Centre de Mathématiques Laurent Schwartz
Ecole Polytechnique

SEMINAIRE

Equations aux Dérivées Partielles

2007-2008

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Séminaire É. D. P. (2007-2008), Exposé n^{o} XVIII, 14 p.

<http://sedp.cedram.org/item?id=SEDP_2007-2008____A18_0>

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Séminaire *Équations aux dérivées partielles*
Centre de mathématiques Laurent Schwartz, 2007-2008
Exposé n° XVIII, 1-14

Propagation of analytic singularities for the Schrödinger Equation

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1 Introduction

Let us consider the solution of the Schrödinger equation on $\mathbb{R}^n$,

$$(\mathrm{Sch}) : \qquad \begin{cases} i\frac{\partial u}{\partial t} = Hu; \\ u_{|t=0} = u_0, \end{cases}$$

where H is a perturbation of the Laplacian operator $H_0 := -\frac{1}{2}\Delta$, and u_0 is in $L^2(\mathbb{R}^n)$ (or, more generally, in some Sobolev space).

The problem we are interested in consists in understanding the relationship between the singularities of $u(t)$ (for any $t > 0$ fixed) and (some simple property of) the initial data u_0.

As it is well known, the regularity of u_0 is not enough to insure that of $u(t)$, as illustrated in the example where one takes $H = H_0$ and $u_0 = (-2i\pi)^{-\frac{n}{2}}e^{-i|x|^2/2}$. Since, for $t \neq 0$, the distributional kernel of e^{-itH_0} is $(2i\pi t)^{-\frac{n}{2}}e^{i|x-y|^2/2t}$, we see that $u(t)$ just coincides with $v(t-1)$, where v solves the same Schrödinger equation with initial date $v(0) = \delta$ (the Dirac measure at $x = 0$). In particular, $u(1) = \delta$ is singular, while $u(0)$ is analytic.

In the literature, this phenomenon is referred as to the "infinite propagation speed" of singularities, and the natural questions one may ask are: Where does the possible singularities come from? Is there any way to read it easily on u_0? Can we analogously read the possible regularity of $u(t)$ on u_0?

Actually, there are mainly two types of results on this problem, that we describe now.

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Regularizing effects

Essentially, these are results that give necessary conditions on u_0 so that $u(t)$ is regular for some fixed $t > 0$. There has been many works on this (see, e.g., [CKS, Do1, Do2, GiVe, HaKa1, HaKa2, KaWa, KRY, KaSa, KaTa, KaYa, KPV, MRZ, Na1, RoZu1, RoZu2, RoZu3, Wu, Yaj1, Yaj2, Yam, Ze]), but the papers that have most motivated our study are those by Wunsh [Wu] (in the C^∞ framework) and Robbiano-Zuily [RoZu1, RoZu2, RoZu3] (in the analytic framework).
In these papers, a new notion of wave front set is introduced (the so-called "scattering quadratic wave front set"), that propagates with finite speed, and gives information both on the decay of u at infinity and on its regularity.
Later on, in [Na1, Na2, MNS1], it has been possible to give a simplified (and somehow, more general) version of these results, by introducing in an easiest way the notion of "homogeneous wave front set", that appeared afterwards to mainly coincide with that of scattering quadratic wave front set (see [It, Mec]).
Note that these phenomenon of regularizing effects is also closely related to the Strichartz estimates, that were recently studied by many authors (see, e.g., [StTa, RoZu4, BGT, BoTz]).

Characterization through the free evolution

In contrast with the previous type of results (where only a sufficient condition for the regularity of $u(t)$ was obtained), here a necessary and sufficient condition (in terms of some wave front set of u_0) is given, in order to know if a non-trapping point $(x,\xi) \in T^*\mathbb{R}^n\backslash 0$ is or is not in the wave front set of $u(t)$. These results are mainly contained in the papers [HaWu, Na2, Na3] (in the C^∞ context) and [MNS2] (in the analytic context).

In [HaWu], A. Hassel and J. Wunsch has obtained a characterization of the wave front set of the solution to the Schrödinger equation, in terms of the oscillations of the initial data near infinity (or near the boundary in the more general case of a so-called *scattering manifold*). More precisely, assuming that the metric is globally nontrapping, they show that the wave front set of $e^{-itH}u_0$ is determined by the so-called *scattering wave front set* of $e^{i|x|^2/2t}u_0$. (If the metric is not nontrapping, the result remains valid in the backward-non-trapped set for $t > 0$, and in the forward-non-trapped set for $t < 0$.) The proof is based on the construction of a global parametrix for the kernel of the Schrödinger propagator e^{-itH}, and requires a considerable amount of microlocal machinery (such as the scattering calculus of pseudodifferential operators, introduced by R.B. Melrose [Mel]).

For the asymptotically short-range flat metric case, Nakamura [Na2] gave simpler proof based on a Egorov-type argument, later extended to long-range type perturbations of the Laplacian in [Na3].

Here, the result that we would like to present more in details, is that of [MNS2], which can be considered as a generalization of [Na2] to the analytic framework.

2 Notations and Main Result

We set

$$H = \frac{1}{2}\sum_{j,k=1}^{n} D_j a_{j,k}(x) D_k + \frac{1}{2}\sum_{j=1}^{n}(a_j(x)D_j + D_j a_j(x)) + a_0(x)$$

on $\mathcal{H} = L^2(\mathbb{R}^n)$, where $D_j = -i\partial_{x_j}$, and we suppose the coefficients $\{a_\alpha(x)\}$ satisfy to the following assumptions. For $\nu > 0$ we denote

$$\Gamma_\nu = \big\{ z \in \mathbb{C}^n \;\big|\; |\mathrm{Im}\, z| < \nu\langle \mathrm{Re}\, z\rangle \big\}.$$

Assumption A. For each α, $a_\alpha(x) \in C^\infty(\mathbb{R}^n)$ is real-valued and can be extended to a holomorphic function on Γ_ν with some $\nu > 0$. Moreover, for $x \in \mathbb{R}^n$, the matrix $(a_{j,k}(x))_{1\le j,k\le n}$ is symmetric and positive definite, and there exists $\sigma \in (0,1]$ such that,

$$\begin{aligned} &\big|a_{j,k}(x) - \delta_{j,k}\big| \le C_0\langle x\rangle^{-1-\sigma}, \quad j,k = 1,\dots,n,\\ &\big|a_j(x)\big| \le C_0\langle x\rangle^{-\sigma}, \qquad\qquad j = 1,\dots,n,\\ &\big|a_0(x)\big| \le C_0\langle x\rangle^{1-\sigma}, \end{aligned}$$

for $x \in \Gamma_\nu$ and with some constant $C_0 > 0$.

In particular, H is essentially selfadjoint on $C_0^\infty(\mathbb{R}^n)$, and we use the same letter H for its unique selfadjoint extension on $L^2(\mathbb{R}^n)$.

We denote by $p(x,\xi) := \frac{1}{2}\sum_{j,k=1}^{n} a_{j,k}(x)\xi_j\xi_k$ the principal symbol of H, and by $H_0 := -\frac{1}{2}\Delta$ the free Laplace operator. For any $(x,\xi) \in \mathbb{R}^{2n}$, we also denote by $(y(t;x,\xi), \eta(t;x,\xi)) = \exp tH_p(x,\xi)$ the solution of the Hamilton system,

$$\frac{dy}{dt} = \frac{\partial p}{\partial \xi}(y,\eta),\ \frac{d\eta}{dt} = -\frac{\partial p}{\partial x}(y,\eta), \tag{2.1}$$

with initial condition $(y(0),\eta(0)) = (x,\xi)$.

Note that the requirement on the $a_j's$ is to decay like $\langle x\rangle^{-\sigma}$ only, and that a_0 may even increase at infinity. We still use the term "short-range" perturbation because, for this problem, the relevant decay is that of each term of the total symbol of $H - H_0$ *divided by* $\langle\xi\rangle^2$, as $\langle x\rangle \sim \langle\xi\rangle \to +\infty$.

We say that a point $(x_0,\xi_0)\in T^*\mathbb{R}^n\backslash 0$ is forward non-trapping (respectively backward non-trapping) when $|y(t,x_0,\xi_0)|\to\infty$ as $t\to+\infty$ (resp. as $t\to-\infty$). In that case, there exist $x_+(x_0,\xi_0),\xi_+(x_0,\xi_0)\in\mathbb{R}^n$ (resp. $x_-(x_0,\xi_0),\xi_-(x_0,\xi_0)\in\mathbb{R}^n$), such that,

$$|x_+(x_0,\xi_0)+t\xi_+(x_0,\xi_0)-y(t,x_0,\xi_0)|\to 0 \text{ as } t\to+\infty,$$

(resp. $|x_-(x_0,\xi_0)+t\xi_-(x_0,\xi_0)-y(t,x_0,\xi_0)|\to 0$ as $t\to-\infty$),
and we set $S_+(x_0,\xi_0):=(x_+(x_0,\xi_0),\xi_+(x_0,\xi_0))$ (respectively $S_-(x_0,\xi_0):=(x_-(x_0,\xi_0),\xi_-(x_0,\xi_0)))$ that corresponds to the forward (resp. backward) classical wave map. We also denote by FNT (resp. BNT) the set of forward (resp. backward) non-trapping points.

Our main result is,

Theorem 2.1. *Suppose Assumption A. Then,*

(i) For any $t<0$, one has,

$$WF_a(e^{-itH}u_0)\cap FNT=S_+^{-1}(WF_a(e^{-itH_0}u_0)); \tag{2.2}$$

(ii) For any $t>0$, one has,

$$WF_a(e^{-itH}u_0)\cap BNT=S_-^{-1}(WF_a(e^{-itH_0}u_0)). \tag{2.3}$$

Remark 2.2. In the particular case where the metric is globally non-trapping, this result gives a complete characterization of the analytic wave front set of $u(t)$ in terms of that of $e^{-itH_0}u_0$.

Remark 2.3. For $\operatorname{Re}\mu>0$, if one sets $T_\mu(z)=\int e^{-\mu(z-y)^2/h}u(y)dy$, a direct computation gives,

$$T_\mu(e^{-itH_0}u_0)(z)=\left(\frac{h}{h+i\mu t}\right)^{\frac{n}{2}}\int e^{-\frac{\mu}{h+i\mu t}(z-y)^2/2}u_0(y)dy, \tag{2.4}$$

and, as it is well known (see, e.g. [Ma]), T_μ can be used instead of T for determining the analytic wave front set of a distribution. Then, for $t>0$, if we set $v_0(y):=e^{i|y|^2/2t}u_0(y)$ and take $\mu=t^{-1}(t^{-1}+ih)$, one can deduce from (2.4) that a point (x_0,ξ_0) is not in $WF_a(e^{-itH_0}u_0)$, if and only if there exists $\delta>0$ such that the quantity

$$\mathcal{T}v_0(\tilde{x},\tilde{\xi};h):=\int e^{i(\frac{\tilde{x}}{h}-y)\tilde{\xi}-h(\frac{\tilde{x}}{h}-y)^2/2}v_0(y)dy$$

is uniformly $\mathcal{O}(e^{-\delta/h})$ as $h\to 0_+$ and $(\tilde{x},\tilde{\xi})$ stays in a neighborhood of $(-\xi_0,x_0)/t$. This naturally leads to a notion of wave front set (say, $\widetilde{WF}_a(v_0)$) that, in many aspects, looks rather similar to that of analytic homogeneous wave front set introduced in [MNS1]. We did not check it in details, but we strongly suspect that it corresponds to the analytic version of the scattering wave front set used in [HaWu] (see also [RoZu3]). In any case, this notion permits us to state our results in a way very similar to that of [HaWu], namely (for the FNT case),

$$(x_0,\xi_0)\in WF_a(e^{-itH}u_0)\cap FNT \iff \frac{1}{t}f\circ S_+(x_0,\xi_0)\in \widetilde{WF}_a(e^{i|y|^2/2t}u_0),$$

where $f(x,\xi) := (-\xi, x)$ is the canonical map of the Fourier transform.

3 Sketch of proof

We explain the proof for the forward non-trapping case only (the backward non-trapping case being similar).
Replacing u_0 by $e^{itH}u_0$, and then changing t to $-t$, we see that we have to prove that for any $t>0$ and $(x_0,\xi_0)\in FNT$, one has the equivalence,

$$(x_0,\xi_0)\in WF_a(u_0)\iff S_+(x_0,\xi_0)\in WF_a(e^{itH_0}e^{-itH}u_0).$$

Following [Na2], we set $v(t) := e^{itH_0}e^{-itH}u_0$, that solves the system,

$$i\frac{\partial v}{\partial t} = L(t)v \quad ; \quad v(0)=u_0. \tag{3.1}$$

Here,

$$L(t) = e^{itH_0}(H-H_0)e^{-itH_0} = L_2(t)+L_1(t)+L_0(t), \tag{3.2}$$

with,

$$\begin{aligned} L_2(t) &:= \frac{1}{2}\sum_{j,k=1}^{n} D_j(a_{j,k}^W(x+tD_x)-\delta_{j,k})D_k \\ L_1(t) &:= \frac{1}{2}\sum_{\ell=1}^{n}(a_\ell^W(x+tD_x)D_\ell + D_\ell a_\ell^W(x+tD_x)) \\ L_0(t) &:= a_0^W(x+tD_x), \end{aligned}$$

where we have denoted by $a^W(x,D_x)$ the usual Weyl-quantization of a symbol $a(x,\xi)$, defined by,

$$a^W(x,D_x)u(x) = \frac{1}{(2\pi)^n}\int e^{i(x-y)\xi}a((x+y)/2,\xi)u(y)dyd\xi.$$

In order to describe the analytic wave-front set of v, we introduce its FBI transform Tv defined by,

$$Tv(z,h) = \int e^{-(z-y)^2/2h}v(y)dy,$$

where $z \in \mathbb{C}^n$ and $h > 0$ is a small extra-parameter. Then, Tv belongs to the Sjöstrand space $H^{loc}_{\Phi_0}$ with $\Phi_0(z) := |\mathrm{Im}\ z|^2/2$ (see [Sj]), and a point (x,ξ) is not in $WF_a(v)$ if and only if there exists some $\delta > 0$ such that $Tv = \mathcal{O}(e^{(\Phi_0(z)-\delta)/h})$ uniformly for z close enough to $x - i\xi$ and $h > 0$ small enough. By Cauchy-formula, this is also equivalent to the existence of some $\delta' > 0$ such that $\|e^{-\Phi_0/h}Tv\|_{L^2(\Omega)} = \mathcal{O}(e^{-\delta'/h})$ for some complex neighborhood Ω of $x - i\xi$.
Since T is a convolution operator, we immediately observe that $TD_{x_j} = D_{z_j}T$. However, in order to study the action of $L(t)$ after transformation by T, we need the following key-lemma that will allow us to enter the framework of Sjöstrand's microlocal analytic theory. Mainly, this lemma tells us that, if f is holomorphic near Γ_ν, then, the operator $\tilde{T} := T \circ f^W(x + thD_x)$ is a FBI transform with the same phase as T, but with some symbol $\tilde{f}(t,z,x;h)$.

Lemma 3.1. *Let f be a holomorphic function on Γ_ν, verifying $f(x) = \mathcal{O}(\langle x\rangle^\rho)$ for some $\rho \in \mathbb{R}$, uniformly on Γ_ν. Let also K_1 and K_2 be two compact subsets of $\mathbb{R}^n$, with $0 \notin K_2$. Then, there exists a function $\tilde{f}(t,z,x;h)$ of the form,*

$$\tilde{f}(t,z,x;h) = \sum_{k=0}^{1/Ch} h^k f_k(t,z,x), \tag{3.3}$$

where f_k is defined, smooth with respect to t and holomorphic with respect to (z,x) near $\Sigma := \mathbb{R}_t \times \{(z,x)\,;\ \mathrm{Re}\ z \in K_1,\ |\mathrm{Re}\ (z-x)| + |\mathrm{Im}\ x| \leq \delta_0,\ \mathrm{Im}\ z \in K_2\}$ with $\delta_0 > 0$ small enough, and such that, for any $u \in L^2(\mathbb{R}^n)$, one has,

$$Tf^W(x+thD_x)u(z,h) = \int_{|x-\mathrm{Re}\ z|<\delta_0} e^{-(z-x)^2/2h}\tilde{f}(t,z,x,h)u(x)dx + \mathcal{O}(\langle t\rangle^{\rho_+} e^{(\Phi_0(z)-\varepsilon)/h}),$$

for some $\varepsilon = \varepsilon(u) > 0$ and uniformly with respect to $h > 0$ small enough, z in a small enough neighborhood of $K := K_1 + iK_2$, and $t \in \mathbb{R}$. (Here, we have set $\rho_+ = \max(\rho, 0)$.)
Moreover, the f_k's verify,

$$f_0(t,z,x) = f(x + it(z-x))\,;$$
$$|\partial^\alpha_{z,x} f_k(t,z,x)| \leq C^{k+|\alpha|+1}(k+|\alpha|)!\langle t\rangle^\rho,$$

for some constant $C > 0$, and uniformly with respect to $k \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}^{2n}_+$, and $(t,z,x) \in \Sigma$.

Thanks to this lemma, and using again Sjöstrand's theory of microlocal analytic singularities [Sj], we deduce the existence of an analytic second-order pseudodifferential operator $Q(t,h)$ on $H^{loc}_{\Phi_0}(\mathbb{C}^n\backslash\{\mathrm{Im}\ z = 0\})$, such that,

$$TL(t) = Q(t,h)T.$$

Moreover, the symbol of $Q(t,h)$ is mainly given by,

$$\begin{aligned} q(t,h;z,\zeta) &= \frac{h^{-2}}{2}\sum_{j,k=1}^{n}(a_{j,k}(z+i\zeta+th^{-1}\zeta)-\delta_{j,k})\zeta_j\zeta_k \\ &\quad +h^{-1}\sum_{\ell=1}^{n}a_\ell(z+i\zeta+th^{-1}\zeta)\zeta_\ell+a_0(z+i\zeta+th^{-1}\zeta). \end{aligned}$$

Then, applying T to (3.1), multiplying it by h^2, and changing the time-scale by setting $s:=t/h$, we obtain the new evolution equation,

$$ih\frac{\partial Tv}{\partial s}=B(s,h)Tv \quad ; \quad Tv(0)=Tu_0, \tag{3.4}$$

where $B(s,h)$ is an analytic pseudodifferential operator of order 0 (still in the sense of [Sj]), acting on $H^{loc}_{\Phi_0}(\mathbb{C}^n\backslash\{\mathrm{Im}\, z=0\})$, with symbol $b(s,h)$ verifying,

$$b(s,h)\sim\sum_{k\geq 0}h^k b_k(s)$$

(in the sense of analytic symbols), with

$$b_0(s;z,\zeta)=\frac{1}{2}\sum_{j,k=1}^{n}(a_{j,k}(z+i\zeta+s\zeta)-\delta_{j,k})\zeta_j\zeta_k,$$

and,

$$\begin{aligned} &b_1(s;z,\zeta)=\mathcal{O}(\langle s\rangle^{-\sigma});\\ &b_k(s;z,\zeta)=\mathcal{O}(\langle s\rangle^{1-\sigma}) \text{ for } k\geq 2, \end{aligned}\tag{3.5}$$

uniformly with respect to $s>0$, and locally uniformly with respect to $z\in\mathbb{C}^n\backslash\{\mathrm{Im}\, z=0\}$ and ζ close enough to $-\mathrm{Im}\, z$.
Let us recall from [Sj] that the quantization of such a symbol $b(s,h;z,\zeta)$ on $H^{loc}_{\Phi_0}$ is given by,

$$B(s,h)w(z;h)=\frac{1}{(2\pi h)^n}\int_{\gamma(z)}e^{i(z-y)\zeta/h}b(s,h;z,\zeta)w(y)dyd\zeta,$$

where $\gamma(z)$ is a complex contour of the form,

$$\gamma(z)\,:\,\zeta=-\mathrm{Im}\, z+iR(\overline{z-y})\ ;\ |y-z|<r,$$

with $R>0$ is fixed large enough, and $r>0$ can be taken arbitrarily small. In particular, denoting by $B_0(s)$ the quantization of $b_0(s)$, we deduce from (3.5) that $B(s,h)$ can be written as,

$$B(s,h)=B_0(s)+hB_1(s,h),$$

where $B_1(s,h)$ admit a symbol uniformly $\mathcal{O}(\langle s\rangle^{-\sigma}+h\langle s\rangle^{1-\sigma})$, for $s>0$, z in a compact subset of $\mathbb{C}^n\backslash\{\mathrm{Im}\, z=0\}$, and $(y,\zeta)\in\gamma(z)$.
Then, for $z_0\in\mathbb{C}^n\backslash\{\mathrm{Im}\, z=0\}$ and $\varepsilon_0>0$, if we set,

$$L^2_{\Phi_0}(z_0,\varepsilon_0):=L^2(\{|z-z_0|<\varepsilon_0\};e^{-2\Phi_0/h}d\mathrm{Re}\, z\, d\mathrm{Im}\, z)\cap H_{\Phi_0}(|z-z_0|<\varepsilon_0),$$

we see that $B_1(s,h)$ is a bounded operator from $L^2_{\Phi_0}(z_0,\varepsilon_0)$ to $L^2_{\tilde\Phi_0}(z_0,\varepsilon_0/2)$, and its norm can be estimated in terms of the supremum of its symbol. Thus, here we obtain,

$$\|B_1(s)\|_{\mathcal{L}\left(L^2_{\Phi_0}(z_0,\varepsilon_0);L^2_{\Phi_0}(z_0,\varepsilon_0/2)\right)}=\mathcal{O}(\langle s\rangle^{-\sigma}+h\langle s\rangle^{1-\sigma})=\mathcal{O}(\langle s\rangle^{-\sigma}), \quad (3.6)$$

uniformly with respect to $h>0$ small enough and $|s|\leq T/h$ ($T>0$ fixed arbitrarily).

3.1 The Flat Case

Let us first consider the case where $p=p_0:=\xi^2/2$, and show how we can easily deduce the result from (3.4). In that case, we have $B_0=0$ and $(x_+(x_0,\xi_0),\xi_+(x_0,\xi_0))=(x_0,\xi_0)$ for all $(x_0,\xi_0)\in T^*\mathbb{R}^n\backslash 0$. Setting $z_0=x_0-i\xi_0$ and $w=Tu$, Equation (3.4) gives,

$$i\partial_s w(s)=B_1(s,h)w(s)\ \text{ in } H_{\Phi_0}(|z-z_0|<\varepsilon_0), \quad (3.7)$$

with $\varepsilon_0>0$ fixed small enough.
Let us denote by $\tilde\Phi_0=\tilde\Phi_0(z,\overline{z})$ a smooth real-valued function defined near $z=z_0$, such that $|\tilde\Phi_0-\Phi_0|$ and $|\nabla_{(z,\overline{z})}(\tilde\Phi_0-\Phi_0)|$ are small enough, and verifying,

$$\tilde\Phi_0\geq\Phi_0 \text{ in } \{|z-z_0|\leq\varepsilon_0\}; \quad (3.8)$$
$$\tilde\Phi_0=\Phi_0 \text{ in } \{|z-z_0|\leq\varepsilon_0/4\}; \quad (3.9)$$
$$\tilde\Phi_0>\Phi_0+\varepsilon_1 \text{ in } \{|z-z_0|\geq\varepsilon_0/2\}, \quad (3.10)$$

for some $\varepsilon_1>0$. Then, by changing the contour defining $B_1(s)$ to a singular contour (see [Sj], Remarque 4.4), we know that $B_1(s)$ is also bounded from $L^2_{\tilde\Phi_0}(z_0,\varepsilon_0)$ to $L^2_{\tilde\Phi_0}(z_0,\varepsilon_0/2)$, and its norm on these space verifies the same estimate (3.6) as on $L^2_{\Phi_0}$.

Now, by (3.7), we have,

$$\partial_s\|w(s)\|^2_{L^2_{\tilde\Phi_0}(z_0,\varepsilon_0/2)}=2\mathrm{Im}\,\langle B_1(s)w(s),w(s)\rangle_{L^2_{\tilde\Phi_0}(z_0,\varepsilon_0/2)},$$

and thus, by Cauchy-Schwarz inequality and (3.6),

$$\left|\partial_s\|w(s)\|^2_{L^2_{\tilde\Phi_0}(z_0,\varepsilon_0/2)}\right|=\mathcal{O}(\langle s\rangle^{-\sigma})\|w(s)\|^2_{L^2_{\tilde\Phi_0}(z_0,\varepsilon_0)}. \quad (3.11)$$

Then, using (3.10) and the fact that $\|v(t)\|_{L^2} = \|u_0\|_{L^2}$ does not depend on t, we also have the estimate,

$$\|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0)} = \|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} + \mathcal{O}(e^{-\varepsilon_1/h}),$$

that, inserted into (3.11), gives,

$$\left|\partial_s \|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)}\right| \leq C\langle s\rangle^{-\sigma}\|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} + Ce^{-\varepsilon_1/h},$$

with some constant $C > 0$. Setting $g(s) := C\int_0^s \langle s'\rangle^{-\sigma}ds'$, and using Gronwall's lemma, we finally obtain,

$$\|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} \leq e^{g(s)}\|w(0)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} + C\int_0^s e^{g(s)-g(s')-\varepsilon_1/h}ds';$$
$$\|w(0)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} \leq e^{g(s)}\|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} + C\int_0^s e^{g(s')-\varepsilon_1/h}ds'.$$

Then, replacing s by t/h and observing that $g(s) = \mathcal{O}(\langle s\rangle^{1-\sigma}) = \mathcal{O}(h^{\sigma-1})$, the equivalence $(x_0,\xi_0) \notin WF_a(u_0) \Longleftrightarrow (x_0,\xi_0) \notin WF_a(u(t))$ follows immediately. Therefore, in that case we have proved,

$$WF_a(e^{-itH}u_0) = WF_a(e^{-itH_0}u_0).$$

3.2 The general case

In order to get rid of $B_0(s)$, in the general case we construct a Fourier integral operator $F(s,h)$ on H_{Φ_0,z_0}, verifying,

$$\begin{cases} ih\partial_s F(s,h) - B_0(s,h)F(s,h) \sim \mathcal{O}(h); \\ F\,|_{s=0} = I. \end{cases}$$

More precisely, we look for $F(s,h)$ of the form,

$$F(s)v(z) = \frac{1}{(2\pi h)^n}\int_{\gamma_s(z)} e^{i(\psi(s,z,\eta)-y\eta)/h}v(y)dyd\eta, \tag{3.12}$$

where $\gamma_s(z)$ is a convenient contour and ψ is a holomorphic function that must solve the system (eikonal equation),

$$\begin{cases} \partial_s\psi + b_0(s,z,\nabla_z\psi) = 0; \\ \psi\,|_{s=0} = z.\eta. \end{cases} \tag{3.13}$$

The construction of $\psi(s)$ for small s just follows from standard Hamilton-Jacobi theory. Then, the extension to larger values of s can be made by using the classical flow R_s of $b_0(s)$, that is related to the Hamilton flow of p through the formula,

$$R_s = \kappa\circ\exp(-sH_{p_0})\circ\exp sH_p\circ\kappa^{-1}, \tag{3.14}$$

where $\kappa(x,\xi)=(x-i\xi,\xi)$ is the complex canonical transformation associated with T.

In that way, we find a solution of (3.13) $\psi(s,\zeta,\eta)$, defined for $s\in\mathbb{R}$, z close to $z_0:=x_0-i\xi_0$ (where $(x_0,\xi_0)\in FNT$ is fixed arbitrarily), and η close to ξ_0. One also has the relation,

$$(z,\nabla_z\psi(s,z,\eta))=R_s(\nabla_\eta\psi(s,z,\eta),\eta), \tag{3.15}$$

which means that ψ is a generating function of the complex canonical transformation R_s. In other words, the operator $F(s,h)$ defined by (3.12) quantizes the canonical relation R_s, and, setting $z_s:=\pi_zR_s(z_0,\xi_0)$ (where $\pi_z:(z,\zeta)\mapsto z$), one can show that for any $\varepsilon_0>0$ small, $F(s,h)$ acts as,

$$F(s)\,:\,H_{\Phi_0}(|z-z_0|<\varepsilon_0)\to H_{\Phi_0}(|z-z_s|<\varepsilon_1), \tag{3.16}$$

for some $\varepsilon_1=\varepsilon_1(\varepsilon_0)>0$. A priori, ε_1 also depends on s, but as a matter of fact, since R_s tends to $R_\infty:=\kappa\circ S_+\circ\kappa^{-1}$ on a neighborhood of (z_0,ξ_0) as $s\to+\infty$, one can prove that $F(s;h)$ admits a limit $F_\infty(h)$, too, that is a FIO quantizing R_∞. Then, the action (3.16) remains valid for $0\le s\le+\infty$ (with $z_\infty:=\pi_zR_\infty(z_0,\xi_0)$), and ε_1 can be taken independent of s.

Now, by construction, for $s\in\mathbb{R}$, $F(s)$ verifies,

$$ih\partial_sF(s)-B_0(s)F(s)=hF_1(s),$$

where $F_1(s)\,:\,H_{\Phi_0}(|z-z_0|<\varepsilon_0)\to H_{\Phi_0}(|z-z_s|<\varepsilon_1)$ is of the form,

$$F_1(s)v(z)=\frac{1}{(2\pi h)^n}\int_{\gamma_s(z)}e^{i(\psi(s,z,\eta)-y\eta)/h}f_1(s,z,\eta;h)v(y)dyd\eta,$$

with f_1 is an analytic symbol that is $\mathcal{O}(\langle s\rangle^{-1-\sigma})$ as $s\to\infty$.

In the same way, for any y close enough to z_0, we can define a Fourier integral operator $\tilde F(s)$ of the form,

$$\tilde F(s)v(y):=\frac{1}{(2\pi h)^n}\int_{\tilde\gamma_s(y)}e^{i(y\eta-\psi(s,z,\eta))/h}v(z)dzd\eta,$$

(where $\tilde\gamma_s(y)$ is again a convenient contour), such that $\tilde F(s)$ maps $H_{\Phi_0}(|z-z_s|<\varepsilon_0)$ into $H_{\Phi_0}(|z-z_0|<\varepsilon_1)$, and verifies,

$$ih\partial_s\tilde F(s)+\tilde F(s)B=h\tilde F_1(s), \tag{3.17}$$

where $\tilde F_1(s)\,:\,H_{\Phi_0}(|z-z_s|<\varepsilon_0)\to H_{\Phi_0}(|z-z_0|<\varepsilon_1)$ is a FIO with same phase as $\tilde F(s)$ and symbol $\tilde f_1=\mathcal{O}(\langle s\rangle^{-1-\sigma})$.

Now, setting,

$$\tilde w(s)=\tilde F(s)Tu(hs)\in H_{\Phi_0}(|z-z_0|<\varepsilon_1),$$

by (3.4) and (3.17), we see that $\tilde{w}$ verifies,

$$i\partial_s \tilde{w}(s) = \left[\tilde{F}(s)B_1(s) + \tilde{F}_1(s)\right] Tu(hs).$$

Moreover, since $A(s) := F(s)\tilde{F}(s)$ is an elliptic pseudodifferential operator on H_{Φ_0,z_s}, by taking a parametrix $\tilde{A}(s)$, we have,

$$Tu(hs) = \tilde{A}(s)F(s)w(s) \text{ in } H_{\Phi_0}(|z - z_s| < \varepsilon), \tag{3.18}$$

(for some $\varepsilon > 0$ independent of s), and thus, we obtain,

$$i\partial_s \tilde{w}(s) = \tilde{B}_1(s)\tilde{w}(s). \tag{3.19}$$

in $H_{\Phi_0}(|z - z_0| < \varepsilon')$, where $\tilde{B}_1(s) := \left[\tilde{F}(s)B_1(s) + \tilde{F}_1(s)\right] \tilde{A}(s)F(s)$ is a pseudodifferential operator on $H_{\Phi_0}(|z - z_0| < \varepsilon')$ with the same properties as $B_1(s)$ when $s \to +\infty$.

Thus, we are reduced to a situation completely similar to that of the flat case, and, if for instance $(x_0, \xi_0) \notin WF_a(u_0)$, the same arguments show that,

$$\|w(s)\|_{L^2_{\Phi_0}(z_0,\delta)} \le Ce^{-\delta/h},$$

for some positive constant δ independent of $h > 0$ small enough and $s \in [0, T/h]$. As a consequence, using (3.18) and the fact that $\tilde{A}(s)F(s)$ is uniformly bounded from $L^2_{\Phi_0}(z_0, \delta)$ to $L^2_{\Phi_0}(z_s, \delta')$ for some $\delta' > 0$, we obtain (with some new constant $C > 0$),

$$\|Tu(hs)\|_{L^2_{\Phi_0}(z_s,\delta')} \le Ce^{-\delta/h}.$$

Replacing s by t/h with $t > 0$ fixed, and observing that $z_{t/h}$ tends to $\kappa \circ S_+(x_0, \xi_0)$ as $h \to 0_+$, we conclude that $S_+(x_0, \xi_0) \notin WF_a(u(t))$. The converse can be seen in the same way, and thus Theorem 2.1 is proved.

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