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# Hamiltonian structure for one and two dimensional completely integrable systems. Two dimensional completely integrable systems associated with eigenvalue problems and complete description of their meromorphic solutions

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S E M I N A I R E  S U R

LES EQUATIONS NON-LINEAIRES

- I -

# HAMILTONIAN STRUCTURE FOR ONE AND TWO DIMENSIONAL COMPLETELY INTEGRABLE SYSTEMS. TWO DIMENSIONAL COMPLETELY INTEGRABLE SYSTEMS ASSOCIATED WITH EIGENVALUE PROBLEMS AND COMPLETE DESCRIPTION OF THEIR MEROMORPHIC SOLUTIONS

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## § 0. INTRODUCTION

This paper contains description of general approach to Hamiltonian completely integrable systems in one space, one time (x,t) or two-space, one time (x,y,t) dimensions. Also significant part of this paper contains detailed investigation of many concrete two(space) dimensional completely integrable systems and complete analysis of their meromorphic solutions.

In § 1 we explain "formal variational calculus" of Gelfand, Dikij and give the definition of Poisson-Gardner brackets for one (space) dimensional Hamiltonian systems.

In § 1 we discuss in details proper definition of "complete integrability" one one- and two-dimensional systems. We consider the existence of wide class of meromorphic solutions and systems of equations associated with some algebra of commutative differential operators.

In § 2 we present infinite systems of non-linear equations in derivatives $\frac{\partial}{\partial x}, \frac{\partial}{\partial t} : (C)_k : k = 0,1,2,\ldots$ the meromorphic solutions $u_k(x,t)$ of which have evolution of poles $a_i = a_i(t)$ according to Hamiltonian H with potential $Gx^{-2}$.

Using this system we describe exactly evolution of poles of all higher Korteweg-de Vries equation $u_t = Q_n(u,u_x,\ldots,u_{x\ldots x})$. As it was conjectured by Airault, McKean, Moser [8] the evolution of poles is given by Hamiltonian $J_{2n+1}$ with restriction grad H = 0.

In § 3 we give brief discussion of Zakharov-Shabat equation for n = 4, m = 2. Detailed picture is given in the paper of D.V. Chudnovsky "Meromorphic solutions of two-dimensional equations with algebraic laws of conservation", part B, (this volume) [14].

The most interesting part is § 4. In this part we present convenient form of two-dimensional completely integrable systems of Zakharov-Shabat form

$$\frac{dL_m}{dt} - \frac{dL_n}{dy} = [L_n, L_m] \quad ,$$

for m = 3, n = 4,5. For this case the character of meromorphic solutions either is connected with potential $Gx^{-2}$ (for n = 4 and in some cases of n = 5), or can lead to absolutely new completely integrable many-particle system (e.g. some cases of n = 5).

As an Appendix to § 4 we give results of Manin and Kypershmidt on Benney equation.

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§1. We will discribe a formal algebraic approach leading to the complete integrability of equations connected with algebraic curves.

We have the following notations. Let $B$ be the associative ring with unit over $Q$ with differentiation $\partial: B \to B$; $C$ the set of constant $C = \{b \in B: \partial b = 0\}$. $B[D]$ be the algebra of differential operators over $B$, i.e. the algebra generated by $B$ and one additional element $D$ with relation $Db - bD = \partial b = b'$ for any $b \in B$. Then for $n \geq 0$, $B_n[D]$ be the set of differential operators of order $\geq n$ from $B[D]$, $B_0[D] = B$ and $B[D] = U_n B_n[D]$.

The most interesting case is when $B$ is the ring generated by finite set of functions and their derivatives. Now for $P \in B$ we write $P^{(j)} = \partial^j P$: $j = 0,1,2,\ldots$ . We suppose

1.0: There are such $u_1 \ldots, u_q \in B$ that

$$B = C[u_r^{(j)}: r = 1 \ldots q,\ j = 0,1,2,\ldots]. \tag{1.1}$$

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This B is called a differential algebra of free type with the generators $u_1,\ldots,u_q$. Now in B there are defined C-differentiations $\partial/\partial u_r^{(j)}$

$$\frac{\partial}{\partial u_r^{(j)}}: B \to B, \quad \text{as } B = C[u_r^{(j)}], \tag{1.2}$$

and we can define also gradient or partial variational derivative:

$$\frac{\delta}{\delta u_i} = \sum_{j=0}^{\infty} (-1)^j \partial^j \cdot \frac{\partial}{\partial u_i^{(j)}} : \quad i = 1,\ldots,q \tag{1.3}$$

and we put

$$\frac{\delta}{\delta \bar{u}} = \left(\frac{\delta}{\delta u_1}, \ldots, \frac{\delta}{\delta u_q}\right): B \to B^q. \tag{1.4}$$

In these notations Gelfand, Dikij and Manin have developed [1], [2], [4] the "formal variational calculus" and have shown how to introduce in general the Hamiltonian structure on the nonlinear evolutionary equations.

First of all we have Gelfand's

Proposition 1.1: Let C = k [and $\partial k = \{0\}$], then $\operatorname{Ker} \delta/\delta\bar{u} = \operatorname{Im} \partial + k$ for a given B. So for $b \in B = k[u_i^{[j]}]$, $\delta/\delta\bar{u} b = 0$, if b is a derivative of some element of $k[u_i^{(j)}]$.

We can put $\bar{B} = B/\partial B$ and for $P,Q \in B$, $P \sim Q$ or $P = Q \pmod{\partial B}$,

if $P - Q \in \mathrm{Im}\ \partial$.

For B-module N, the elements of $N^q$ we understand as vector-column of height q and i-th coordinate of $\bar{P} \in N^q$ being $P_i$; $\bar{P}^t$ will be the vector row $(P_1,\ldots,P_q)$. The operator $^{-t}$ of transponing we'll also use to $q \times q$ matrices from $M_q(B[\partial])$. Analogically $\delta P/\delta \bar{u}$ be the column from $\delta P/\delta u_i$ and the $\delta P/\delta \bar{u}^{-t}$ the row. For $\bar{P} \in B^q$, $\bar{Q} \in N^q$, the $\bar{P}^t\bar{Q}$ is the "scalar p roduct" $\Sigma_{i=1}^{q} P_i Q_i \in N$ etc.

Then on $\bar{B} = B/\partial B$ there is a natural structure of Lie algebra with brackets [ , ]:

Theorem (Lax-Gelfand-Gardner)1.2: Let $A \in M_q[k][\partial]$ be the skew symmetric operator, i.e. for all $\bar{P},\bar{Q} \in B^q$ we have $\bar{P}^t A\bar{Q} \sim \bar{Q}^t A\bar{P}$ (mod $\partial B$). We define bracket [ , ]: $\bar{B} \times \bar{B} \to \bar{B}$ by the formula

(1.5) for $P^* \equiv P \pmod{\partial B}$, $Q^* \equiv Q \pmod{\partial B}$;

$$[P^*,Q^*] = \frac{\delta P}{\delta \bar{u}^{-t}} A \cdot \frac{\delta Q}{\delta \bar{u}} \pmod{\partial B}.$$

Then the brackets $[P^*,Q^*]$ satisfy Jacobi identity and so make $\bar{B} = B/\mathrm{Im}\ \partial$ a Lie algebra.

The brackets [ , ] are defined because of Proposition 1.1.

Gardner [3] was the first to realize what the Poisson brackets in the infinite-dimensional cases were. The theorem 1.2 in the complete generality both for the cases of scalar

and vector functions belongs to Gelfand-Dikij and Gelfand-Manin -Shubin [1], [2], [4].

Why is it interesting? Because this enables us to interpret all evolutionary equations as Hamiltonian. For example, let's consider the one-dimensional situation and vector function $\bar{u} = \bar{u}(x,t)$. If $A = A[\partial/\partial x]$ then for the system

$$\frac{\partial \bar{u}}{\partial t} = A[\partial/\partial x] \times \frac{\delta Q}{\delta \bar{u}}. \tag{1.6}$$

For functional $I_{Q^*} = \int Q\, dx$ ($Q \in B$) [which are defined as class of equivalence $Q^*$ of $Q$, up to full derivatives of elements of $B$ (i.e. $\partial B$), and exist analytically for: a) rapidly decreasing $u_r(x,t)$ for $|x| \to \infty$; b) for periodical $u_r(x,t)$ on $x$; c) for quasi-periodical functions $u_r(x,t)$ as mean values] this means

$$\frac{\partial \bar{u}}{\partial t} = A \cdot \frac{\delta}{\delta \bar{u}} I_{Q^*}. \tag{1.6}'$$

Now we have the Liouville formula in infinite dimensional situations:

$$\frac{\partial I_{P^*}}{\partial t} = I_{[P^*,Q^*]} \quad \text{for } I_{P^*} = \int P\, dx\,, \; P \in B. \tag{1.7}$$

For example, the fact that $I_{P^*}$ is a first integral of (1.6)' means that

$$I_{[P^*,Q^*]} = 0$$

or

$$[P^*,Q^*] = 0.$$

This is equivalent to

$$\frac{\delta P}{\delta \bar{u}}{}^{t} A \cdot \frac{\delta Q}{\delta \bar{u}} \in \partial B, \tag{1.8}$$

or

$$\frac{\delta P}{\delta \bar{u}}{}^{t} A \cdot \frac{\delta Q}{\delta \bar{u}} = \frac{d}{dx} R_{P,Q} \tag{1.8}'$$

for $R_{P,Q} \in k[u_r^{(j)}]$.

In fact there is a deep connection between nonstationary Hamiltonian problem (1.6) and the corresponding time independent (stationary) problem corresponding to the case $\partial\bar{u}/\partial t \equiv 0$. This connection will be shown below.

Now we can already consider special Hamiltonian systems (1.6) and show how they are connected with "isospectral deformations" and Lax pairs. In fact these systems having the Lax pairs have infinitely many first integrals $I_{P^*}$ and so they are called completely integrably or nearly completely integrable. We must say that only for Hamiltonian with $n$ degrees of freedom the existence of $n$ commuting (in involution) and

functionally independent first integrals implies real complete integrability and allow to obtain formulae for solutions through hyperelliptic integrals. In infinite dimensional situations the existence of infinite numbers of conservation laws shows nothing [take for example the product of an ergodic finite dimensional system and a completely integrable one].

In this paper we will try to consider mainly focusing at particular equations, which are the good properties of two dimensional equations that are supposed to be completely integrable. As a natural test for complete integrability we take:

a) the existence of good "multisoliton" solutions;

b) the existence of additional conservation laws.

By multisoliton solutions we do not mean precisely localized and elastically interacting soliton shape like solutions. Our point of view is slightly different from the ordinary framework of inverse scattering.

In fact the main class of solutions which we consider is the class of meromorphic solutions $u(x,y,t)$ as functions on $x$ in the complex plane for $(y,t) \in [y_0,y_1] \times [t_0,t_1]$. For different reasons algebraic as well as analytical methods of inverse scattering produces mainly meromorphic solutions.

These are "multisoliton" and rational solutions of course; solutions expressed through $\theta$-function corresponding to generalized finitely lacunary quasipotentials and many others. As meromorphic solutions are described by positions of their poles our criteria of complete integrability can be reformulated in the following way:

Problem: To find non-linear equations in $\vec{u}(x,y,t)$ such that the evolution of poles $a_i(y,t)$ correspond to solvable (completely integrable) many particle problems.

If we can solve a non-linear equation at least for the meromorphic case, then we are absolutely sure that such an equation is a good candidate for complete integrability, as for example it has very good generalized multisoliton solutions.

Remark 1: The idea of pole interpretation on the example of the KdV belongs to Kruskal but even now the precise evolution of poles for arbitrary n-soliton solutions is not absolutely clear.

Remark 2: Pole and only poles interpretation produces, for example, complete integrability of Benjamin-Ono equation, where poles correspond to particles with two charges +i and -i.

In general all known two dimensional systems suspicious

on complete integrability arise from conditions of commutativity of two operators

$$[L_1, L_2] = 0,$$

where $L_1$ and $L_2$ are differential operators in $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial t$. Then the condition of their commutativity is expressed as a system of non-linear partial differential equations on coefficients of operators $L_1$, $L_2$. There are different forms of presenting conditions of commutativity of operators especially for operators in many dimensions. For example it can be the condition of existence of common eigenfunction for the operators $L_1, L_2$ corresponding to all eigenvalues or to one prescribed eigenvalue. The latter condition is less restrictive then the condition of commutativity of $L_1$ and $L_2$. However this situation is not so carefully examined (this is the "less completely integrable case").

We have more or less information for the strong condition of commutativity $[L_1, L_2] = 0$. In this situation non-trivial equations arise only in the case when $L_1$ and $L_2$ are at most linear operators on $\partial/\partial t$ and $\partial/\partial y$. In this situation, we can write

$$L_1 = L_n - \frac{\partial}{\partial t} \quad , \quad L_2 = L_m - \frac{\partial}{\partial y}$$

and the equation, which is called Zaharov-Shabat equation has the form

$$(2\ \text{dim}) \quad [L_n, L_m] = \frac{\partial L_m}{\partial t} - \frac{\partial L_n}{\partial y} \ .$$

For this system we know of the existence of a lot of solutions arising from algebraic curves. Existence of these solutions obtained by Burchnall, Chaudy, Novikov, McKean, Trubowitz, Krichiver, is explained easily by the fact that each of the Zaharov-Shabat systems has a lot of invariant submanifolds. These invariant manifolds have the form

$$[L_n, Q] = 0 \qquad \text{and} \qquad [L_m, Q] = 0$$

for appropriate differential operators $Q$. These invariant manifolds are simply $(y,t)$ independent parts of algebraic conservation laws existing for $y$ or $t$ independent Zaharov-Shabat systems (these one dimensional $(x,t)$-dimensional systems are called Lax systems). However such solutions are very specific for general two dimensional solutions. Moreover, for the two dimensional case we have no precise sense of the spectral problem corresponding to arbitrary "quasipotentials" (by quasipotentials we always understand coefficients of the operators $L_n$ and $L_m$). There are even more difficult questions:

a) What is the sense of Hamiltonian structure for two dimensional equations?

b) What are the solutions analogical to soliton solutions with respect to genericity;

c) Does there exist a good analogue of finitely lacunary potentials?

We (specialist, not necessarily author ) start to collect the information on a), b), c) only for some special two-dimensional equations and in particular, when $m = 2$, $n = 3$, however we still have no idea of how to answer b) or c) (cf. Manin, Kupersehmidt [13] for a) and Benney equation)(see [19], [20]).

There is a lack of information for one dimensional Lax systems. For $m = 2,3$ and y independent case, we have some good results, but generally speaking, the situation with $\min(m,n) > 2$ is unclear. Especially when this corresponds to the case when $m$ and $n$ are not relatively prime. Then even the problem of finding solutions corresponding to albebraic curves becomes extremely difficult and we still don't know what kind of $\theta$-functions arise here. Moreover for non-relatively prime $m$ and $n$ we don't even have good multisoliton formulae and there are even several points of view on evolution and interaction of such multi-soliton solutions (that clearly exist!). Because of the lack of a precise analytical inverse scattering theory of higher order

multi-dimensional equation we restrict ourselves to good solutions in the whole complex plane- meromorphic solutions. The first difficult case will be m = 3, n = 5.

## §2. Meromorphic Solutions of Two-Dimensional Equations and Their Poles.

We know by [12], [8], [14] that the evolution of poles of several one-dimensional equations (e.g. KdV, Boussinesq,...) is connected with the Hamiltonian

$$H_{\wp} = \frac{1}{2} \sum_{i \in I} y_i^2 + G \sum_{i \neq j} \wp(x_i - x_j)$$

for the Weierstrass elliptic function $\wp(X)$ and with the corresponding first integrals $J_n = \frac{1}{n} \operatorname{tr}(L^n)$: $n = 1,2,\ldots$ of $H_{\wp}$ described in [15] and [12].

These one-dimensional systems have Lax' form

$$\frac{dL_1}{dt} = [L_1, L_2],$$

and so are included in more general two-dimensional equation (2 dim). In this paper we shall confirm some natural conjuectures about meromorphic solutions of (2 dim) and obtain new solutions in terms of elliptic functions.

In general the idea of the pole interpretation and the

establishment of a connection with the Hamiltonian $H_\wp$ can be described along the lines of the following general scheme [9].

We consider the following special class of meromorphic functions, residues and poles of which are expressed in terms of the variables $(x_i, \dot{x}_i)$ of $H_\wp$:

$$\hat{u}_k(z,t) = \sum_{i \in I} \frac{\partial J_{K+1}}{\partial \dot{x}_i} \wp(z - x_i), \quad k = 0,1,2,\ldots . \tag{2.1}$$

There exists a special sequence of differential equations connected with (21) :

Theorem 2.1: Let $u_{k,\underbrace{x,\ldots,x}_{m}}$ have the weight $k + m + 2$. Then there exist polynomials $\Omega_k(u_0,\ldots,u_{k-1})$ in $u_0, u_{0,x}, \ldots, u_1, u_{1x}, \ldots, u_{k-1}, \ldots, u_{k-1,xx\ldots}, \ldots$ of degree two and having all the monomials of weight $k + 3$ such that the system of equations

$$(C)_k \qquad u_{k,t} + u_{k+1,x} + \frac{d}{dx} \Omega_k(u_0,\ldots,u_{k-1}) = 0, \quad k = 0,1,\ldots$$

satisfies the following properties:

1) the functions $\hat{u}_k$ satisfy (C) if and only if $x_i = x_i(t)$ move according to $H_\wp$;

2) if $u_k(x,t)$ satisfy (C) and are meromorphic functions with poles of order 2, then $u_k = \hat{u}_k$.

Here are the first few $\Omega_k$:

$$\Omega_0 = 0,\ \Omega_1 = -\frac{G}{2}u_0^2 + \frac{G}{12}u_{0xx},\ \Omega_2 = -Gu_0u_1 + \frac{G}{6}u_{1xx},$$

$$\Omega_3 = -\frac{G}{2}u_1^2 - Gu_0u_2 - \frac{G^2}{8}u_{0x}^2 - \frac{G^2}{12}u_0u_{0xx} + \frac{G^2}{120}u_{0xxxx} + \frac{G}{4}u_{2xx}, \ \ldots .$$

Corollary : If in the system (C) we put $u_n \equiv 0$, then the system $(C)_1 - (C)_{n-1}$ possesses infinitely many polynomial conservation laws.

The first such non-trivial system coincides with the Boussinesq equation [10] $u_{tt} + (u^2)_{xx} + u_{xxxx} = 0$. In general the system (C) describes one of the scheme of approximations of two-dimensional shallow water equation [11].

In fact the infinite chain system (C) is connected with Lax or Zaharov-Shabat system corresponding to the Schrödinger operator $L_2 = d^2/dx^2 + u$. Let us consider the Zaharov-Shabat system corresponding to the case $\min(n,m) = 2$, i.e.

$$[L_2, L_n] = \frac{\partial L_n}{\partial t} - \frac{\partial L_2}{\partial y} .$$

If we consider polynomials $\Omega_k(u_0,\ldots,u_{k-1})$ in $u_0, u_{0,x}, \ldots, u_1, u_{1x}, \ldots, u_{k-1}, \ldots, u_{k-1,xx}, \ldots$: $k = 0,1,2,\ldots$, then the following system

$$(C_n)\begin{cases} u_{k,t} + u_{k+1,x} + \dfrac{d}{dx}\,\Omega_k(u_0,\ldots,u_{k-1}) = 0\colon\ k = 0,\ldots,n-3 \\ \text{and} \\ u_{n-2,t} + u_{0,y} + \dfrac{d}{dx}\,\Omega_{n-2}(u_0,\ldots,u_{k-1}) = 0. \end{cases}$$

is equivalent to the Zaharov-Shabat system

$$(L_{2,n})\ [L_2,L_n] = \frac{\partial L_n}{\partial t} - \frac{\partial L_2}{\partial y}\ .$$

Here the coefficients of polynomials $L_2$, $L_n$ are expressed in terms of functions $u_k$. For example $u = -2u_0$ and if $L_n = \dfrac{d^n}{dx^n} + v_{n-2}\dfrac{d^{n-2}}{dx^{n-2}} + \ldots + v_0$, then $v_{n-2} = -nu_0$, $v_{n-3} = -n(n-2)/2\ u_{0,x} - n/4\ u_1,\ldots$ . Thus using the main property of the chain system (C) and putting $u_{n-1,x} = u_{0,y}$ we obtain the following corollary*:

Pole evolution for Zaharov-Shabat system $(L_{2,n})$ for each of the functions $u$, $u_k$ or $v_j$ coincides with the evolution of the particle system $a_i(y,t)$, where in $t$ direction evolution is according to $J_2 = H$ and in $y$ direction according to $J_n$.

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*For the first time the infinite chain system (C) together with the pole interpretation of meromorphic solutions and formulation of conservations laws was given in D. V. Chooduovsky, Infinite chains of non-linear equations of evolution associated with one-dimensional many body problems I. N.A.M.S. 24, no. 4 (1977), A-387.

Here for general $n$ the coupling constant $G$ for normalization condition must be chosen as

$$G = -4.$$

Consequence. If we consider the Lax system, i.e. $y$ or $t$ independent, then the evolution of poles is described by $J_2$ or $J_n$, respectively, with restrictions grad $J_n = 0$ or grad $J_2 = 0$, correspondingly.

For example let us consider a n-th order higher Kd V equation. This equation as usual [1], [5], [7] has the form

$$u_t = Q_n(u, u_x, \ldots, \underbrace{u_{x\ldots x}}_{2n+1}): \quad n = 1,2,\ldots.$$

This equation is equivalent to the Lax representation

$$\frac{dL_2}{dt} = [L_2, L_{2n+1}],$$

where $L_2 = \frac{d^2}{dx^2} + u$ and $L_{2n+1} = \frac{d^{2n+1}}{dx^{2n+1}} + v_{2n-1}\frac{d^{2n-1}}{dx^{2n-1}} + \ldots + v_0$,

then for $u = -2\sum_{i\in I}(x - a_i)^{-2}$ the evolution of poles $a_i = a_i(t)$ is according to $J_{2n+1}$ with restriction grad $J_2 = 0$ i.e

$$\sum_{j\neq i}(a_i - a_j)^{-3} = 0: \quad i \in I.$$

This was well known for $n = 1$ for ordinary KdV [12], [8] and also was proved in 1976 [12], [8] for a second KdV. Thus

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we confirm our general conjecture for the case min(m,n) = 2.

Mcreover infinite chain systems give us the possibility to examine two dimensional laws of conservation. What do we understand by two dimensional law of conservation? If we have one component two dimensional function u = u(x,y,t) then by law of conservation we understand

$$\frac{\partial}{\partial t} P(u) + \frac{\partial}{\partial x} R(u) = 0,$$

where P(u) and R(u) are polynomials in u, $u_x$,..., etc. But this is a one-dimensional conservation law and it is possible to introduce two dimensional laws of conservation:

$$\frac{\partial}{\partial t} P(u) + \frac{\partial}{\partial x} R(u) + \frac{\partial}{\partial y} T(u) = 0,$$

where now P(u), R(u), T(u) are polynomials in u, $u_x$, $u_{xx}$,.. u, $u_x$, $u_{xx}$,...,$u_y$,$u_{yy}$,.... Of course now triviality conditions are different. In the first case we assume that P(u) is not a full derivative in x , in the second case we assume that P(u) is not a full derivative in x,y.

Infinite chain (C) allows us to find for any n > 2 and any system $(L_{2,n})$ of two-dimensional equations infinitely many two-dimensional conservation laws. Unfortunately in this case we don't know what precisely Hamiltonian structure means and what is the condition of the involution of these

conservation laws.

There is another possibility to obtain two dimensional conservation laws for Kadomzev-Petviashvili and similar two-dimensional equation: this is a Backlung transformation arising from a spectral problem for $L_n$ and $L_m$.

In general in order to obtain two-dimensional equations with algebraic laws of conservation or with Backlung transformation it is unnecessary to suppose that two operators commute. As it was mentioned before it is quite enough that they have common eigenfunction corresponding to only one common eigenvalue. This gives us the possibility to investigate the cases when two operators have higher order derivatives on $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial y}$.

§3. In general it is impossible by analogy to generalize soliton theory from KdV equations to other infinite-dimensional completely integrable systems. First of all we have no solution of inverse scattering problem for operators of order more than 2 (unlike with Schrödinger). Secondly we still cannot solve the Gelfand-Levitan equation for this case. We do not know exactly multisoliton solutions and we don't know how they interact. Our equations as usual have the form

$$\frac{dL_n}{dt} - \frac{dL_m}{dy} = [L_n, L_m]$$

or, for the one dimensional case

$$\frac{dL_n}{dt} = [L_n, L_m].$$

In this case we know that for $m$ and $n$ non relatively prime, the solution in general cannot be written down through some $\theta$-function of an algebraic curve. In this case multisoliton behavior is also different from those of KdV. E.g. for $n = 4$, $m = 2$ there can appear "gaizeron" and multisolitons with non-elastic interaction (connected solitons). However in any case it is possible to find explicitly all meromorphic (rational, etc) solutions of one and two-dimensional systems. This is based on the fact that for the general system $(L_{n,m})$ the evolution of poles $a_i = a_i(t)$ is goverened on $y$ by $J_m$ and on $t$ by $J_n$. This result is proved for any meromorphic solution, when $\max(n,m) = 2$. So in this case we have extremely simple formula for meromorphic solutions. Corresponding solutions can be called solitons because they are of course asymptotically stable and naturally generic in the class of asymptotical states. For the case $\min(m,n) > 2$ we have proved that for any many particle system $a_i = a_i(t)$ goverened in $y$ by $J_n$ in $t$ by $J_m$ we

get meromorphic solutions of $(L_{n,m})$ with poles $a_i = a_i(t)$. However for $\min(m,n) \geq 3$ and $\max(m,n) \geq 5$ there arise meromorphic solutions with different behavior of poles corresponding to another completely integrable many particle system.

If we consider operators in the form

$$L_2 = \frac{d^2}{dx^2} + u, \quad L_4 = \frac{d^4}{dx^4} + u_2 \frac{d^2}{dx^2} + u_1 \frac{d}{dx} + u_0$$

then the equation

$$\frac{dL_2}{dy} - \frac{dL_4}{dt} = [L_2, L_4]$$

is equivalent to a system of equations for $u_0$, $u_1$, $u_2$ in x,y,t. We'll write this equation in a more convenient form. For

$$u_1 = u_{2x} + \frac{\bar{u}_1}{2},$$

$$u_0 = \frac{5}{12} u_{2xx} + \frac{\bar{u}_{1x}}{4} + \frac{1}{8} u_2^2 + \frac{\bar{u}_0}{4}$$

we have the system

$$\begin{cases} u_{2t} = -\bar{u}_{1x} \\ \bar{u}_{1t} = \bar{u}_{0x} + \frac{1}{3} u_{2xxx} + u_2 u_{2x} \\ \bar{u}_{0t} - 2u_{2y} = \frac{2}{3} \bar{u}_{1xxx} + \bar{u}_{1x} u_2 + \bar{u}_1 u_{2x}. \end{cases}$$

From an analytical point of view this system of equations possesses excellent meromorphic solutions. However for real $x$ the shape of the corresponding soliton traveling wave solution differs completely from those of KdV. For example, general soliton solutions consist of connected pieces with different amplitude, but the same velocity. We present examples Manin [16] of one-soliton solution that can be called "geizeron". This solution corresponds to the usual one dimensional case

$$\mu_1 = 2a^2 \frac{\text{ch } 2ax - \cos 2ax - 2 \sin 4a^2t}{\text{ch } 2ax + \cos 2ax + 2 \cos 4a^2t}$$

$$\mu_2 = -2a \frac{\text{sh } 2ax - \sin 2ax}{\text{ch } 2ax + \cos 2ax + 2 \cos 4a^2t}$$

$$u_2 = -4\mu_{2x}, \; u_1 = -6\mu_{2xx} - 4\mu_{1x} + 4\mu_2\mu_{2x};$$

$$u_0 = -4\mu_{2xxx} - 6\mu_{1xx} + 8(\mu_{2x})^2 + 4(\mu_1\mu_2)_x + 6\mu_2\mu_{2xx} - 4\mu_{2x}\mu_2^2.$$

This solution:

a) decreases on infinity

b) is periodic on $t$ with period $\pi/2a^2$, but

c) when $\cos 4a^2t = -1$ soliton gives explosion at the origin $x = 0$.

However it is possible to write down the expression for multisoliton and general meromorphic solutions of our system. These solutions in general have the form

$$\bar{u}_2 = -4 \sum_{i \in I} \wp(x - a_i);$$

$$\bar{u}_1 = -4 \sum_{i \in I} a_{it} \wp(x - a_i)$$

$$\bar{u}_0 = -4 \sum_{i \in I} (a_{it}^2 + G \sum_{j \neq i} \wp(a_i - a_j))\wp(x - a_i)$$

for $G = -4$. Here poles $a_i$ move in $y$ according to $J_4$ and in $t$ according to $J_2$, for $G = -4$. In particular

$$a_{itt} = 4 \sum_{j \neq i} \wp'(a_i - a_j): \; i \in I.$$

§4. Completely integrable systems associated with a linear operator of third order.

4.1. We have already considered the case $\min(n,m) = 2$ and now we can come to the case $\min(n,m) = 3$. This case displays some very interesting particular features.

A. First of all any Lax equation

$$\frac{dL_m}{dt} = [L_n, L_m]$$

or Zaharov-Shabat equation

$$\frac{\partial L_m}{\partial t} - \frac{\partial L_n}{\partial y} = [L_n, L_m]$$

is the system of non-linear equations for

$$\min\{m - 1,\ n - 1\}$$

coefficients of $L_m$ or $L_n$ depending on $m < n$ or $n < m$ respectively. Here we assume at least that $n$ and $m$ do not contain another number as a factor. Thus for $\min(m,n) = 2$ we have a non-linear equation only for one function $u$. For the Lax system and $m = 2$ these are simply higher KdV equations. For $m > 2 = n$ we obtain Lax systems resembling a Bonsinesq equation but not having evolutionary form. So a question arises whether it is possible to find some Lax system with

$\min(n,m) > 2$ and having evolutionary form

$$u_t = Q'(u,u_x,\dots)$$

for Q'-polynomial in $u$ and derivatives in $x$.

This must correspond to the situation where $m < n$ and among $n - m$ coefficients of $L_n$ there is only one independent. In this direction there was found in 1974 by Sawada and Kotera (Prog. Theor. Phys. 51 no 5(1974)) and then by Caudrey, Dodd and Gibbon (Proc. R.S. LOND A351 (1976)) a new evolutionary equation of 5th order, which possesses n-soliton solutions. Later it was shown by Satsuma and Kaup (J. Phys. Soc. Japan v 43 (1977) no. 2) and Dodd, Gibbon (Proc. R. Soc. LOND A358 (1977), 287-296) that this equation in fact corresponds to the Lax system with $m = 3$, $n = 5$ where $L_3 = \frac{d^3}{dx^3} + u$ and the equation (second type KdV equation) has the form (up to scalar transformation)

$$(E_1) \qquad u_t + 45u^2u_x + 15(u_xu_{xx} + uu_{xxx}) + u_{xxxxx} = 0$$

at the time as the usual second KdV equation has the form

$$u_t + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x + u_{xxxxx} = 0.$$

In our paper (N. Cim. v 40B(1977)no 2) it was given precise pole interpretation for equation $(E_1)$ and it was

mentioned that pole interpretation here leads to many new particle problems. Here we'll give general descriptions of the two-dimensional Zaharov-Shabat generalization of $(E_1)$ and the precise form of a new many particle problem on the line having additional first integrals. For this we consider Zaharov-Shabat systems for $m = 3$, $n = 4,5$.

The possibility of finding a Lax system with $2 < m < n$ and with only one nonzero coefficient in $L_m$ at least for $m = 3$, $n = 5$ gives us hope that it will be possible to construct a really new evolutionary of first order in $t$ equations.

Pole interpretation suggests the following interesting conjecture.

Conjecture: There are completely integrable, in the sense of multi-soliton solutions and algebraic conservation laws, evolutionary equations

$$u_t = Q_n(u, u_x, \ldots, \underbrace{u_{x\ldots x}}_{2n+1})$$

for $Q_n$ polynomial in $u, u_x, \ldots, \underbrace{u_{x\ldots x}}_{2n+1}$ of weight $2n + 3$, where the number of non-equivalent up to scalar transformation equations for given $n$ is not less than $n$ (equal $n$).

Here by weight we understand that $u$ has weight 2 and each space derivative adds weight one.

A KdV equation gives one example and a second type KdV equation gives the second for $n \geq 2$.

4.2. We consider the Za harov-Shabat system in the first non-trivial case, when $\min\{n,m\} > 2$. This means that we consider the case $n = 3$, $m = 4$ and

$$(L_{n,m}) \qquad \frac{dL_m}{dt} - \frac{dL_n}{dy} = [L_n, L_m]$$

for operators $L_3$ and $L_4$ of orders 3 and 4, respectively. If we denote

$$(4.1) \qquad L_3 = \frac{d^3}{dx^3} + u_1 \frac{d}{dx} + u_0,$$

$$(4.2) \qquad L_4 = \frac{d^4}{dx^4} + v_2 \frac{d^2}{dx^2} + v_1 \frac{d}{dx} + v_0.$$

Then the two-dimensional equation

$$(4.3) \qquad \frac{dL_3}{dt} - \frac{dL_4}{dy} = [L_3, L_4]$$

is equivalent to the following system of equations for $u_i, v_j$: $i = 0,1$, $j = 0,1,2$:

$$(4.4) \qquad \begin{cases} 3v_2' - 4u_1' = 0; \\ 3v_2'' + 3v_1' - 6u_1'' - 4u_0' = 0; \end{cases}$$

(4.4 cont.)
$$\begin{cases} v_2''' + 3v_1'' + 3v_0' + u_1 v_2' - 4u_1''' - 6u_0'' \\ \quad - 2v_2 u_1' = v_{2t}; \\ v_1''' + 3v_0'' + u_1 v_1' - u_1^{IV} - 4u_0''' - v_2 u_1'' \\ \quad - 2v_2 u_0' - v_1 u_1' = v_{1t} - u_{1y}; \\ v_0''' + u_1 v_0' - u_0^{IV} - v_2 u_0'' - v_1 u_0' = v_{0t} - u_{0y}. \end{cases}$$

Now it is possible to exclude $v_0, v_1, v_2$:

(4.5)
$$\begin{cases} v_2 = \frac{4}{3} u_1; \\ v_1 = \frac{2}{3} u_1' + \frac{4}{3} u_0; \\ v_0' = \frac{4}{9} u_{1t} + \frac{2}{9} u_1''' + \frac{2}{3} u_0'' + \frac{4}{9} u_1 u_1'. \end{cases}$$

Here we omit possible constants, appearing in (4.5) after solution (4.4) in general, since these constants are absolutely not important.

We define

(4.6)
$$\bar{u}_0 = u_0 - \frac{u_1'}{2}.$$

Then, taking into accoutn (4.4), (4.5) and (4.6) we obtain the following simple system of equations for $\bar{u}_0, u_1$:

(4.7)
$$-\frac{2}{3}\bar{u}_0''' - \frac{4}{3}(\bar{u}_0 u_1)' = \frac{4}{3}\cdot\bar{u}_{0t} - u_{1y};$$

$$(4.8)\quad \frac{1}{18}u_1^{V} - \frac{1}{3}u_0^{IV} + \frac{4}{27}(u_1^3)' - \frac{2}{3}(\bar{u}_0^2)' -$$

$$- \frac{2}{3}(\bar{u}_0 u_1)'' - \frac{1}{9}u''_{1t} + \frac{1}{6}(u_1'^2)' + \frac{1}{3}(u_1 u_1'')'$$

$$= \frac{4}{9}\int u_{1tt}dx + \frac{2}{3}\bar{u}'_{0t} - \bar{u}_{Cy} - \frac{u'_{1y}}{2}.$$

Taking into account (4.7) into (4.8) we simplify (4.7)-(4.8) to such system

$$(4.9)\quad \text{or}\quad (\bar{\mathcal{L}}_{3,4})\quad \begin{cases} -\frac{2}{3}\bar{u}_0''' - \frac{4}{3}(\bar{u}_0 u_1)' = \frac{4}{3}\bar{u}_{0t} - u_{1y}; \\ \frac{1}{18}u_1^{V} + \frac{4}{27}(u_1^3)' - \frac{2}{3}(\bar{u}_0^2)' - \frac{1}{9}u''_{1t} + \frac{1}{6}(u_1'^2)' \\ \qquad + \frac{1}{3}(u_1 u_1'')' - \frac{4}{9}\int u_{1tt}dx + \bar{u}_{0y} = 0. \end{cases}$$

Now we find the class of meromorphic solutions of $(\bar{\mathcal{L}}_{3,4})$. The form of these solutions is the following

$$(4.10)\qquad u_1 = -3\sum_{i\in I}(x - a_i)^{-2}$$

and

$$(4.11)\qquad \bar{u}_0 = \sum_{i\in I} e_i (x - a_i)^{-2}$$

or for an arbitrary Weirstrass elliptic function $\wp(z)$:

$$(4.12)\qquad \begin{cases} u_1 = -3\sum_{i\in I}\wp(x - z_i); \\ \bar{u}_0 = \sum_{i\in I} e_i\,\wp(x - a_i). \end{cases}$$

Then the system of equations for $a_i, e_i$: $i \in I$ equivalent to $(\bar{\mathcal{L}}_{3,4})$ is in fact the system, induced by $J_3$ and $J_4$. We have for the cannonical variables $(a_i, b_i)$: $i \in I$ and two Hamiltonian flows

$$(4.13) \qquad \overline{J_3} = -3 \cdot \{\frac{1}{3} \sum_{i \in I} b_i^3 - \sum_{i \neq j} (b_i + b_j) \varphi(a_i - a_j)\};$$

$$(4.14) \qquad \overline{J_4} = 4\{\frac{1}{4} \sum_{i \in I} b_i^4 - \sum_{i \neq j} (b_i^2 + b_i b_j + b_j^2) \varphi(a_i - a_j)$$
$$+ \sum_i \sum_{\{j,k\} \not\ni i} \varphi(a_i - a_j) \varphi(a_i - a_k)$$
$$+ \frac{1}{2} \sum_{i \neq j} \varphi^2(a_i - a_j)\},$$

in other words in our traditional notations

$$(4.15) \qquad \begin{cases} \overline{J_3} = (-3) J_3; \quad \overline{J_4} = 4 J_4 \text{ and} \\ \\ G = -1. \end{cases}$$

Then, for $\bar{u}_0, u_1$ from (4.12) the satisfiability of (4.9)-$(\bar{\mathcal{L}}_{3,4})$ is equivalent to the fact that $a_i$ are governed on t according to $\overline{J_3}$ and on y according to $\overline{J_4}$:

Lemma 1: For

$$(4.12) \qquad u_1 = -3 \sum_{i \in I} \varphi(x - a_i),$$
$$u_0 = \sum_{i \in I} e_i \varphi(x - a_i),$$

the system $(\bar{\mathcal{L}}_{3,4})$ is equivalent to

(4.16) $$e_i = 3b_i$$

and

$$a_{it} = \frac{\partial \overline{J_3}}{\partial b_i}, \quad b_{it} = -\frac{\partial \overline{J_3}}{\partial a_i};$$

(4.17)

$$a_{iy} = \frac{\partial \overline{J_4}}{\partial b_i}, \quad b_{ij} = -\frac{\partial \overline{J_4}}{\partial a_i}: \quad i \in I.$$

This system (4.16)-(4.17) is, of course, consistent and in terms of $a_i$ and $e_i$ can be written in precise form for $\wp(x) = x^{-2}$

$$a_{it} = -\frac{1}{3}e_i^2 + 3\sum_{j\neq i}(a_i - a_j)^{-2};$$

$$a_{iy} = \frac{4}{27}e_i^3 - \frac{4}{3}\sum_{j\neq i}(2e_i + e_j)(a_i - a_j)^{-2};$$

$$e_{it} = 6\sum_{j\neq i}(e_i + e_j)(a_i - a_j)^{-3}: \quad i \in I$$

and for arbitrary $\wp(x)$, $(a_i - a_j)^{-2}$ must be substituted for $\wp(a_i - a_j)$ and $(a_i - a_j)^{-3}$ for $\wp'(a_i - a_j)/(-2)$.

In other words, pole interpretation for $n = 3$, $m = 4$ is consistent with our conjecture on the connection of meromorphic solutions with many-particle Hamiltonian $J_n$: $n \geq 2$.

4.3. We consider the case m = 3, n = 5. Then the equations

$$(L_{3,5}) \qquad \frac{\partial L_5}{\partial y} - \frac{\partial L_3}{\partial t} = [L_3, L_5]$$

for $L_3 = \frac{\partial^3}{\partial x^3} + u_1 \frac{\partial}{\partial x} + u_0$, $L_5 = \frac{\partial^5}{\partial x^5} + v_3 \frac{\partial^3}{\partial x^3} + v_2 \frac{\partial^2}{\partial x^2} + v_1 \frac{\partial}{\partial x} + v_0$

have the following form

$$(L_{3,5}) \quad \begin{cases} 3v_3' - 5u_1' = 0 \\ 3v_3'' + 3v_2' - 10u_1'' - 5u_0' = 0; \\ v_3''' + 3v_2'' + 3v_1' + u_1 v_3' - 10u_1''' - 10u_0'' - 3u_1' v_3 = v_{3y}; \\ v_2''' + 3v_1'' + 3v_0' + u_1 v_2' - 5u_1^{IV} - 10u_0''' - 3u_1'' v_3 - \\ \quad - 3u_0' v_3 - 2u_1' v_2 = v_{2y}; \\ v_1''' + 3v_0'' + u_1 v_1' - u_1^{V} - 5u_0^{IV} - u_1''' v_3 - 3u_0'' v_3 - \\ \quad - 2u_0' v_2 - u_1'' v_2 - u_1' v_1 = v_{iy} - u_{1t}; \\ v_0''' + u_1 v_0' - u_0^{V} - u_0''' v_3 - u_0'' v_2 - u_0' v_1 = v_{0y} - u_{0t}. \end{cases}$$

Here we find an extremely important fact: the conditions

$$(R) \qquad u_0 = 0, \quad v_0 = 0$$

gives us an invariant submanifold for $(L_{3,5})$. Then

$$v_3 = \frac{5}{3} u_1;$$

$$v_2 = \frac{5}{3} u_1';$$

$$v_1 = \frac{10}{9} u_1'' + \frac{5}{9} u_1^2 + \frac{5}{9} \int u_{1y} dx.$$

Then instead of $(L_{3,5})$ under the restrictions (R) we have the following two-dimensional equation

$$(\Lambda) \qquad \frac{d}{dx}(u_{1xxxx} + 5u_1 u_{1xx} + \frac{5}{3} u_1^3) =$$

$$= 5u_{1xxy} + 5 \int u_{1yy} dx + 5 \frac{d}{dx}(u_1 \int u_{1y} dx) - 9u_{1t}.$$

This system is equivalent to

$$(\Lambda) \qquad \frac{dL_5'}{dy} - \frac{dL_3'}{dt} = [L_3', L_5']$$

for

$$L_3' = \frac{d^3}{dx^3} + u_1 \frac{d}{dx} .$$

For the y-independent case we obtain a simply Lax system equivalent to $(E_1)$-second type KdV equation.

In fact equation $(\Lambda)$ possesses not only multisoliton solutions but also excellent general elliptic and arbitrary meromorphic solutions.

Lemma: If $u_1 = u_1(x,y,t)$ is the meromorphic solution of $(\Lambda)$ for $(y,t) \in [y_0,y_1] \times [t_0,t_1]$, then

$$u_1 = \sum_{i\in I} (-6)\wp(x - a_i)$$

for some $\wp(x)$, at least for $\wp(x) = x^{-2}$. Here the function

$$u_1 = \sum_{i\in I} (-6)\wp(x - a_i)$$

satisfy $(\Lambda)$ iff

$$(\Lambda_1) \qquad a_{iyy} = -\sum_{j\neq i} 6(a_{iy} + a_{jy})\wp'(a_i - a_j)$$

$$+ 72 \sum_{\{j,k\}\not\ni i} (\wp'(a_i - a_j)\wp(a_i - a_k)$$

$$+ \wp'(a_i - a_k)\wp(a_i - a_j): \quad i \in I;$$

$$(\Lambda_2) \qquad a_{it} = -\frac{5}{9}a_{iy}^2 + 40 \sum_{\{j,k\}\not\ni i} \wp(a_i - a_j)\wp(a_i - a_k)$$

$$+ \frac{10}{3} \sum_{j\neq i} (a_{iy} + a_{jy})\wp(a_i - a_j): \quad i \in I.$$

This system is consistent and $(\Lambda_1)$ gives new many-particle problems having many invariant submanifolds.

We must mention that in the y-independent case the system $(\Lambda_1)$ - $(\Lambda_2)$ was written in [12] as pole interpretation of equation $(E_1)$. The new system $(\Lambda_1)$ corresponds to many-particle problems not governed by any Hamiltonian $J_n$.

Appendix to §4.

There exists one two-dimensional system of equations, having physical sense and having Hamiltonian form and two-dimensional laws of conservations. This is the so-called theory of long waves in shallow water: Benney equations:

$$(\text{Sh } W_1) \quad \begin{cases} u_t + uu_x - u_y \int_0^y u_x|_{y=\eta}\, d\eta + h_x = 0 \\ \cdot \qquad h_t + \left(\int_0^h u dy\right)_x = 0 \end{cases}$$

x-horizontal, $y \geq 0$-vertical coordinates, $u(x,y,t)$-horizontal component of velocity at $(x,y)$; $h(x,t)$-the height of a free surface over $(x,0)$ at $t$.

For $A_n(x,t) = \int_0^h u^n(x,y,t)dy$, this system is equivalent to

$$A_{nt} + A_{n+1x} + nA_{n-1}A_{0x} = 0: \quad n \geq 0.$$

According to Benney D. J. [Studies App. Math., Lll, No 1, (1973), 45-50] and Miura R.M. [Studies App. Math. LIII, No 1, (1974), 45-56] there are conservation laws for this system:

$$H_{n,t} + F_{n,x} = 0: \quad n \geq 0;$$

$$H_n \in Q[A_0,\ldots,A_n]; \quad F_n \in Q[A_0,\ldots,A_{n+1}]$$

and

$$\bar{H}_{n,t} + \bar{F}_{n,x} + (\bar{H}_n v)_y = 0: \quad n \geq 1, \; v = -\int_0^y u_x d\eta.$$

Many recurrent formulae for $H_n$, $\bar{H}_n$ were given by Manin Y.M., Kuperschmidt B. A. (Funct. Anal. Appl. v.11, No3, (1977)]. Simultaniously Manin-Kuperschmidt established a Hamiltonian structure for this equation for the y-independent case. Then we have the Hamiltonian form

$$(\mathrm{Sh}\ W_2) \qquad \begin{cases} u_t = -(u^2/2 + h)_x, \\ h_t = -(uh)_x \end{cases}$$

or the Hamiltonian system with skewsymmetric operator

$B = \begin{pmatrix} 0 & \partial/\partial x \\ \partial/\partial x & 0 \end{pmatrix}$ and $H = -(h^2/2 + hu^2/2)$.

However we must add the following observation, made first by G. B. Whitham [Linear and Nonlinear Waves, 1974]. The system of equations (Sh $W_2$) is linearized by the hodograph transformation

$$u \longrightarrow t$$

$$h \longrightarrow x$$

$$x_u = \frac{-h_t}{J}, \quad x_h = \frac{u_t}{J}, \quad t_u = \frac{h_x}{J}, \quad t_h = -\frac{u_x}{J},$$

$$j = u_t h_x - h_t u_x.$$

Analytical solutions for (Sh $W_2$) or for (Sh $W_1$) in an $A_n$ representation can also be obtained by using the chain system (C) if we put in this system $G \to 0$.

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