F(z)$, then from (11) for finite I it follows (10) with $\Phi(z) = 2\Psi(z)\Psi'(z) + (|I| - 2)F(z)$.

As Ψ and F are odd, eq. (13) coincides with (3) and (12) coincides with (8). Thus for any zeta-function $\zeta(x)$ and the corresponding Weierstrass function $\wp(x)$ the system

$$\dot{x}_i = \sum_{j=1, j\neq i}^{i} \{a\zeta(x_i - x_j) + \omega(x_i - x_j)\}, \qquad i = 1,2,\ldots,N,$$

implies

$$(14) \quad \begin{cases} \ddot{x}_i = \displaystyle\sum_{j=1, j\neq i}^{i} V(x_i - x_j), & i = 1,2,\ldots,N, \\ V(z) = [a\zeta(z) + \omega z][-a\wp(z) + \omega]N - \frac{1}{2}(N - 2)a^2\wp'(z). \end{cases}$$

System (14) is Hamiltonian with the Hamiltonian

$$H = \frac{1}{2}\{\sum_{i=1}^{N} \dot{x}_i^2 - \sum_{i\neq j} [N[a\zeta(x_i - x_j) + \omega(x_i - x_j)]^2 - a^2(N-2)\wp(x_i - x_j)]\}$$

We would like to attract attention to the evident relation of this Hamiltonian, as well as to the functional equations (12), (13) with the quantum-mechanical problems of Sutherland and Calogero [10], [11].

Corollary 2.6: Any system of equations

$$\dot{x}_i = a \sum_{j\neq i, j\in I} (x_i - x_j)^{-1} \qquad i \in I \tag{15}$$

implies

$$\ddot{x}_i = -2a^2 \sum_{j\neq i, j\in I} (x_i - x_j)^{-3} \qquad i \in I.$$

For a finite system,

$$\dot{x}_i = \sum_{j=1, j\neq i}^{N} [a(x_i - x_j)^{-1} + \omega(x_i - x_j)], \quad i = 1,2,\ldots,N$$

implies

$$\ddot{x}_i = \sum_{j=1, j\neq i}^{N} [-2a^2 (x_i - x_j)^{-3} + \omega^2 N(x_i - x_j)], \qquad i = 1,2,\ldots,N.$$

2.4: The problem of the connection of (10) with (11) for even $\Psi(z)$ can be treated analogously.

Proposition 2.7: Any solution of the system of equations

$$\dot{x}_i = k \sum_{j\neq i, j\in I} \wp(d(x_i - x_j)), \qquad i \in I,$$

$$\sum_{j\neq i, j\in I} \wp'(d(x_i - x_j)) = 0$$

is a trajectory of the system with the Hamiltonian

$$H_0 = \frac{1}{2}\sum_{i\in I} \dot{x}_i^2 + \frac{k^2 d}{2}\sum_{i\neq j} \wp^2(d(x_i - x_j)).$$

Indeed

$$\ddot{x}_i = k^2 d \sum_{j\neq i, j\in I} \wp'(d(x_i - x_j))\{\sum_{k\in I, k\neq i,j} \wp(d(x_i - x_k)) - \wp(d(x_j - x_k))\}$$

$$= k^2 d \sum_{j_1\neq j_2, j_1\neq i, j_2\neq i} \{\wp(d(x_i - x_{j_1}))\wp'(d(x_{j_2} - x_{j_1}))$$

$$- \wp(d(x_i - x_{j_2}))\wp'(d(x_{j_2} - x_{j_2}))\}$$

$$= k^2 d \sum_{j_1\neq i} \wp(d(x_i - x_{j_1}))\{\sum_{j_2\neq j, j_2\neq i} \wp'(d(x_{j_2} - x_{j_1}))\}$$

$$= -k^2 d \sum_{j\neq i} \wp(d(x_i - x_j))\wp'(d(x_i - x_j)).$$

Thus the particles satisfying (2) with the restriction $\Sigma_{j\neq i, j\in I} (x_i - x_j)^{-3} = 0$, $i \in I$, can be considered as one-dimensional particles interacting via the Maxwell potential x^{-4}.

Now we exhibit the connection between the many-body problems (2) and the KdV equation.

<u>Proposition 2.8</u>: If $u(x,t) = \Sigma_{v\in I} (x - a_v)^{-2}$, then $u(x,t)$

satisfies the equation $u_t = 12cuu_x - cu_{xxx}$ if and only if

$$\dot{a}_v = -12c \sum_{v_1 \neq v, v_1 \in I} (a_v - a_{v_1})^{-2} \qquad v \in I,$$

and

$$\sum_{v_1 \neq v, v_1 \in I} (a_v - a_{v_1})^{-3} = 0.$$

To prove this it suffices to apply the law of addition for $\wp(x) = x^{-2}$ [12] to uu_x.

Proposition 2.9: If $u(x,t) = \Sigma_{v \in I} \wp[d(x-a_v)]$, then $u(x,t)$ satisfies the equation $u_t = 12cd^2uu_x - cu_{xxx}$ if and only if

$$\dot{a}_v = -12cd^2 \sum_{v_1 \neq v, v_1 \in I} \wp[d(a_v - a_{v_1})], \qquad v \in I,$$

$$\sum_{v_1 \neq v, v_1 \in I} \wp'[d(a_v - a_{v_1})] = 0, \qquad v \in I.$$

2.5: Thus, for $u(x,t) = \Sigma_{i \in I} \wp(x - a_i)$ the satisfiability of the KdV equation is equivalent to the system

$$(2') \qquad \dot{a}_i = -12c \sum_{j \neq i, j \in I} \wp(a_i - a_j)$$

with the algebraic restrictions

$$M: \sum_{j \neq i, j \in I} \wp'(a_i - a_j) = 0, \qquad i \in I.$$

It is easy to verify that (2') describe the Hamiltonian flow induced by J_3 with

$$H_{\mathcal{P}} = \frac{1}{2}\sum_{i \in I} b_i^2 + G\sum_{i \neq j} \mathcal{P}(a_i - a_j),$$

$$M: \text{grad } H_{\mathcal{P}} = 0$$

and

$$J_3 = \frac{1}{3}\sum_{i \in I} b_i^3 + G\sum_{i \neq j} (b_i + b_j)\mathcal{P}(a_i - a_j)$$

with G = -12c.

2.6: Now let us consider the pole expansions of solutions of the mKdV equation.

Proposition 2.10: If $u(x,t) = \Sigma_{v \in I}\, c_v(x - a_v)^{-1}$, then the equation of mKdV type $u_t = 6du^2u_x - du_{xxx}$ is satisfied if and only if

$$(16)\qquad \begin{cases} \dot{a}_v = -3d \displaystyle\sum_{v_1 \in I, v_1 \neq v} (a_v - a_{v_1})^{-2}, \\ \displaystyle\sum_{v_1 \neq v, v_1 \in I} c_{v_1}(a_v - a_{v_1})^{-1} = 0 \end{cases}$$

for all $v \in I$ and $c_v = \pm 1$, $v \in I$.

2.7: Thus it appears that certain solutions of the KdV and mKdV equations, whose evolution may be obtained by the inverse-scattering method, are precisely related to certain solutions of the one-dimensional many-body problem with potentials x^{-4} or $\wp^2(x)$. This raises the following question: which other kinds of potentials besides $\alpha^2(x)$, where $\alpha(x)$ is a solution of the Calogero functional equation [5], generate an integrable many-body problem of type [3], [4]. In this connection we note that there exists an immediate generalization of the Calogero equation, namely

$$\alpha(x)\alpha'(y)-\alpha'(x)\alpha(y) = \alpha(x+y)[V_1(y)-V_1(x)]+\alpha'(x+y)[V_2(y)-V_2(x)],$$

which is satisfied by $\alpha(x) = V_2(x) = \wp(x)$, $V_1(x) = \wp'(x)$.

Thus one could hope that for the potential $\wp^2(x)$ there is also an analogue of the Lax representation. This is true to some extent. Consider the equivalent of the KdV equation:

$$(17)\quad \begin{cases} \dot{x}_v = k \sum_{v_1 \neq v, v_1 \in I} \wp(x_v - x_{v_1}) & v \in I, \\ \sum_{v_1 \neq v, v_1 \in I} \wp'(x_v - x_{v_1}) = 0, & v \in I. \end{cases}$$

According to proposition 2.7, all the solutions of this system are trajectories of the Hamiltonian

$$H_0 = \frac{1}{2} \sum_{i \in I} \dot{x}_i^2 + \frac{1}{2} k^2 \sum_{i \neq j} \wp^2(x_i - x_j).$$

Proposition 2.11: There exists a Lax representation

$\dot{L} = [L,A]$ for system (17) in which the matrices L and A are of the form

$$L_{ij} = \wp(x_i - x_j), \qquad A_{ij} = k\wp'(x_i - x_j), \qquad i \neq j,$$

$$L_{ii} = 2\sum_{j\neq i} \wp(x_i - x_j), \quad A_{ii} = 0.$$

In fact,

$$\dot{L}_{ij} = k\wp'(x_i - x_j)\{\sum_{k\neq i,j} \wp(x_i - x_k) - \wp(x_j - x_k)\},$$

$$\sum_{s\neq i,j} L_{is}A_{sj} - A_{is}L_{sj} + A_{ij}(L_{ii} - L_{jj})$$

$$= k\wp'(x_i - x_j)\{\sum_{s\neq i,j} \wp(x_i - x_s) - \wp(x_j - x_s)\} \text{ etc.}$$

Thus the Lax representation follows from the law of addition for $\wp(x)$ (including the special case $\wp(x) = x^{-2}$). Note that these calculations can be considered as analytical variants of the considerations of [15], [23]. Certainly the Lax representation for the KdV equation and (17) are equivalent.

2.8: The complete description of the meromorphic solutions of KdV having the form $u(x,t) = \Sigma_{i\in I}\, \wp(x - a_i)$ for finite I can be given. For the finite set I

$$M = \{(a_i): \text{grad } J_2 = 0\}$$

is non-void, if and only if $|I| = \frac{n(n+1)}{2}$. In this situation the corresponding M has the dimension n and is an invariant manifold for all J_m. In fact to this class corresponds exactly a n-lacunary potential $n(n+1)\wp(x)$, which arises in the theory of the Lamé equation. Really, all the potentials of the form $u(x) = \Sigma_{i\in I} \wp(x - a_i)$ for such I and $(a_i) \in M$ are periodical n-band potentials and, conversely, all the doubly periodical n-band potentials have the form $\Sigma_{i\in I} \wp(x - a_i)$ for some $(a_i) \in M$. The last result belongs to Moser and McKean [8].

Moreover the answer to the question posed in [8], as to whether the pole interpretation for higher order KdV equations [20], [24], is connected with J_{2n+1}, can be given. In fact, the evolution of system of poles of solutions of the n-th order KdV equation [20], [24] is governed by J_{2n+1} with constrants M: grad $H_\wp = 0$. For example, for the second KdV equation

$$(18) \qquad u_t + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x + u_{xxxxx} = 0,$$

and $u(x,t) = -2\,\Sigma_{i\in I} \wp(x - a_i)$ we obtain the many-body problem related with J_5:

$$\dot{a}_i = 120\{(\sum_{j\neq i} \wp(a_i - a_j))^2 + \sum_{j\neq i} \wp^2(a_i - a_j)\}, \qquad i \in I$$

with restrictions M.

In this context instead of a clever conjecture of [8], Khrichiver[15] had made a wrong conjecture about the coincidence of the action J_i on M with the motion of poles of the i-th KdV which corresponds in fact to J_{2i+1}.

Generally speaking in review [15], interesting because of the reproduction of the results on the algebra of differential equations, obtained in general form by Burchnall and Chaund y [25], Krichiver makes some claim for two-dimensional KdV equation. But for the 2-dimensional KdV equation as well as for 1-dimensional the motion of the poles was completely described in §10 of the paper [1].

2.9: There is another variant of the second KdV equation [18], [19] (see below):

$$(19) \qquad u_t + 45u^2u_x + 15(u_xu_{xx} + uu_{xxx}) + u_{xxxxx} = 0,$$

different from the usual second KdV equation (18). In the pole interpretation, (19) gives the system of equations

$$\dot{a}_i = 180\{(\sum_{j\neq i}\wp(a_i-a_j))^2 - \sum_{j\neq i}\wp^2(a_i-\alpha_j)\}$$

with restrictions

$$\{\sum_{j\neq i}\wp(a_i-a_j)\}\cdot\{\sum_{j\neq i}\wp'(a_i-a_j)\} = \sum_{j\neq i}\wp(a_i-a_j)\wp'(a_i-a_j).$$

This system is new (see also two-dimensional variant of (19)).

§3. HIGHER BURGERS-HOPF EQUATIONS AND THEIR POLE INTERPRETATION.

There are natural generalizations of Burgers-Hopf equation $u_x = 2uu_x + u_{xx}$ [17] . These equations called in [1], [21] higher Burgers-Hopf equations play a very important role for solutions of two-dimensional equations.

Higher (n-th order) Burgers-Hopf equation can be described as

$$(3.1)\quad u_t = \frac{\partial}{\partial x}\{\frac{\partial^n}{\partial x^n}[\exp(\int u dx)]\cdot\exp(-\int u dx)\} = BH_n[u].$$

These equations are generated as follows:

$$BH_n[u] = \frac{d}{dx} C_n[u]$$

and

$$C_{n+1}[u] = \frac{d}{dx} C_n[u] + u\cdot C_n[u] \quad , \quad C_0[u] = 1.$$

Then $C_n[u]$ and so $BH_n[u]$ are the polynomials in $u, u_x, \ldots$. The first terms are the following:

$BH_1[u] = u_x,$

$$BH_2[u] = 2uu_x + u_{xx},$$

$$BH_3[u] = 3u^2u_x + 3uu_{xx} + 3u_x^2 + u_{xxx},$$

$$BH_4[u] = 4u^3u_x + 18uu_x^2 + 9u^2u_{xx} + 10u_xu_{xx} + 4uuu_{xxx} + u_{xxxx}, \ldots .$$

In fact the main property of higher BH equations is that they can be linearized by Hopf-Cole substitution $u = \frac{\partial}{\partial x}(\log \varphi)$ [17]:

<u>Proposition 3.1</u>: If $u = \frac{\partial}{\partial x}(\log \varphi)$, then $u_t = BH_n[u]$ iff

(3.2) $$\varphi_t = \underbrace{\varphi_{x\ldots x}}_{u} + \lambda^n \varphi \quad \text{for some} \quad \lambda \in C.$$

The pole interpretation of BH_n is connected with J_n from §1, (1.4) for $\mathscr{P}(x) = x^{-2}$, $G = -1$.

<u>Theorem 3.2</u>: 1) For meromorphic functions $u(x,t) = \Sigma_{i\in I}(x-a_i)^{-}$ the fulfilment of the equation

$$u_t = BH_n[u]$$

is equivalent to the fulfilment of the system

(3.3) $$-\dot{a}_i = n! \sum_{\{j_1,\ldots,j_{n-1}\}\not\ni i} (a_i - a_{j_1})^{-1} \ldots (a_i - a_{j_{n-1}})^{-1}, \quad i$$

2) Any system (3.3) is embedded into the system with Hamiltonian J_n in (1.4) for $\mathscr{P}(x) = x^{-2}$, $G = -1$. The system (3.3) is equivalent to J_n under the following invariant restriction

(3.4) $$b_i = -\sum_{j\neq i} (a_i - a_j)^{-1} : i \in I.$$

Moreover for finite I, the trajectories of J_n on which all the integrals J_n vanish, $J_m = 0$, are precisely the solutions of the system (3.3).

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