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On a stochastic Korteweg-de Vries equation with homogeneous noise

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1 Introduction and motivations

Our aim here is an attempt to describe the dynamical properties of the solutions of the Korteweg-de Vries (KdV) equation in the presence of a certain type of random perturbations, depending on the space and time variables.

It is indeed well known that the KdV equation

$$\partial_t u + \partial_x^3 u + \partial_x(u^2) = 0 \tag{1.1}$$

describes the motion of unidirectional weakly nonlinear waves at the surface of water under some specific scaling conditions on the surface waves, relating their wavelength, their amplitude, and the depth of the fluid. A rigorous derivation of this asymptotic equation starting from the free surface Euler equations for the fluid can be found in [8]. In this physical context, there are essentially two kinds of situations that could asymptotically lead to a KdV equation with noise, even though, up to know, there is no rigorous derivation of such an asymptotic model starting from the full water wave problem. The first situation is the case in which the pressure field at the surface of the water is non homogeneous, and modeled by a stationary space-time process with small correlation length compared to the wavelength of the surface waves. In this case, one could expect to get an asymptotic model of the form of a KdV equation with a random forcing term, white in time and possibly also in space. The second situation is the case where the bottom topography is non homogeneous, and again modeled with the use of a stationary random process in the space variable. Among the large literature dealing with modeling of the water wave problem with non constant bottom topography, the case when the bottom is modeled by a stationary ergodic process with small correlation length (compared to the surface waves) has been studied in [2], [10] and [14]. It essentially leads to a KdV equation with an additional term of the form $(\partial_x u)\dot{\eta}(\omega,t)$, where $\dot{\eta}$ is a white noise in time that does not have any spatial dependence. In this case, the solution of the perturbed equation may be simply written in terms of the solution of the deterministic equation, in a frame moving with velocity $\dot{\eta}$, that is it has the simple form $u(t, x-\eta(t))$, with u a solution of (1.1).

The existence of solutions to equation (1.1) perturbed with a stochastic forcing, close to space-time white noise is much less obvious. The question was studied in [5] in the case $x \in \mathbb{R}$ and in [6] in the spatially periodic case. However when the noise is close to space-time white noise the solutions are not spatially regular, they are not even processes of the space variable x and the question of their qualitative behavior is still widely open. When the noise has sufficiently regular spatial correlations, however, the solutions lives in the energy space, which may help in studying their dynamical behavior. This fact has been used e.g in [3] where the exit time of a neighborhood of a randomly modulated soliton was studied.

Let us mention some related works using methods linked to integrability properties of equation (1.1). This is the case of [9] where inverse scattering methods are used to describe the qualitative behavior of the solution of the perturbed equation with some specific (multiplicative) noise, and of [11] where the KdV-Burgers equation is viewed, as the viscosity and the amplitude of the noise both tend to zero, as a small perturbation of an infinite dimensional Hamiltonian system, in action-angle variables.

Finally, let us recall the pioneering remark by Wadati [17], from which our motivation to study the present problem originated. Consider the following perturbed KdV equation

$$\partial_t u + \partial_x^3 u + \partial_x(u^2) = \dot{W}(t) \tag{1.2}$$

in which $\dot{W}(t)$ is a one-dimensional white noise, that is the time derivative of a real valued standard Brownian motion. Then, using the Galilean invariance of the KdV equation, it is easy to compute the solution u of equation (1.2) in terms of the solution U of equation (1.1) :

$$u(t,x) = U\left(t, x - \int_0^t W(s)ds\right) + W(t).$$

In the particular case where the solution U is a soliton solution of the KdV equation, which we recall is given by $U(t,x) = \varphi_c(x-ct)$ with the localized profile

$$\varphi_c(x) = \frac{3c}{2\cosh^2\left(\sqrt{c}x/2\right)}, \tag{1.3}$$

the expression above allows to get the asymptotic behavior in time of the spatial maximum of the expected solution. Indeed,

$$\mathbb{E}(u(t,x)) = \mathbb{E}(\varphi_c(x - ct - \int_0^t W(s)ds)) = \int_{\mathbb{R}} \varphi_c(x-ct-y)\mu_t(dy)$$

and since μ_t, the distribution of the (real valued) random variable $\int_0^t W(s)ds$, is a centered Gaussian distribution with variance $t^3/3$, one gets

$$\mathbb{E}(u(t,x)) = \frac{\sqrt{3}}{\sqrt{2\pi t^3}} \int_{\mathbb{R}} \varphi_c(x-ct-y) e^{-\frac{3y^2}{2t^3}} dy$$

from which we deduce, thanks to the localization of the profile φ_c, that as time goes to infinity,

$$\max_{x\in\mathbb{R}} \mathbb{E}(u(t,x)) \leq Ct^{-3/2},$$

i.e. the soliton diffuses with a rate $t^{-3/2}$.

The question was then : is it possible to recover the same kind of results in the case of a perturbed equation of the form (1.2), but in which the noise depends both on the space and time variables ? Of course, in that case no explicit expression of the solution is available in terms of the solution of the unperturbed equation, and one can only hope to get results on the soliton diffusion in the limit where the amplitude of the noise tends to zero. A first attempt to study the problem was the motivation of [3], where the KdV equation with an additive space-time noise was studied. We failed to get an asymptotic rate of diffusion in that case, but it appears that homogeneity (or stationarity) in space of the noise can help to get such a result. However, spatial stationarity and additivity of the noise are incompatible as long as the solution lives in an homogeneous Sobolev space, as the energy space $H^1(\mathbb{R})$ for the KdV equation (see [5] for an explanation of this fact). This is the reason why we will here consider a multiplicative equation.

2 The multiplicative homogeneous noise

The equation we consider may be written in Itô form as

$$du + (\partial_x^3 u + \partial_x(u^2))dt = \varepsilon u dW \tag{2.1}$$

where $\varepsilon > 0$, u is a random process defined on $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, W is an infinite dimensional Wiener process on $L^2(\mathbb{R})$ with covariance $Q = \phi\phi^*$, ϕ being a convolution operator on $L^2(\mathbb{R})$ defined by

$$\phi f(x) = \int_{\mathbb{R}} k(x - y) f(y) dy, \quad f \in L^2(\mathbb{R}). \tag{2.2}$$

We will assume that the convolution kernel k is in $H^1(\mathbb{R}) \cap L^1(\mathbb{R})$, so that the solutions will almost surely have finite energy (see Theorem (2.1) below). Considering a complete orthonormal system $(e_i)_{i\in\mathbb{N}}$ in $L^2(\mathbb{R})$, we may alternatively write W as

$$W(t, x) = \sum_{i\in\mathbb{N}} \beta_i(t) \phi e_i(x),$$

$(\beta_i)_{i\in\mathbb{N}}$ being a family of independent real valued Brownian motions. Note that the correlation function of the process W is given by

$$\mathbb{E}(W(t, x)W(s, y)) = c(x - y)(s \wedge t), \quad x, y \in \mathbb{R}, \quad s, t > 0,$$

where

$$c(x) = \int_{\mathbb{R}} k(x + z)k(z)dz.$$

Let us recall the existence and uniqueness result for solutions of (2.1) which was proved in [4].

Theorem 2.1 *Let $\varepsilon > 0$; assume that the kernel k of the noise satisfies $k \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$, $s = 0$ or 1. Then for any u_0 in $H^s(\mathbb{R})$, there is a unique adapted solution u^ε with paths almost surely in $C(\mathbb{R}^+; H^s(\mathbb{R}))$ of equation (2.1), satisfying $u^\varepsilon(0, x) = u_0(x)$ a.s. Moreover, $u \in L^2(\Omega; C(\mathbb{R}^+; L^2(\mathbb{R})))$.*

Note that the paths of the solution are not sufficiently regular in time to allow us to define pathwise the product of $\partial_t W$ by u^ε : typically, for a fixed x, $u^\varepsilon(.,x)$ belongs to H^s for any $s < 1/2$, while $\partial_t W(.,x)$ belongs to H^s for any $s < -1/2$; hence the use of the stochastic calculus is necessary to define this product. Here, as mentioned above, we use the Itô definition for the product.

The local existence result in Theorem 2.1 is obtained with the use of Bourgain's type spaces, based on space-time Sobolev spaces and on the group associated with the linear (Airy) equation

$$\partial_t u + \partial_x^3 u = 0.$$

The time regularity used for these spaces is however not the standard one : it is less than 1/2, as imposed by the time regularity of the Brownian motion. In order to get the globalization of the solutions in time, we use the evolution of the mass

$$m(u) = \frac{1}{2} \int_{\mathbb{R}} (u(x))^2 dx \tag{2.3}$$

and of the energy (or Hamiltonian)

$$H(u) = \frac{1}{2} \int_{\mathbb{R}} (\partial_x u)^2 dx - \frac{1}{3} \int_{\mathbb{R}} u^3 dx, \tag{2.4}$$

which are both conserved quantities for the deterministic equation (1.1). Their evolution for the solutions of the stochastic equation (2.1) is obtained thanks to the Itô formula.

3 Dynamics of the stochastic equation

From now on we fix $c_0 > 0$ and we take as the initial state $u^\varepsilon(0,x) = \varphi_{c_0}(x)$. Let us recall that the soliton family is a two parameter family of solutions given by $\{\varphi_c(. + x_0),\ c > 0,\ x_0 \in \mathbb{R}\}$. Our aim is then to describe the main part of the corresponding solution of the stochastic equation as a soliton whose parameters – the velocity c and the phase x_0 – may have been shifted.

In order to explain how we must then write the solution, we have to recall briefly the arguments of the proof of orbital stability for φ_{c_0} in the deterministic equation, which were originally given by Benjamin in [1]. The proof relies on the use of the functional $Q_{c_0}(u) = H(u) + c_0 m(u)$, defined for

$u \in H^1(\mathbb{R})$, as a Lyapunov functional. Note that φ_{c_0} is a critical point of Q_{c_0}, i.e. it is a stationary solution of the evolution equation written in the frame moving with velocity c_0. Then it is not difficult to see that

$$L_{c_0} := Q''_{c_0}(\varphi_{c_0}) = -\partial_x^2 + c_0 - 2\varphi_{c_0}$$

is a positive operator on $L^2(\mathbb{R})$ (with domain $H^2(\mathbb{R})$) when restricted to the orthogonal in $L^2(\mathbb{R})$ of the space spanned by φ_{c_0} and $\partial_x\varphi_{c_0}$. Writing then the solution of the deterministic equation (1.1) (with initial state u_0 close to φ_{c_0}) as

$$u(t, x + x(t)) = \varphi_{c_0}(x) + \eta(t, x)$$

with $x(t)$ chosen in such a way that $(\eta, \partial_x\varphi_{c_0}) = 0$, then one gets

$$\begin{aligned} &Q_{c_0}(u(t, x + x(t))) - Q_{c_0}(\varphi_{c_0}) \\ &= Q_{c_0}(u_0) - Q_{c_0}(\varphi_{c_0}) \\ &= \tfrac{1}{2}(L_{c_0}\eta, \eta) + o(\|\eta\|^2_{H^1}) \end{aligned}$$

from which the bound $\delta\|\eta\|^2_{H^1} \leq Q_{c_0}(u_0) - Q_{c_0}(\varphi_{c_0})$ would follow, with a positive constant δ, provided that η is initially sufficiently small, if one had in addition the orthogonality relation $(\eta, \varphi_{c_0}) = 0$. This relation is not satisfied in general, but the conservation of m for the solution u of equation (1.1) allows to prove that (η, φ_{c_0}) is a second order term in $\|\eta\|_{L^2}$, and hence to get the above estimate on η which leads to the orbital stability of φ_{c_0}.

Let us now come back to the solution of the stochastic equation (2.1) with initial state φ_{c_0}, and let us write the corresponding solution as

$$u^\varepsilon(t, x) = \varphi_{c^\varepsilon(t)}(x - x^\varepsilon(t)) + \varepsilon\eta^\varepsilon(t, x - x^\varepsilon(t)), \tag{3.1}$$

with the aim of keeping the remaining term $\varepsilon\eta^\varepsilon$ small as long as possible. In view of the above arguments it is natural to require

$$(\eta^\varepsilon(t), \varphi_{c_0}) = (\eta^\varepsilon(t), \partial_x\varphi_{c_0}) = 0. \tag{3.2}$$

If α is a parameter that measures the smallness of the remaining term $\varepsilon\eta^\varepsilon(t)$, then we can prove the following estimate on the stopping time τ^ε_α up to which the decomposition (3.1)-(3.2) is valid, showing that this stopping time is of the order of ε^{-2}.

Theorem 3.1 *Let u^ε be, as defined above, the solution of (2.1) with $u^\varepsilon(0) = \varphi_{c_0}$, and let $\alpha > 0$ be sufficiently small. Then there are random processes $x^\varepsilon(t)$ and $c^\varepsilon(t)$ with values resp. in $\mathbb{R}$ and $\mathbb{R}^+$, which are semi-martingales (i.e. solutions of stochastic differential equations), defined a.s. for $t \leq \tau_\alpha^\varepsilon$, where τ_α^ε, is a stopping time, and such that (3.1) and (3.2) hold a.s. for $t \leq \tau_\alpha^\varepsilon$. Moreover, a.s. for $t \leq \tau_\alpha^\varepsilon$, $|\varepsilon\eta^\varepsilon(t)|_{H^1(\mathbb{R})} \leq \alpha$ and $|c^\varepsilon(t) - c_0| \leq \alpha$.*

In addition, there is a constant $C_\alpha > 0$, such that for any $T > 0$, there exists $\varepsilon_0 > 0$, with, for each $\varepsilon < \varepsilon_0$,

$$\mathbb{P}(\tau_\alpha^\varepsilon \leq T) \leq \exp\left(-\frac{C(\alpha)}{\varepsilon^2 T \|k\|_{H^1}^2}\right). \tag{3.3}$$

Note that in [3], the same result was obtained for the KdV equation with an additive noise, but the bound on the probability of the exit time was of the form $\mathbb{P}(\tau_\alpha^\varepsilon \leq T) \leq C_\alpha \varepsilon^2 T$. The exponential bound, in the additive case, was obtained in [7], thanks to the use of exponential martingale estimates. The same method works in the multiplicative case of equation (2.1).

Note also that the orthogonality conditions given in (3.2) are not the only possible way of keeping the remaining term $\varepsilon\eta^\varepsilon(t)$ small as long as possible, although they are the most convenient to estimate the exit time τ_α^ε (see the argument on the orbital stability at the beginning of section 3).

4 A central limit theorem

Let us come back to the deterministic equation (1.1) and write the solution u (with initial data u_0 close to φ_{c_0} in $H^1(\mathbb{R})$) as

$$u(t,x) = \varphi_{c_0}(x - c_0 t) + v(t, x - c_0 t).$$

Then the linearized equation for v is

$$\partial_t v = \partial_x L_{c_0} v, \text{ with } L_{c_0} = Q''_{c_0}(\varphi_{c_0}). \tag{4.1}$$

It is well known that the spectrum of the operator $\partial_x L_{c_0}$ in $L^2(\mathbb{R})$ (with domain $H^3(\mathbb{R})$) is entirely located on the imaginary axis. The continuous spectrum fills all the axis $i\mathbb{R}$, and the only eigenvalue is $\lambda = 0$, with a (generalized) null space spanned by $\partial_x \varphi_{c_0}$ and $\partial_c \varphi_{c_0}$, which satisfy

$$\partial_x L_{c_0} \partial_x \varphi_{c_0} = 0, \quad \partial_x L_{c_0} \partial_c \varphi_{c_0} = -\partial_x \varphi_{c_0},$$

as easily obtained by differentiating the equation for φ_c with respect to x and c. Those two zero modes correspond to small changes in the location and velocity of the solitary wave, and give rise to solutions of (4.1) which are respectively constant and linearly growing in time. This was pointed out in particular in [12], where the asymptotic stability of the family of solitary waves is studied. Formally, these two modes generate a two dimensional "center manifold" for equation (4.1), on which the dynamics can be written thanks to the reduced two-dimensional system of ordinary differential equations

$$\begin{cases} \dot{x}_1 &= 0 \\ \dot{x}_2 &= -x_1. \end{cases}$$

Now, coming back to the stochastic equation (2.1), and proceeding as above, that is writing the solution as

$$u^\varepsilon(t,x) = \varphi_{c_0}(x - c_0 t) + v^\varepsilon(t, x - c_0 t),$$

linearizing the equation for v^ε and projecting on the "center manifold", one would formally be led to the reduced system of stochastic differential equations

$$\begin{cases} dx_1 &= \varepsilon dW_1 \\ dx_2 &= -x_1 dt + \varepsilon dW_2 \end{cases}$$

where (W_1, W_2) is a $\mathbb{R}^2$-valued Brownian motion corresponding to the (spectral) projection of the process $\varphi_{c_0} W$ on the "center manifold " spanned by the two modes $(\partial_x \varphi_{c_0}, \partial_c \varphi_{c_0})$ (see below for more precisions). It follows that

$$x_2(t) = \varepsilon W_2(t) - \varepsilon \int_0^t W_1(s) ds$$

(recall that the solution starts from the soliton solution φ_{c_0}). Hence, one can see that, asymptotically for large T and small ε, and for $\delta > 0$ sufficiently small, there is a constant C such that

$$\mathbb{P}(x_2(T) > \delta) \leq C e^{-\frac{\delta}{\varepsilon^2 T^3}};$$

this corresponds to the probability that the projection of v^ε on the center manifold is larger than δ. Hence this strongly suggests that without using the modulation parameters, the solution will not stay close to the soliton solution on a time larger than $\varepsilon^{-2/3}$, to be compared to the order ε^{-2} obtained in Theorem 3.1.

Actually the modulation parameters may be changed in order to ensure that η^ε do not contain any component on the "center manifold" defined above, as least at first order in ε. More precisely, Setting $\tilde{x}^\varepsilon(t) = x^\varepsilon(t) - \varepsilon B(t)$, with B some well chosen Brownian motion, then we can write

$$u^\varepsilon(t,x) = \varphi_{c^\varepsilon(t)}(x - \tilde{x}^\varepsilon(t)) + \varepsilon\tilde{\eta}^\varepsilon(t, x - \tilde{x}^\varepsilon(t))$$

for $t \le \tau_\alpha^\varepsilon$ with the same stopping time τ_α^ε as in Theorem 3.1, and then the following theorem holds for the order one term $\tilde{\eta}^\varepsilon$, in which $Q = I - P$, and P denotes the spectral projection (with respect to the operator $\partial_x L_{c_0}$) on the generalized null space spanned by $\partial_x \varphi_{c_0}$ and $\partial_c \varphi_{c_0}$.

Theorem 4.1 *For any $T > 0$, $\tilde{\eta}^\varepsilon$ defined above satisfies*

$$\mathbb{E}\Big(\sup_{t\le\tau_\alpha^\varepsilon\wedge T} \|\tilde{\eta}^\varepsilon(t)\|_{H^1}^2\Big) \le C(T,\alpha,c_0).$$

Moreover, $\tilde{\eta}^\varepsilon$ converges to η, as ε tends to zero, in $L^2(\Omega, L^\infty(0, T\wedge\tau_\alpha^\varepsilon; L^2(\mathbb{R})))$, with η solution of the linear equation

$$d\eta = \partial_x L_{c_0}\eta dt + Q(\varphi_{c_0} d\tilde{W}), \quad \eta(0) = 0.$$

Here, $\tilde{W}(t,x) = W(t, x + c_0 t)$ corresponds to the process written in the frame moving with velocity c_0. Note that due to our stationarity assumption on W, the processes W and $\tilde{W}$ have the same distribution. It follows that η is a centered Gaussian process, that we may write as

$$\eta(t) = \int_0^t e^{-\partial_x L_{c_0}(t-s)} Q\varphi_{c_0} d\tilde{W}(s). \tag{4.2}$$

Now, since the projection Q eliminates the secular modes, it is expected that the operator $\partial_x L_{c_0}$ is in some sense a dissipative operator, when restricted to the image of Q. It was proved in [12] that this is indeed the case, provided that the functions are considered in the weighted space

$$H_a^1 = \{v \in H^1(\mathbb{R}),\ e^{ax}v \in H^1(\mathbb{R})\}$$

for $a > 0$ sufficiently small, depending on c_0, where H_a^1 is endowed with the norm

$$\|v\|_{H_a^1} = \|e^{ax}v\|_{H^1}.$$

More precisely, it is proved in [12] that there are constants $C > 0$ and $b > 0$, depending on a and c_0, such that for any $w \in H_a^1$,

$$\|e^{-(\partial_x L_{c_0})t} Q w\|_{H_a^1} \leq C e^{-bt} \|w\|_{H_a^1}. \tag{4.3}$$

Using this result, it is then not difficult to prove that the covariance of the process $w_a(t,x) = e^{ax}\eta(t,x)$ has a uniformly bounded trace, for $t \in \mathbb{R}^+$. Indeed, using that

$$\tilde{W}(t,x) = \sum_{i\in\mathbb{N}} \tilde{\beta}_i(t)\phi e_i(x),$$

for some family $(\tilde{\beta}_i)_{i\in\mathbb{N}}$ of independent real valued Brownian motions, and ϕ as in (2.2), we may easily estimate, thanks to (4.3) and Lemma 2.6 in [4]

$$\begin{aligned} \mathrm{tr}(\mathrm{cov} w_a(t)) &= \sum_{j\in\mathbb{N}} \int_0^t \|e^{ax} e^{-\partial_x L_{c_0}(t-s)} Q[\varphi_{c_0}\phi e_j]\|_{H^1}^2 ds \\ &\leq C \left(\int_0^t e^{-2b\sigma} d\sigma \right) \sum_{j\in\mathbb{N}} \|e^{ax}[\varphi_{c_0}(k * e_j)]\|_{H^1}^2 \\ &\leq C(b) \|\varphi_{c_0}\|_{H_a^1}^2 \|k\|_{H^1}^2. \end{aligned}$$

Note that $\|\varphi_{c_0}\|_{H_a^1} < +\infty$ for $a < \sqrt{c_0}$ (see (1.3). We easily deduce that for each $a > 0$ sufficiently small, the random variable $w_a(t)$ converges weakly to a centered Gaussian measure as t tends to infinity, or equivalently, the process $\eta(t)$ defined in (4.2) converges weakly to

$$\eta_\infty = \int_{-\infty}^0 e^{-\partial_x L_{c_0} s} Q \varphi_{c_0} d\tilde{W}(s)$$

in the H_a^1-norm.

5 Modulation equations and soliton diffusion

Let us come back to the dynamics on the center manifold, which is given by the modulation equations on the parameters $\tilde{x}^\varepsilon$ and c^ε; at order one in ε the system of equations may be written as

$$\begin{cases} d\tilde{x}^\varepsilon = c_0 dt + \varepsilon B_1 dt + \varepsilon dB_2 + o(\varepsilon) \\ dc^\varepsilon = \varepsilon dB_1 + o(\varepsilon) \end{cases}$$

where $(B_1, B_2) = P(\varphi_{c_0}\tilde{W})$, P being the spectral projection on the center manifold defined in Section 4. Note that (B_1, B_2) is a two-dimensional Brownian motion. If we keep only the terms of order one in ε, i.e. we consider the solution $(X^\varepsilon(t), C^\varepsilon(t))$ of the system of SDEs

$$\begin{cases} dX^\varepsilon = c_0 dt + \varepsilon B_1 dt + \varepsilon dB_2 \\ dC^\varepsilon = \varepsilon dB_1, \end{cases}$$

then $(X^\varepsilon(t) - c_0 t, C^\varepsilon(t) - c_0)$ is a centered Gaussian vector, and it is easy to compute its covariance matrix. Denoting then by μ_t^ε its distribution, we may compute

$$\max_{x\in\mathbb{R}} \mathbb{E}\left(\varphi_{C^\varepsilon(t)}(x - X^\varepsilon(t))\right) = \max_{x\in\mathbb{R}} \iint_{\mathbb{R}^2} \varphi(x-y)\mu_t^\varepsilon(dy, dc)$$

and we get after a few computations, asymptotically as t goes to infinity,

$$\max_{x\in\mathbb{R}} \mathbb{E}\left(\varphi_{C^\varepsilon(t)}(x - X^\varepsilon(t))\right) \sim C\varepsilon^{-1/2}t^{-5/4}.$$

Note that for the original solution, this has only a meaning in the limit when t goes to infinity and ε goes to zero with $t = o(\varepsilon^{-2})$.

We could not obtain such a result for the additive equation studied in [3], although the numerical computations in [13] and [15] suggest that diffusion of the soliton also occurs with an additive noise, but with possibly a different rate.

Let us finally mention the result of [16], where equation (2.1) is studied, but with periodic boundary conditions for the space variable x. Although it is not specified that the noise is homogeneous in x, the result of [16] could be stated in our setting, under the assumption that for some constant α with $2\alpha^2 > \|k\|_{L^2}^2$,

$$\|k * |v|^2\|_{L^2}^2 \geq \alpha^2 \|v\|_{L^2}^4, \quad \text{for all} \quad v \in L^2, \tag{5.1}$$

where k is the convolution kernel associated to the operator ϕ defining the process W (see Section 2). It is then proved in [16] that for any $\varepsilon > 0$, the solution $u^\varepsilon(t)$ of (2.1) with periodic boundary conditions tends to zero a.s. in L_x^2 as t goes to infinity. Note that on the opposite, $\mathbb{E}(\|u^\varepsilon(t)\|_{L^2}^2)$ is exponentially growing in time, as may be easily seen with an application of the Itô formula. The result does not apply directly to the case $x \in \mathbb{R}$, since it is easy to see that there is no function k in $L^2(\mathbb{R})$ satisfying (5.1) for any $v \in L^2(\mathbb{R})$. However, the asymptotic behavior in time of the solution of (2.1) for a fixed ε is still an open problem.

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