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KLEIN PARADOX AND SUPERRADIANCE FOR THE CHARGED KLEIN-GORDON FIELD

ALAIN BACHELOT

I. Introduction

In this paper, we deal with the gyroscopic Klein-Gordon equation

$$\left(\partial_t - iA(x)\right)^2 u - \partial_x^2 u + V(x)u = 0, \;\; t \in \mathbb{R}, \;\; x \in \mathbb{R}, \tag{I.1}$$

with the main hypotheses

$$V(x) \longrightarrow 0, \;\; A(x) \longrightarrow a_{\pm}, \;\; x \longrightarrow \pm\infty, \;\; a_- \neq a_+. \tag{I.2}$$

Fundamental examples of such an equation arise in General Relativity for the propagation of waves on space-times of Black-Hole type with an electrostatic charge. Other one-dimensional field equations with step-like perturbations have been studied : the existence of a Scattering Operator that is unitary, was established for the Dirac system by S.N.M. Ruijsenaars and P.J.M. Bongaarts [16], and for the Schrödinger equation by E.B. Davies and B. Simon [6]. The key point for both these equations is the conservation of the L^2 norm. The situation drastically differs for the Klein-Gordon equation (I.1) since the conserved energy

$$E(u,t) := \int \mid \partial_t u(t,x) \mid^2 + \mid \partial_x u(t,x) \mid^2 + \left[V(x) - A^2(x)\right] \mid u(t,x) \mid^2 dx \tag{I.3}$$

is not always positive. In particular, when A satisfies the steplike hypothesis (I.2), the set of modes is finite dimensional, but there exists no finite codimensional subspace of Cauchy data, on which this energy is positive. This is the root of the so called *Klein paradox*. Nevertheless we shall be able to describe the asymptotic behaviours of the solutions of (I.1), and to prove the existence of a Scattering Operator the norm of which is always *strictly larger* than one : this is the *superradiance*. Furthermore, in some situations, there exist solutions polynomially increasing in time (*hyperradiant* modes). Recently, the γ-bursts have been attributed to the superradiance of the charged black-holes (R. Ruffini [15]). Since we cannot use this energy to get some asymptotic estimates, we construct the spectral representation for the harmonic equation, then we establish the existence of the Scattering Operator the symbol of which has a norm strictly larger than 1. The detail of the proofs will appear in [3].

II. Spectral Decomposition

We investigate the harmonic Klein-Gordon equation :

$$\frac{d^2}{dx^2}y + \left[k - A(x)\right]^2 y - V(x)y = 0, \;\; x \in \mathbb{R}, \;\; k \in \mathbb{C}. \tag{II.1}$$

We assume that the potentials satisfy :

$$A \in L^\infty(\mathbb{R};\mathbb{R}), \;\; V \in L^\infty(\mathbb{R};\mathbb{R}), \tag{II.2}$$

and there exist $\alpha > 0$, $a_\pm \in \mathbb{R}$, $a_- < a_+$, such that :

$$\int_{\mathbb{R}^\pm} (\mid A(x) - a_\pm \mid + \mid V(x) \mid) e^{\alpha|x|} dx + \sup_{0<|h|<1} \int_{-\infty}^{\infty} \left| \frac{A(x+h) - A(x)}{h} \right| e^{\alpha|x|} dx < \infty. \tag{II.3}$$

We start by constructing suitable Jost functions, taking the different asymptotics as $x \to \pm\infty$, into account. For any $k \in \mathbb{C}$, $\Im k > -\frac{\alpha}{2}$ (resp. $\Im k < \frac{\alpha}{2}$), there exists unique functions $f^\pm_{in(out)}(k;x) \in C^1(\mathbb{R}_x)$, solutions of (II.1) and satisfying $\lim_{x\to\pm\infty} f^\pm_{in(out)}(k;x) - e^{\pm(\mp)i(k-a_\pm)x} = 0$. Moreover, for each $x \in \mathbb{R}$, they are analytic functions of $k \in \mathbb{C}$, $\Im k > -\frac{\alpha}{2}$ (resp. $\Im k < \frac{\alpha}{2}$). The following Wronskians do not depend of x :

$$W_{in(out)}(k) := [f^+_{in(out)}, f^-_{in(out)}](k).$$

Since W_{in} is an analytic function of $k \in \mathbb{C}$, $\Im k > -\alpha/2$, the set of its zeros is locally finite, and each of them is of finite multiplicity. We introduce

$$\sigma_p := \{k \in \mathbb{C};\ \ \Im k > 0,\ \ W_{in}(k) = 0\},$$

$$\sigma_{ss} := \{k \in \mathbb{R};\ \ \ W_{in}(k) = 0\},$$

$$\mathcal{R} := \{k \in \mathbb{C};\ \ -\frac{\alpha}{2} < \Im k < 0,\ \ W_{in}(k) = 0\}.$$

The elements of σ_p are the *eigenvalues* or *normal modes*, and the elements of $\mathcal{R}$ are the *resonances* or *quasinormal modes*. The *Klein zone* is the open interval $I_K :=]a_-, a_+[$. We shall see that the asymptotic behaviour of the solutions of the Klein-Gordon equation with step-like potential A is very peculiar, and justifies to call *superradiant modes* the real frequencies in $I_K \setminus \sigma_{ss}$, and *hyperradiant modes* the elements of σ_{ss}, that play the role of the *spectral singularities* of the quadratic pencils with short range complex potential [4]. In the simple example of the step potential $A_0(x) = a\mathbf{1}_{]-\infty,0]}(x)$, $a \in \mathbb{R}^*$, $V = 0$, we easily find $W_{in}(k) = i(2k - a)$, hence $\sigma_p = \mathcal{R} = \emptyset$, $\sigma_{ss} = \{\frac{a}{2}\}$. There exists cases where there is no hyperradiant mode. For instance if we choose $A_1(x) = 1 - \tanh(x)$ or $A_2(x) = 1$ when $x < 0$, $A_2(x) = 0$ when $x > 1$, $A_2(x) = 1 - x$ when $0 \leq x \leq 1$, and $V(x) = 0$, we can compute $W_{in}(\kappa)$ by using some formal calculus system. We get frightful combinations of hypergeometric functions for A_1, and Bessel functions for A_2 and the investigation of the possible roots of the equation $W_{in}(\kappa) = 0$ seems to be rather delicate. A numerical evaluation of $\mid W_{in}(\kappa) \mid$ using the Maple system, clearly shows that $\sigma_{ss} = \emptyset$ for both these potentials. In the general case, we prove that σ_p and σ_{ss} are finite sets and

$$\sigma_{ss} \subset I_K. \tag{II.4}$$

For the Schrödinger equation, i.e. $A = 0$, we know that the multiplicity of the zeros of W_{in} is simple. This is proved in [7] for the short range potentials V, and in [5] for the steplike case. Unlike this situation, when $A \neq 0$, the multiplicity $m(k) \in \mathbb{N}^*$ of $k \in \mathbb{C}$, defined by

$$\frac{d^l}{dk^l} W_{in}(k) = 0,\ \ 0 \leq l \leq m(k) - 1,\ \ \frac{d^{m(k)}}{dk^{m(k)}} W_{in}(k) \neq 0,$$

can be strictly larger that 1. As an example, we choose $A_3(x) = \mathbf{1}_{]-\infty,0]\cup[\frac{\pi}{3},\frac{2\pi}{3}]}(x)$, $V = 0$. By tedious but elementary calculations, we check that $W_{in}\left(\frac{1}{2}\right) = W'_{in}\left(\frac{1}{2}\right) = 0$. We put :

$$\nu := \max_{\kappa\in\sigma_{ss}} (m(\kappa))\ \ if\ \ \sigma_{ss} \neq \emptyset,\ \ \nu := 0\ \ if\ \ \sigma_{ss} = \emptyset. \tag{II.5}$$

We introduce the transmission coefficients $T^{\pm}(\kappa)$ and the reflection coefficients $R^{\pm}(\kappa)$, defined for $\kappa \in \mathbb{R} \setminus \sigma_{ss}$, by :

$$\kappa \neq a_{\pm}, \Rightarrow T^{\pm}(\kappa) := \frac{1}{\tau^{\pm}_{out}(\kappa)}, \quad R^{\pm}(\kappa) := \frac{\rho^{\mp}_{out}(\kappa)}{\tau^{\mp}_{out}(\kappa)}, \quad R^{\pm}(a_{\pm}) = -1, \quad T^{\pm}(a_{\mp}) = 0. \tag{II.6}$$

These quantities describe the propagation of the field as $x \to \pm\infty$:

$$f^{\pm}_{in} = T^{\mp} f^{\mp}_{out} - R^{\pm} f^{\pm}_{out}.$$

$R^{\pm}(\kappa)$ and $T^{\pm}(\kappa)$ are analytic functions on $\mathbb{R}_{\kappa} \setminus \sigma_{ss}$ and satisfy

$$\frac{\kappa - a_{\pm}}{\kappa - a_{\mp}} \mid T^{\pm}(\kappa) \mid^2 + \mid R^{\pm}(\kappa) \mid^2 = 1, \tag{II.7}$$

$$\mid T^{+}(\kappa) T^{-}(\kappa) - R^{+}(\kappa) R^{-}(\kappa) \mid = 1, \tag{II.8}$$

$$\kappa \in \mathbb{R} \setminus I_K \Longrightarrow \mid R^{\pm}(\kappa) \mid \leq 1, \tag{II.9}$$

$$\kappa \in I_K \setminus \sigma_{ss} \Longrightarrow \mid R^{\pm}(\kappa) \mid > 1, \tag{II.10}$$

$$\kappa \to \kappa_j \in \sigma_{ss} \Longrightarrow \mid R^{\pm}(\kappa) \mid, \mid T^{\pm}(\kappa) \mid \to \infty. \tag{II.11}$$

We emphasize that when κ is outside the Klein zone, the reflection coefficient is not greater than one as in the usual case of the decaying potential (i.e. $a_{\pm} = 0$). But when κ is a superradiant mode, $\mid R^{\pm}(\kappa) \mid$ is strictly larger than one, but finite : this is the phenomenon of *superradiance* of the Klein-Gordon fields (II.10). At last $T^{\pm}$ and $R^{\pm}$ diverge at the hyperradiant modes. The situation differs for the Dirac or Schrödinger equations, for which the reflection is total in the Klein zone (*i.e.* $T = 0$, $R = 1$, see [6], [16]).

We construct the distorded Fourier transforms. Given $f \in C^{\infty}_0(\mathbb{R}_x)$, $\varphi \in C^{\infty}_0(\mathbb{R}_{\kappa})$ we put :

$$F^{\pm}_{in(out)}(f)(k) := \int_{-\infty}^{\infty} f^{\pm}_{in(out)}(k; x) f(x) dx, \quad k \in \mathbb{C}, \quad +(-)\Im k \geq 0. \tag{II.12}$$

$F^{\pm}_{in(out)}$ is well defined from $C^{\infty}_0(\mathbb{R}_x)$ to $L^2(\mathbb{R}_{\kappa})$, but, unlike the short range case, $a_{\pm} = 0$, a problem arises for the low frequencies $\kappa = a_{\pm}$, with loss of regularity, when we want to define $F^{\pm}_{in(out)}$ on $L^2(\mathbb{R}_x)$. To overcome this difficulty, it is necessary to use the weighted L^2-spaces, $L^2_s(\mathbb{R}) := L^2(\mathbb{R}_y, (1+y^2)^s dy)$, $s \in \mathbb{R}$.

Proposition II.1. *$F^{\pm}_{in(out)}$ that is defined from $C^{\infty}_0(\mathbb{R}_x)$ to $\cap_n [H^n \cap L^2_n](\mathbb{R}_{\kappa})$, has a continuous extension :*

1. *from $L^2_{\frac{1}{2}+\delta}(\mathbb{R}_x)$ to $H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}_{\kappa}) \cap \mathcal{E}'(\mathbb{R}_{\kappa}) + H^{\frac{1}{2}+\delta}(\mathbb{R}_{\kappa})$, for any $\delta, \varepsilon > 0$;*
2. *from $L^2_1(\mathbb{R}_x)$ to $L^2(\mathbb{R}_{\kappa})$;*
3. *from $H^1 \cap L^2_1(\mathbb{R}_x)$ to $L^2_1(\mathbb{R}_{\kappa})$.*

There exists no continuous extension from $L^2_{\frac{1}{2}}(\mathbb{R}_x)$ to $\mathcal{D}'(\mathbb{R}_{\kappa})$. For any $\delta > 0$ there exists no continuous extension from $L^2_{1-\delta}(\mathbb{R}_x)$ to $L^2(\mathbb{R}_{\kappa})$.

We now introduce the inverse distorded Fourier transforms, defined for $\varphi \in C^{\infty}_0(\mathbb{R}_{\kappa})$ by :

$$\Phi^{\pm}_{in(out)}(\varphi)(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} f^{\pm}_{in(out)}(\kappa; x) \varphi(\kappa) d\kappa, \quad x \in \mathbb{R}. \tag{II.13}$$

Lemma II.2. *There exists a constant $C > 0$, a function $C(s)$, such that for all $s \in \mathbb{R}$, $-1 \le p \le 1$, $\varphi \in C_0^\infty(\mathbb{R}_\kappa)$, we have*

$$\| \Phi^{\pm}_{in(out)}(\varphi) \|_{H^p(\mathbb{R}^{\pm}_x)} \le C \| \varphi \|_{L^2_p(\mathbb{R}_\kappa)}, \quad \| \Phi^{\pm}_{in(out)}(\varphi) \|_{L^2_s(\mathbb{R}^{\pm}_x)} \le C(s) \| \varphi \|_{H^s(\mathbb{R}_\kappa)} .$$

Moreover $\Phi^{\pm}_{in(out)}$ is a bounded operator from $\mathcal{E}'(\mathbb{R}_\kappa)$ to $H^2_{loc}(\mathbb{R}_x)$.

We are now ready to state the resolution of the identity.

Theorem II.3. *There exists complex numbers $c_{\lambda,l}$, for $\lambda \in \sigma_p$, $0 \le l \le m(\lambda) - 1$, with $c_{\lambda,m(\lambda)-1} \neq 0$, such that for all $f \in L^2_s(\mathbb{R}_x)$, $s > \max(\frac{1}{2}, \nu - \frac{1}{2})$, where ν is defined by (II.5), we have for $p = 0, 1$:*

$$pf = \Phi^{\pm}_{in}\left(\frac{i\kappa^p}{W_{in}(\kappa + i0)} F^{\mp}_{in}(f)\right) - \Phi^{\pm}_{out}\left(\frac{i\kappa^p}{W_{out}(\kappa - i0)} F^{\mp}_{out}(f)\right)$$
$$+ \sum_{\lambda \in \sigma_p} \sum_{l=0}^{m(\lambda)-1} c_{\lambda,l} \partial_k^l \left(k^p f^{\pm}_{in}(k;x) F^{\mp}_{in}(f)\right)(k = \lambda) + \overline{c_{\lambda,l}} \partial_k^l \left(k^p f^{\pm}_{out}(k;x) F^{\mp}_{out}(f)\right)(k = \overline{\lambda}).$$

III. Scattering

In this section, we investigate the asymptotic behaviours in time of the solutions of the charged Klein-Gordon equation with the assumptions (II.2), (II.3) :

$$(\partial_t - iA(x))^2 u - \partial_x^2 u + V(x)u = 0, \quad t \in \mathbb{R}, \ x \in \mathbb{R}, \tag{III.1}$$

$$u(t = 0, x) = u_0(x), \quad \partial_t u(t = 0, x) = u_1(x). \tag{III.2}$$

It is convenient to introduce the following Hilbert space :

$$H^1_s(\mathbb{R}) := \left\{ f \in L^2_s(\mathbb{R}), \ \ f' \in L^2_s(\mathbb{R}) \right\}, \quad s \in \mathbb{R}, \quad \| f \|^2_{H^1_s} := \| f \|^2_{L^2_s} + \| f' \|^2_{L^2_s(I)} .$$

The Cauchy problem is well solved : For any $u_0 \in H^1_s(\mathbb{R})$, $u_1 \in L^2_s(\mathbb{R})$, $s \in \mathbb{R}$, there exists a unique solution $u \in C^0(\mathbb{R}_t; H^1_s(\mathbb{R}_x)) \cap C^1(\mathbb{R}_t; L^2_s(\mathbb{R}_x))$ of (III.1), (III.2). To give a representation of the solution involving the distorded Fourier transforms, we introduce the operators

$$E^{\pm}_{in(out)} : (u_0, u_1) \mapsto E^{\pm}_{in(out)}(u_0, u_1)(k) := kF^{\pm}_{in(out)}(u_0) - iF^{\pm}_{in(out)}(u_1 - 2iAu_0),$$

and the Hilbert space of initial data, where ν is defined by (II.5) :

$$X := H^1_{\max(\nu,1)}(\mathbb{R}_x) \times L^2_{\max(\nu,1)}(\mathbb{R}_x). \tag{III.3}$$

Proposition III.1. *For any $(u_0, u_1) \in X$, the solution u is expressed by :*

$$u(t) = \Phi^{\mp}_{in}\left(\frac{ie^{i\kappa t}}{W_{in}(\kappa + i0)} E^{\pm}_{in}(u_0, u_1)\right) - \Phi^{\mp}_{out}\left(\frac{ie^{ikt}}{W_{out}(\kappa - i0)} E^{\pm}_{out}(u_0, u_1)\right)$$
$$+ \sum_{\lambda \in \sigma_p} \sum_{l=0}^{m(\lambda)-1} c_{\lambda,l} \partial_k^l \left(e^{ikt} f^{\mp}_{in}(k;x) E^{\pm}_{in}(u_0, u_1)\right)(k = \lambda) + \overline{c_{\lambda,l}} \partial_k^l \left(e^{ikt} f^{\mp}_{out}(k;x) E^{\pm}_{out}(u_0, u_1)\right)(k = \overline{\lambda})$$

where the constants $c_{\lambda,l}$ are defined in Theorem II.3.

We denote $< t > := (1 + t^2)^{\frac{1}{2}}$. The energy estimates for the solutions are the following :

Theorem III.2. *There exist $C > 0$, $N \in \mathbb{N}$, such that for any $(u_0, u_1) \in X$, we have :*
$\| (u(t), \partial_t u(t)) \|_{H^1 \times L^2}$

$$\leq C\Big(\| (u_0, u_1) \|_X + \sum_{\kappa \in \sigma_{ss}} \sum_{l=0}^{m(\kappa)-1} < t >^{m(\kappa)-l+\frac{1}{2}} \sum_{\substack{\sharp=+,- \\ \flat=in,out}} \left| \frac{d^l}{dk^l} E^{\sharp}_{\flat}(u_0, u_1)(\kappa) \right|$$

$$+ \sum_{\lambda \in \sigma_p} \sum_{l=0}^{m(\lambda)-1} < t >^{m(\lambda)-l-1} \sum_{\sharp=+,-} \left| \frac{d^l}{dk^l} E^{\sharp}_{in}(u_0, u_1)(\lambda) \right| e^{-\Im(\lambda)t} + \left| \frac{d^l}{dk^l} E^{\sharp}_{out}(u_0, u_1)(\overline{\lambda}) \right| e^{\Im(\lambda)t} \Big),$$

$$\| (u(t), \partial_t u(t)) \|_X \leq C < t >^N e^{\gamma |t|} \| (u_0, u_1) \|_X, \quad \gamma := \max_{\lambda \in \sigma_p} \Im \lambda.$$

By a microlocalization near the hyperradiant modes, we now construct solutions of finite energy with polynomial behaviour in time.

Theorem III.3. *For all $\kappa \in \sigma_{ss}$, $l \leq m(\kappa) - 1$, there exist $u_0, u_1 \in C_0^\infty(\mathbb{R}_x)$, such that for any $x \in \mathbb{R}$, we have :*

$$u(t, x) = t^{m(\kappa)-l-1} e^{i\kappa t} f^{\mp}_{in}(\kappa; x) + o\left(t^{m(\kappa)-l-1}\right), \ t \to -\infty; \ \ u(t, x) = o\left(t^{m(\kappa)-l-1}\right), \ t \to +\infty.$$

To investigate the scattering states, we must avoid the usual modes and the hyperradiant ones. Hence we introduce the following subspaces of finite codimension in X :

$$X_{in(out)} := \left\{ (u_0, u_1) \in X; \forall k \in \sigma_p \cup \sigma_{ss}, \forall l < m(k), \ \frac{d^l}{dk^l} E^{\pm}_{in(out)}(u_0, u_1)(k(\overline{k})) = 0 \right\},$$

$$X_{scatt} := X_{in} \cap X_{out}. \tag{III.4}$$

$X_{in(out)}$ and X_{scatt} are well defined, and they are Hilbert subspaces of X, invariant under the action of the group $G(t) : \ (u_0, u_1) \mapsto (u(t), \partial_t u(t))$.

We now arrive at an important result of this work : the solutions with Cauchy data in X_{scatt} are asymptotically free. We emphazise that the conserved energy of such solutions given by (I.3) can be negative. In fact we do not use this conservation law to get our scattering theory.

Theorem III.4. *For any $(u_0, u_1) \in X_{scatt}$ there exists unique $u^{\pm}_{in(out)} \in H^1(\mathbb{R})$ such that*

$$\lim_{t \to -(+)\infty} \| u(t, x) - \left(e^{+(-)ia_- x} u^-_{in(out)}(t - (+)x) + e^{-(+)ia_+ x} u^+_{in(out)}(t + (-)x) \right) \|_{H^1(\mathbb{R}_x)}$$

$$+ \| \partial_t u(t, x) - \left(e^{+(-)ia_- x} \left(u^-_{in(out)} \right)' (t - (+)x) + e^{-(+)ia_+ x} \left(u^+_{in(out)} \right)' (t + (-)x) \right) \|_{L^2(\mathbb{R}_x)} \tag{III.5}$$

u_0, u_1, $u^{\pm}_{in(out)}$ are bound by the following relations :

$$u^{\pm}_{in(out)} = +(-)\mathcal{F}^{-1} \left(\frac{i}{W_{in(out)}(\kappa + (-)i0)} E^{\mp}_{in(out)}(u_0, u_1) \right),$$

$$u_p = \Phi^{\pm}_{in} \left((i\kappa)^p \mathcal{F}\left(u^{\pm}_{in} \right) \right) + \Phi^{\pm}_{out} \left((i\kappa)^p \mathcal{F}\left(u^{\pm}_{out} \right) \right), \ \ p = 0, 1,$$

$$\| u^{\pm}_{in(out)} \|_{H^1(\mathbb{R})} \leq C \| (u_0, u_1) \|_X,$$

$$\forall \kappa \in \mathbb{R} \setminus \sigma_{ss}, \quad \begin{pmatrix} \mathcal{F}\left(u^+_{out}\right)(\kappa) \\ \mathcal{F}\left(u^-_{out}\right)(\kappa) \end{pmatrix} = \begin{pmatrix} R^+(\kappa) & T^+(\kappa) \\ T^-(\kappa) & R^-(\kappa) \end{pmatrix} \begin{pmatrix} \mathcal{F}\left(u^+_{in}\right)(\kappa) \\ \mathcal{F}\left(u^-_{in}\right)(\kappa) \end{pmatrix}. \qquad \text{(III.6)}$$

We introduce the Wave Operators

$$\mathbb{W}_{in(out)} : (u_0, u_1) \longmapsto \left(u^+_{in(out)}, u^-_{in(out)}\right).$$

Corollary III.5. *$\mathbb{W}_{in(out)}$ is a one-to-one, continuous operator from X_{scatt} onto a subspace $Y_{in(out)}$ of $H^1(\mathbb{R}_s) \times H^1(\mathbb{R}_s)$. Moreover the map*

$$\left(u^+(s), u^-(s)\right) \longmapsto \left(\overline{u^+(-s)}, \overline{u^-(-s)}\right)$$

is one-to-one from Y_{in} onto Y_{out}.

This result assures that $\mathbb{W}^{-1}_{in(out)}$ is well defined on $Y_{in(out)}$, and we define the scattering operator by :

$$\mathbb{S} = \mathbb{W}_{out} \left(\mathbb{W}_{in}\right)^{-1}.$$

Since we do not know if $Y_{in(out)}$ is closed, the question of the continuity of $\mathbb{W}^{-1}_{in(out)}$ remains open. Therefore we want to construct continuous inverse Wave Operators formally given by :

$$\mathbf{\Omega}_{in(out)} : \left(u^+_{in(out)}, u^-_{in(out)}\right) \longmapsto (u_0, u_1),$$

such that limit (III.5) is satisfied. When $\sigma_p \neq \emptyset$, the modes associated with an eigenvalue are exponentially decreasing as $t \to +(-)\infty$, hence $\mathbf{\Omega}_{in(out)}$ would be multivalued. Therefore it is natural to assume that there exists no such exponentially damped modes.

Proposition III.6. *When $\sigma_p = \emptyset$, there exists $q \geq 1$, and bounded operators $\mathbf{\Omega}_{in(out)}$ from $\left[H^1_{\max(\nu,1)}(\mathbb{R}) \cap H^1_q(\mathbb{R}^{-(+)})\right]^2$, to $X_{in(out)} \cap D\left(\mathbb{W}_{in(out)}\right)$ such that*

$$\mathbb{W}_{in(out)} \mathbf{\Omega}_{in(out)} = Id \ \ on \ \ \left[H^1_{\max(\nu,1)}(\mathbb{R}) \cap H^1_q(\mathbb{R}^{-(+)})\right]^2.$$

When $\sigma_{ss} \neq \emptyset$, (III.6) shows that the continuity of $\mathbb{S}$ are not clear. Nevertheless we can develop a complete scattering theory when there occurs no usual or hyperradiant mode. We need a subspace of X :

Lemma III.7. *We assume $\sigma_{ss} = \sigma_p = \emptyset$. Then given $(u_0, u_1) \in X$, $E^+_{in}(u_0, u_1), E^-_{in}(u_0, u_1)$ belong to $H^1(\mathbb{R}_\kappa)$ iff $E^+_{out}(u_0, u_1), E^-_{out}(u_0, u_1)$ belong to $H^1(\mathbb{R}_\kappa)$. We put :*

$$X_1 := \left\{(u_0, u_1) \in X;\ E^{\pm}_{in/out}(u_0, u_1) \in H^1(\mathbb{R}_\kappa)\right\},$$

$$\| (u_0, u_1) \|^2_{X_1, in(out)} := \| E^+_{in(out)}(u_0, u_1) \|^2_{H^1(\mathbb{R}_\kappa)} + \| E^-_{in(out)}(u_0, u_1) \|^2_{H^1(\mathbb{R}_\kappa)}.$$

$\| . \|_{X_1, in}$ and $\| . \|_{X_1, out}$ are two equivalent norms for which X_1 is a Hilbert space, invariant under the action of the group $G(t)$, and there exists $C > 0$ such that for all $(u_0, u_1) \in X_1$ we have :

$$\| (u_0, u_1) \|_X \leq C \| (u_0, u_1) \|_{X_1, in(out)}.$$

Moreover we have :

$$H^1 \cap \mathcal{E}'(\mathbb{R}_x) \times L^2 \cap \mathcal{E}'(\mathbb{R}_x) \subset X_1.$$

We introduce the Hilbert spaces :

$$K^{\pm} := \left\{ u \in H^1(\mathbb{R}_x) ; \;\; iu' + a_{\pm} u \in L^2_1(\mathbb{R}_x) \right\}, \;\; \| u \|^2_{K^{\pm}} := \| u \|^2_{H^1} + \| iu' + a_{\pm} u \|^2_{L^2_1} .$$

The scattering theory in the absence of modes, is described by the following :

Theorem III.8. *We assume* $\sigma_{ss} = \sigma_p = \emptyset$. *Then* $(u^+, u^-) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ *belongs to* $Y_{in(out)}$ *iff*

$$u_p^{in(out)} := \Phi^+_{out(in)} \left(\frac{(i\kappa)^p}{\tau^+_{out(in)}(\kappa)} \mathcal{F}\left(u^-\right) \right) + \Phi^-_{out(in)} \left(\frac{(i\kappa)^p}{\tau^-_{out(in)}(\kappa)} \mathcal{F}\left(u^+\right) \right) \in H^1_1(\mathbb{R}_x), \;\; p = 0, 1,$$

and in this case we have :

$$\mathbb{W}_{in(out)}(u_0^{in(out)}, u_1^{in(out)}) = (u^+, u^-).$$

Moreover we have :

$$H^1_1(\mathbb{R}) \times H^1_1(\mathbb{R}) \subset Y_{in(out)} \subset K^+ \times K^-,$$

and $\mathbb{W}_{in(out)}$ *are continuous, one-to-one, operators, from* X_1 *onto* $H^1_1(\mathbb{R}) \times H^1_1(\mathbb{R})$, *and from* X *onto* $Y_{in(out)}$ *endowed with the norm of* $K^+ \times K^-$. *The scattering operator is a continuous, one-to-one operator from* $H^1_1(\mathbb{R}) \times H^1_1(\mathbb{R})$ *onto* $H^1_1(\mathbb{R}) \times H^1_1(\mathbb{R})$, *and from* Y_{in} *onto* Y_{out} *where* $Y_{in(out)}$ *are endowed with the norm of* $K^+ \times K^-$, *or* $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. *This operator has the form :*

$$\mathbb{S} = \mathcal{F}^{-1} \hat{S}(\kappa) \mathcal{F}, \quad \hat{S}(\kappa) := \begin{pmatrix} R^+(\kappa) & T^+(\kappa) \\ T^-(\kappa) & R^-(\kappa) \end{pmatrix}.$$

The scattering matrix $\hat{S}(k)$ *is meromorphic on* $\omega := \{k \in \mathbb{C}; \mid \Im k \mid < \frac{\alpha}{2}\}$ *and* $k \in \omega$ *is a pole of* $\hat{S}$ *iff* $\overline{k}$ *belongs to the set of resonances* $\mathcal{R}$. *Furthermore the scattering is superradiant for the frequencies in the Klein zone :*

$$\kappa \in]a_-, a_+[\Longrightarrow 1 < \mid R^{\pm}(\kappa) \mid, \; \| \hat{S}(\kappa) \|_{\mathcal{L}(\mathbb{C}^2)}, \; \| \left(\hat{S}(\kappa)\right)^{-1} \|_{\mathcal{L}(\mathbb{C}^2)} .$$

We make some comments on these results.

(1) We can easily see that :

$$H^1_1(\mathbb{R}) \times H^1_1(\mathbb{R}) \underset{\neq}{\subset} Y_{in(out)} \underset{\neq}{\subset} K^+ \times K^-,$$

when $A(x) = a_-$ and $V(x) = 0$ for $x << 0$, or when $A(x) = a_+$ and $V(x) = 0$ for $x >> 0$.

(2) When $A = 0$ and V is compactly supported, the Lax-Phillips theory assures that $\hat{S}$ has a meromorphic continuation on $\mathbb{C}_k$, and solution a u has an asymptotic expansion

$$u(t, x) \sim \sum_{\overline{k} \in \mathcal{R}} \sum_{n=0}^{m(k)-1} C(k, n, x) t^n e^{itk}, \;\; t \to +\infty.$$

Several analogous results are known when V is a compactly supported, or short range potential, with an analytic continuation on a conic neighbourhood of $\mathbb{R}_x$ (e.g. [17]). We conjecture a similar expansion for the charged fields considered in this paper.

Since the asymptotic dynamics are $(\partial_t - ia)^2 - \partial_x^2$ as $x \to -\infty$ and $\partial_t^2 - \partial_x^2$ as $x \to +\infty$, it is natural to study the scattering in the Hilbert spaces associated with the energy for these wave equations. Given $c \in \mathbb{R}$, we introduce the Beppo Levi type spaces $BL^1_{(c)}(\mathbb{R})$ defined as the closure of $C_0^\infty(\mathbb{R})$ in the norm

$$f \in C_0^\infty(\mathbb{R}), \quad \| f \|_{BL^1_{(c)}} := \| if' + cf \|_{L^2} .$$

These spaces are not spaces of distributions on $\mathbb{R}$ and the solutions of the wave equations have to be interpreted in the sense of the spectral calculus.

Corollary III.9. *$\mathbb{W}_{in(out)}$ can be extended into a bounded operator from $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ to $BL^1_{(a_+)}(\mathbb{R}) \times BL^1_{(a_-)}(\mathbb{R})$. $\mathbb{S}$ can be extended into an isomorphism on $BL^1_{(a_+)}(\mathbb{R}) \times BL^1_{(a_-)}(\mathbb{R})$ and denoting by $\| \, . \, \|_e$ the norm of $\mathcal{L}\left(BL^1_{(a_+)}(\mathbb{R}) \times BL^1_{(a_-)}(\mathbb{R})\right)$, we have*

$$1 < \| \mathbb{S} \|_e, \quad \| \mathbb{S}^{-1} \|_e .$$

We emphasize that this extended scattering operator is of an unusual type : we do not know if the inverse wave operators $\mathbb{W}^{-1}_{in(out)}$, which are defined from $Y_{in(out)}$ onto X, can be extended from $BL^1_{(a_+)}(\mathbb{R}) \times BL^1_{(a_-)}(\mathbb{R})$ to $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. This situation has already been encountered in the case of space-times with causality violation [2]. The root of this phenomenon is the same : the conserved energy is not definite positive.

IV. An Application in General Relativity

The asymptotic behaviours of classical fields on several important curved space-times of General Relativity, have been the subject of numerous studies. We can mention the works on the scalar equations by the author [1], [2], D. Häfner [9], [10], J-P. Nicolas [13], and on the Dirac system by F. Melnyk [11], [12], J-P. Nicolas [14]. As regards the propagation of the energy, there exists a deep difference between the bosons and the fermions : the L^2 norm of a field with half-integral spin, is conserved, while the conserved energy of the Klein-Gordon field on a curved background is not necessarily positive. In such cases of indefinite conserved energy, the field is allowed to extract energy from a particular region of space-time, for instance the ergosphere of a Kerr black-hole, or the dyadosphere of a charged black-hole. This phenomenon has been described, for the first time, by R. Penrose who proved that a classical particle can enter the ergosphere of a rotating black hole, and come out again with more energy than it originally had. The corresponding effect for integral spin fields is called superradiance [8], [16]. To our knowledge, a rigorous mathematical analysis is missing, and the present study is a first step in this direction since we can apply the results of the previous sections to the superradiant scattering of charged Klein-Gordon fields by a charged black-hole in an expanding universe.

The spin 0 field with mass $m \geq 0$, and charge $e \in \mathbb{R}$, on a lorentzian manifold $(\mathcal{M}, g)$ endowed with an electromagnetic potential $A_\mu dx^\mu$, obeys the Klein-Gordon equation

$$\left(\nabla_\mu - ieA_\mu\right)\left(\nabla^\mu - ieA^\mu\right)\Phi + m^2\Phi + \xi R\Phi = 0, \tag{IV.1}$$

where $R = g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature, and $\xi \in \mathbb{R}$ is a numerical factor. We are concerned with the 3+1 dimensional, spherically symmetric space-time $\mathbb{R}_t \times I_r \times S^2_\omega$, I being a real open interval, that describes a black hole in an expanding universe. In this case the metric can be written as :

$$g_{\mu\nu}dx^\mu dx^\nu = F(r)dt^2 - [F(r)]^{-1}\, dr^2 - r^2 d\omega^2, \tag{IV.2}$$

where $F \in C^2([r_-, r_+])$, $0 < r_- < r_+ < \infty$, is called the lapse function, and satisfies :

$$F(r_-) = F(r_+) = 0, \;\; r_- < r < r_+ \Rightarrow 0 < F(r), \;\; 0 < F'(r_-), \;\; F'(r_+) < 0. \tag{IV.3}$$

r_- is the radius of the Horizon of the Black-Hole, r_+ is the radius of the Cosmological Horizon. The Ricci scalar is given by

$$R = F'' + \frac{4}{r}F' + \frac{2}{r^2}(F-1).$$

We assume that the electromagnetic potential is electrostatic and also spherically symetric :

$$A_\mu dx^\mu = A_t(r)dt, \;\; A_t \in C^1([r_-, r_+]), \;\; A_t(r_-) \neq A_t(r_+). \tag{IV.4}$$

These hypotheses are satisfied, for a suitable choice of the physical parameters, in the important case of a charged black-hole in an expanding universe, for which the DeSitter-Reissner-Nordstrøm metric, and the Maxwell connection, are given by :

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2, \;\;\; A_t(r) = \frac{Q}{r}. \tag{IV.5}$$

Here $0 < M$ and $Q \in \mathbb{R}$ are the mass and the charge of the black-hole, $\Lambda > 0$ is the cosmological constant (see e.g. [11]).

It is convenient to push the horizons away to infinity by putting :

$$x = \frac{1}{F'(r_-)}\left\{ \ln \mid r - r_- \mid - \int_{r_-}^{r} \left[\frac{1}{r-r_-} - \frac{F'(r_-)}{F(r)}\right] dr \right\}.$$

Then $u = r\Phi$ is solution of

$$\left(\partial_t - iA(x)\right)^2 u - \partial_x^2 u - B(x)\Delta_{S^2} u + C(x)u = 0, \;\; t \in \mathbb{R}, \; x \in \mathbb{R}, \;\; \omega \in S^2,$$

with

$$A(x) = eA_t(r), \;\; B(x) = \frac{1}{r^2}F(r), \;\; C(x) = \left(\xi F''(r) + \frac{4\xi+1}{r}F'(r) + \frac{2\xi}{r^2}F(r) - \frac{2\xi}{r^2} + m^2\right) F(r).$$

The conserved energy is given by :

$$E(u) := \int_{\mathbb{R}\times S^2_\omega} \left(\mid \partial_t u(t,.) \mid^2 + \mid \partial_x u(t,.) \mid^2 + B(x) \mid \nabla_\omega u(t,.) \mid^2 + \left[C(x) - A^2(x)\right] \mid u(t,.) \mid^2\right) dx d\omega.$$

The *dyadosphere* $\mathcal{D}_{e,m}$ is defined as the region outside the black hole horizon where the electrostatic energy, associated with the charge e of the field, exceeds the gravitational interacting energy associated with the mass m of the field :

$$\mathcal{D}_{e,m} := \left\{ x \in \mathbb{R}; \; A^2(x) > C(x) \right\} \times S^2_\omega.$$

We remark that, because of the existence of the cosmological horizon, unlike the case of the asymptotically flat space-time for which $F(r) \to 1$ as $r \to +\infty$, $\mathcal{D}_{e,m}$ is never empty, whatever the mass of the field and the gauge transform on A. Furthermore, if $\mid e \mid$ is large enough, we can have $\mathcal{D}_{e,m} = \mathbb{R}_x \times S^2_\omega$.

Taking advantage of the spherical symmetry, we expand $u(t,x,.)$ on the basis of spherical harmonics $Y_{l,m}$ of $L^2\left(S^2_\omega\right)$:

$$u(t,x,\omega)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}u_{l,m}(t,x)Y_{l,m}(\omega).$$

Finally $u_{l,m}$ is solution of the gyroscopic Klein-Gordon equation

$$\left(\partial_t-iA(x)\right)^2u_{l,m}-\partial_x^2u_{l,m}-V(x)u_{l,m}=0,\ \ t\in\mathbb{R},\ x\in\mathbb{R}, \tag{IV.6}$$

with

$$V(x)=l(l+1)B(x)+C(x).$$

Since A and V satisfy

$$\mid A(x)-eA_t(r_\pm)\mid+\mid A'(x)\mid+\mid V(x)\mid\leq Ce^{F'(r_\pm)x},\ \ x\to\pm\infty,$$

we may apply the results of the preceding sections to equation (IV.6). In particular, Theorem III.4 gives a rigorous explanation of the superradiance of charged black-holes, in terms of scattering of spin-0 charged fields ([8], [15]). We leave open the problem of the nature of σ_{ss} and σ_p for the DeSitter-Reissner-Nordstrøm metric (IV.5). Several numerical experiments suggest that these sets are empty, hence we conjecture that there is no hyperradiance in this case.

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