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Malliavin Calculus for a general manifold

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1 Introduction

Let us begin by considering the finite dimensional case. Let us consider a function F from R^N with generic element b (N will become infinite later) into R^d with generic element y. We suppose that F is smooth with bounded derivatives of all orders. We say that the function F is a **submersion in the strong sense**, if its derivative $dF(b)$ is in all b a linear surjection. We can express this fact by introducing the Gram matrix $dF(b)^t dF(b)$ which is a symmetric matrix in R^d and saying that the Gram matrix is strictly positive in all b. If we suppose that our space R^N is endowed with a non degenerate Gaussian law (with in order to simplify a covariance matrix equals to the identity), it is almost equivalent to say that $E[(dF^t dF)^{-p}] < \infty$ for all integers p, if we can control the behaviour at the infinity of the Gram matrix. In this part, we will skip the problem to control the expressions at the infinity, which can be handled by introducing some mollifers. The introduction of such mollifers (in infinite dimension) is the purpose of this work.

Let us consider the law of the random variable F: its law has a smooth density. We can see that by using two following points of view which can be "a priori" different:

-)The first one is Bismut's point of view ([Bi]). Since F is a submersion, $F^{-1}(y)$ is a submanifold of R^N of codimension d, and by using the implicit function theorem, we get an "explicit" expression for the density $p(y)$ of F:

$$p(y) = \int_{F^{-1}(y)} \sqrt{2\pi}^{-N} \exp[-\|b\|^2/2] \sqrt{detdF(b)^t dF(b)}^{-1} d\sigma^y(b) \tag{1.1}$$

$d\sigma^y(b)$ is the Riemannian volume element over $F^{-1}(y)$.

-)The second one is Malliavin's point of view ([Ma]). In order to show that the law of F has a smooth density, it is enough to obtain integration by parts formulae. More precisely, let (α) be a multi-index over R^d. There exists a universal polynomial in the derivatives of F and in $det(dF^t dF)^{-1}$ (where $det(dF^t dF)^{-1}$ appears with an exponent which increases when the length of

(α) increases) such that for all test functions f

$$E[f^{(\alpha)}(F)] = E[L(\alpha)f(F)] \tag{1.2}$$

Let us remark in order to request more and more regularity on the law of F, we need multi-indices of length more and more big such that we request the hypothesis that $E[(dF^t dF)^{-p}] < \infty$ for bigger and bigger integers p. But this point of view is in principle more general than the first point of view because it allows to treat the case when $F^{-1}(y)$ has some singularities.

We can see that when the target space is R and the source is R^N with a big N. We consider as random variable a non degenerate quadratic form Q. $E[(dQ^t dQ)^{-p}]$ is finite for bigger and bigger p when $N \to \infty$, which shows that the law of Q is more and more regular when $N \to \infty$.

We are concerned in this part by an infinite dimensional generalization of this remark, and we will treat in the third part the problem of the estimation of the derivative, which can be handled, as we will see, by using some mollifers. That is, we take $N = \infty$, and we consider the canonical space $C([0,1]; R^m)$ of continuous paths $w_.$ ($B_0 = 0$) in R^m endowed with the uniform topology and the Brownian measure as non degenerate Gaussian measure. There is an underlying Hilbert space, the Cameron-Martin space, H, which is constituted of integrals $\int_0^\cdot h_s ds$ endowed with the Hilbert structure $\int_0^1 |h_s|^2 ds = \|h\|^2$. Formally, the Brownian measure is the measure over H $C\exp[-\|h\|^2/2]dD(h)$ where $dD(h)$ is the formal Lebesgue measure on H. Unfortunately, this leads to some problems of measure theory, and this measure lives in fact over $C([0,1]; R^m)$ instead of H, or on the $1/2 - \epsilon$ path.

Malliavin's point of view works when we consider $C([0,1]; R^m)$. Malliavin established a differential Calculus, where there is no Sobolev imbedding ([Ma]): it is possible to find functionals which belong in infinite dimension to all Sobolev spaces and which are only almost surely defined, unlike the case of the finite dimension. The big rupture of Malliavin Calculus with respect of its preliminary versions (see works of Hida, Albeverio, Fomin, Elworthy..) is namely to complete the differential operations on the Wiener space in all the L^p. Since there is no Sobolev imbedding in infinite dimension, it is possible to find functionals which are only almost surely defined, although they belong to all the Sobolev spaces. The stochastic gradient DF of F is random application from H into the target space. We get by this procedure the notion of first order Sobolev norm $W_{1,p}$ of functionals such that DF belongs in L^p. We can iterate the notion of stochastic derivative, and we get the notion of higher Sobolev spaces $W_{k,p}$. We can interpret the concept of Gram matrix in this situation, and we get the Malliavin matrix $DF^t DF$, which is a random matrix. Malliavin's theorem is the following: if F belongs to all the Sobolev spaces and if the inverse of its Malliavin matrix belong to all the L^p, the law of F has a smooth density with respect to the Lebesgue measure over R^d.

A functional may belong to all the Sobolev spaces and may be only surely defined. The main example of Malliavin for that is the following: we consider a finite dimensional manifold M (not necessarily compact), and some smooth vector fields $X_i, i = 0,..,m$ with **compact** supports in M. Malliavin studies the case of the stochastic differential equation in Stratonovitch sense:

$$dx_t(x) = X_0(x_t(x))dt + \sum_{i>0} X_i(x_t(x)) \circ dw_t^i \tag{1.3}$$

starting from x. Since the vector fields have compact supports, we can perturb dw_t^i into $dw_t^i + \lambda h_t^i dt$, and we get the solution $x_t(\lambda)$ of the deduced stochastic differential equation from (2.3). $x_1(\lambda)$ is almost surely smooth in λ, and we can take its derivative in $\lambda = 0$, by doing the formal computations as if it were an ordinary differential equation instead of a stochastic differential equation. The computations are only almost surely true. This shows that $x_1(x)$ belongs to all the Sobolev spaces of Malliavin Calculus: we have some small modifications which are due to the fact we work over M instead of R^d (We refer to [Me] for this statement). In order to study the regularity of the law of $x_1(x)$, it is enough to study the invertibility in all the L^p of the Malliavin matrix of $x_1(x)$. The inverse of the Malliavin matrix belongs to all the L^p if the weak Hoermander hypothesis is checked in x.. We refer to [N] for a simple proof of this result.

Let us look now at Bismut's point of view. Instead of considering the stochastic differential equation in Stratonovitch sense (1.3), we consider the ordinary differential equation starting from x:

$$dx_t(h) = X_0(x_t(h))dt + \sum_{i>0} X_i(x_t(h))h_t^i dt \tag{1.4}$$

Since the vector fields have compact support, $h \to x_1(h)$ is Frechet smooth from H into M. We can look at if it was a Frechet-submersion in h. In particular, it is a submersion in $h = 0$ if the vector fields X_i, $i \neq 0$ spann the tangent space at x (Elliptic situation).

The importance of the fact that in (1.4) the vector fields have compact support can be seen as follows: if they have no compact supports, the solution $x_t(h)$ of (1.4) can go to infinity with an exit time $\tau(h)$ which is not differentiable. In (1.4), if the vector fields have no compact supports, the exit time $\tau(x)$ of the diffusion of the manifold does not belong in general to the Sobolev spaces of Malliavin Calculus.

The goal of this communication is to remove the boundedness or compactness assumptions in Malliavin Calculus, by using some suitable mollifiers. We get a generalization of the positivity theorem of Ben Arous-Léandre for a compact manifold to a general manifold. This allows us to extend to the non-bounded case some short time asymptotics for hypoelliptic heat-kernels by Malliavin Calculus before in the compact case. We refer to the surveys of Léandre ([L_4], [L_6]), of Kusuoka ([Ku]) and Watanabe ([Wa]) for applications of Malliavin Calculus to heat kernels. Let us remark than the pioneering works about probabilistic methods for heat kernels are the works of Molchanov ([Mo]) in the Riemannian case and of Gaveau in the hypoelliptic case ([Ga]). This communication is a shorter version of [L_{11}] and [L_{13}].

2 A mollifier on the Wiener space

Let us introduce the solution of the stochastic differential equation in Stratonovitch sense, where w_t^i are some independent Brownian motions:

$$dx_t(x) = X_0(x_t(x))dt + \sum_{i>0} X_i(x_t(x)) \circ dw_t^i \tag{2.1}$$

starting from x, which represents the semi-group associated to the Hoermander's type operator $L = X_0 + 1/2\sum_{i>0} X_i^2$. Under weak Hoermander's hypothesis in x, the semi-group $\exp[tL]$ is represented by an heat-kernel $p_t(x,y)$ with respect of the Riemannian measure of the Riemannian manifold. Let us introduce the exit time τ of the manifold. If f is a smooth function on M, we have classically (See [I.W], [Nu]):

$$\int p_1(x,y)f(y)dy = E[f(x_1(x))1_{\tau>1}] \tag{2.2}$$

In general, we cannot apply Malliavin Calculus to the diffusion $x_t(x)$. In order to be able to apply Malliavin Calculus, we introduce the mollifiers of Jones-Léandre ([J.L]) and Léandre ([L$_3$]). We consider a smooth function d from M into R^+, equal to 0 only in x and which tends to ∞ when y tends to infinity, the one compactification point of M. We consider a smooth function over $]-k,k[$ ($k \in R^+$), equals to 1 over $[-k/2,k/2]$ and which behaves as $\frac{1}{(k-y)^{+r}}$ when $y \to k_-$. Outside $]-k,k[$, this function, called $g_k(y)$ is equals to ∞. We suppose that $g_k \geq 1$.

We choose a big integer r. We choose a smooth function from $[1,\infty[$ into $[0,1]$, with compact support, equals to 1 in 1 and which decreases.

The mollifier functional of Jones-Léandre ([J.L]) is

$$F_k = h(\int_0^1 g_k(d(x_s(x)))ds) \tag{2.3}$$

Lemma 2.1: F_k belongs to all the Sobolev spaces in the sense of Malliavin Calculus if r is big enough, and is equal to 1 if $\sup_s d(x_s(x)) \leq k/2$, is smaller to 1 if $\sup_s d(x_s(x)) > k/2$ and is equal to 0 almost surely if $\sup_s d(x_s(x)) \geq k$. Moreover, $F_k \geq 0$.

Proof of the lemma: the support property of F_k comes from the fact that the paths of the diffusion $s \to x_s(x)$ are in fact almost surely Hoelder with an Hoelder exponent strictly smaller than 1/2, instead of being only continuous.

Let us show that F_k belongs to all the Sobolev spaces.

Let us introduce some smooth vector fields X_i^k which are equal to X_i for $d \leq k$ and which are equal to 0 if $d \geq k+1$. We consider the stochastic differential equation in Stratonovitch sense starting from x:

$$dx_t^k(x) = X_0^k(x_t^k(x))dt + \sum_{i>0} X_i^k(x_t^k(x)) \circ dw_t^i \tag{2.4}$$

Since we consider a Stratonovitch equation, its solution is the limit in all the L^p of the solution of the random ordinary differential equation got when we replace the Stratonovich differential dw_t^i by the random ordinary differential of the polygonal approximation of the leading Brownian motion. It is called Wong-Zakai approximation ([I.W]). We put

$$\tilde{F}_k = h(\int_0^1 g_k(d(x_s^k(x)))ds) \tag{2.5}$$

We get clearly $\tilde{F}_k = F_k$. The interest to use the diffusion $x_t^k(x)$ instead of the initial diffusion is that we can apply Malliavin Calculus to it (See [J.L], [L_9]). $\diamondsuit$

3 Posivity theorem for a general manifold

We get by using the mollifier of the previous section a generalization of the positivity theorem of Ben-Arous-Léandre. An abstract version in the bounded case was given by Aida-Kusuoka-Stroock ([A.K.S]). Léandre [L_6] has given a generalization of this theorem for a jump process. Bally and Pardoux ([B.P]) have given an extension of this positivity theorem to a stochastic heat equation. A.Millet and M. Sanz-Solé ([M.S]) have given a generalization of this theorem to the case of a stochastic wave equation. Fournier ([F]) has generalized the theorem of Léandre to the case of a non-linear jump process associated to a Boltzmann equation. Léandre ([L_{12}]) has studied the case of a delay equation on a manifold. All these works were done under the traditional boundedness assumptions of Malliavin Calculus.

The following theorem avoids this assumption. We suppose that weak Hoermander's hypothesis is checked in all points x: the Lie ideal spanned by the vector fields $X_i, i \neq 0$ in the Lie algebra spanned by **all** the vector fields X_i is equal in x to the tangent space in x of the manifold M.

Theorem 3.1: $p_1(x, y) > 0$ if and only there exists an h such that $x_0(h) = x$, $x_1(h) = y$ and $h' \to x_1(h')$ is a submersion in h.

Proof: We introduce the auxiliary measure μ_k:

$$\mu_k : f \to E[F_k f(x_1(x))] \tag{3.1}$$

To the measure μ_k, we can apply Malliavin Calculus. Namely, $\mu_k[f] = E[\tilde{F}_k f(x_1^k(x))]$. In particular μ_k has a smaller density q_k smaller than $p_1(x, y)$. In particular, if there exists a h such that $x_1(h) = y$ and $h' \to x_1(h')$ is a submersion in h, we can find an enough big k such that $q_k(y) > 0$, by the positivity theorem of Ben Arous and Léandre ([BA.L]) in the compact case with the extra-condition that $\tilde{F}_k$ has to be strictly positive. This shows that the condition is sufficient.

In order to show that the condition is necessary, we remark that if $p_1(x, y) > 0$ in y, $q_k(y)$ is still strictly positive for k enough big, because for k big enough, for ϵ small

$$|E[(1_{\tau>1} - F_k)f(x_1(x))]| \leq \epsilon \|f\|_\infty \tag{3.2}$$

where $\|f\|_\infty$ denotes the uniform norm of f.

Therefore, it is enough to apply Ben Arous-Léandre result in the other sense. $\diamondsuit$

Remark: Let us suppose that Hoermander's condition is checked only in x. We can suppose that h is decreasing and that g_k decreases to 1, such that F_k increases to $1_{\tau>1}$. By Malliavin Calculus, μ_k has a density q_k, which increases. Let us consider the function $f = 1_A$ for a set A of measure 0 for the Lebesgue measure over M. We have:

$$\mu_k[f] = 0 \tag{3.3}$$

But

$$\mu_k[f] = E[F_k f(x_1(x))] = 0 \tag{3.4}$$

anf $F_k f(x_1(x))$ increases and tends to $1_{\tau>1} f(x_1(x))$, which is in L^1. We deduce that

$$E[1_{\tau>1} f(x_1(x))] = 0 \tag{3.5}$$

This means that the the law of $x_1(x)$ has a density without to suppose that Hoermander's hypothesis is checked in all points.

4 The main localizing lemma

Let M be a Riemannian manifold with compactification point ∞. Let π be a probability measure on M smoothly equivalent to the Riemannian measure. Let X_i $i = 0, .., m$ smooth vector fields over M. Let us suppose that X_0 is divergence free, the divergence being computed for π **(Hypothesis H(0))**. Let L be the operator:

$$L = X_0 + 1/2 \sum_{i>0} X_i^* X_i \tag{4.1}$$

It can be written under Hoermander's form (see [H])

$$L = \tilde{X}_0 + 1/2 \sum_{i>0} X_i^2 \tag{4.2}$$

Let E be the Dirichlet form associated to the symmetric operator $\sum X_i^* X_i$.

Suppose **(Hypothesis H(1))** the following Nash inequality:

$$\|f\|^2_{L^2(\pi)} \leq C(E(f,f)^{N/N+2}) \tag{4.3}$$

for some N for all f of $L^1(\pi)$ norm equal to 1. Let us consider the Hilbert space H of L^2 functions h_t^i from [0,1] into R^m. We consider the horizontal curve:

$$dx_t(h,x) = \sum_{i>0} X_i(x_t(h,x)) h_t^i dt \ ; \ x_0(h,x) = x \tag{4.4}$$

We suppose **(Hypothesis H(2))** that the solution of (4.4) does not blow up for all x in M and all h in H.

To an horizontal curve $x_.(h,x), h \in H$ we associate its energy $\|h\|^2 = \sum \int_0^1 |h_t^i|^2 dt$. We define

$$d^2(x,y) = \inf_{x_1(h,x)=y} \|h\|^2 \tag{4.5}$$

We suppose **(Hypothesis H(3))**, that the Carnot-Caratheodory distance $(x,y) \to d(x,y)$ is continuous, finite and when one of the two points x or y tends to the compactification point of M, that $d(x,y) \to \infty$.

In the case where the manifold is compact, the strong Hoermander hypothesis (Hypothesis H(4)) in all x implies Hypothesis H(1), H(2) and H(3). We consider the stochastic differential equation in Stratonovitch sense associated to the semi- group $\exp[-tL]$

$$dx_t(x) = \tilde{X}_0(x_t(x))dt + \sum_{i>0} X_i(x_t(x)) \circ dw_t^i \ ; \ x_0(x) = x \tag{4.6}$$

where w_t^i are independent flat Brownian motions (See [I.W]). The law of $x_t(x)$ has the density $p_t(x,y)$ with respect to π. Let $\tau_R(x)$ be the exit time of the ball $B(x,R)$ of radius R and center x for the Carnot-Carathéodory distance.

We get:

Lemma 4.1: Under the previous assumptions, the measure $\mu_{t,R} : f \to E[1_{\tau_r(x)<t} f(x_t(x))]$ has a density $p_{R,t}(x,y)$ with respect to π bounded by $\exp[-C_R/t]$ for $t \le 1$ when $C_R \to \infty$ where $R \to \infty$. This estimate is uniform in all compact of $M \times M$.

Proof: We remark that, by Hoelder inequality and large deviations estimates, that

$$\mu_{t,R}[f] \le C \exp[-C_R/t] (\int_M f(y)^2 p_t(x,y) d\pi(y))^{1/2} \tag{4.7}$$

Moreover the density $p_t(x,y)$ of the diffusion $x_t(x)$ $p_t(x,y)$ with respect of $d\pi(y)$ is smaller than t^{-K} when $t \to 0$ (See [L$_{10}$]). We conclude by using Kolmogorov lemma and the time reversed process.

$\diamondsuit$

5 Varadhan estimates without boundedness assumption

Let us introduce the solution of the stochastic differential equation in Stratonovitch sense, where w_t^i are some independent Brownian motions:

$$dx_t(x) = \tilde{X}_0(x_t(x))dt + \sum_{i>0} X_i(x_t(x)) \circ dw_t^i \tag{5.1}$$

starting from x. Let us introduce the exit time τ of the manifold. If f is a smooth function on M, we have classically (See [I.W], [Nu]):

$$\int p_t(x,y)f(y)dy = E[f(x_1(x))1_{\tau>t}] \tag{5.2}$$

We get by using the method of the proof of [L_1], where we have replaced the role of Malliavin Calculus by the Nash inequality (See [L_{10}])in order to get the rough estimate of $p_t(x,y)$ in Ct^{-K}:

Theorem 5.1: Uniformly over all compact of M

$$\overline{\lim}_{t\to 0} 2t \log p_t(x,y) \leq -d^2(x,y) \tag{5.3}$$

In the sequel of this paper, we will do the following hypothesis **(Hypothesis H(4))**: in the starting point of the diffusion x, the Lie algebra spanned by the $X_i, i>0$ is equal to $T_x(M)$ (Strong Hoermander hypothesis in x).

In general, we cannot apply Malliavin Calculus to the diffusion $x_t(x)$. In order to be able to apply Malliavin Calculus, we introduce a mollifier F_k of the same type of mollifiers of the part 2.

We intoduce the auxiliary measure μ_k:

$$\mu_k : f \to E[F_k f(x_1(x))] \tag{5.4}$$

To the measure μ_k, we can apply Malliavin Calculus. Namely, $\mu_k[f] = E[\tilde{F}_k f(x_1^k(x))]$.

The introduction of this auxiliary measure and the Lemma 4.1 allow us to state the following theorem:

Theorem 5.2: When $t \to 0$ uniformly in y over all compact , we have the following inequality:

$$\underline{\lim}_{t\to 0} 2t \log p_t(x,y) \geq -d^2(x,y) \tag{5.5}$$

Proof: We choose k big enough in (5.4). By the technics of [L_2] and [L_3], the density $q_t^k(x,y)$ of μ_k satisfy to (5.5) over the chosen compact in y of M. Moreover, by Lemma 4.1, it differs of $p_t(x,y)$ by $\exp[-C/t]$ where C is much more bigger than $d^2(x,y)$ where y describes our compact neighborhood.
$\diamondsuit$

6 Asymptotic expansion without boundedness assumption

Let us recall some statements of [L_3]. A bicaracteristic issued of x and of cotangent vector q is the solution of the differential equation on $T^*(M)$:

$$\begin{aligned} d\gamma_t(x,q) &= \sum_{i>0} < q_t, X_i(\gamma_t(x,q)) > X_i(\gamma_t(x,q))dt \\ dq_t &= -\sum_{i>0} < q_t, X_i(\gamma_t(x,q)) >^t \frac{\partial}{\partial x} X_i(\gamma_t(x,q)) q_t dt \end{aligned} \tag{6.1}$$

In order to give a rigorous meaning to this equation, we have imbedded M into R^r such that $T^*(M)$ is embedded into $R^r \times R^r$. We will say that x and y does not belong to the cut-locus of the Carnot-Carathéodory metric if the following conditions are checked:

i) There exists a unique $h \in H$ such that $d^2(x,y) = \|h\|^2$ and such that $x_1(h,x) = y$.

ii) $t \to x_t(h,x)$ is a bicaracteristic $\gamma_t(x,q)$.

iii) $q' \to \gamma_1(x,q')$ is a diffeomorphism of a neighborhood of q in $T^*_x(M)$ into a neighborhood of y.

Under these conditions, we get:

Theorem 6.1:: If x and y do not belong to the cut-locus of the Carnot-Carathéodory metric, we have, when $t \to 0$:

$$(6.2) \qquad p_t(x,y) = \exp[-d^2(x,y)/2t]\sqrt{t}^{-d}(\sum_{i=0}^{N} c_i(x,y)\sqrt{t}^i + O(\sqrt{t}^N))$$

where $c_0(x,y) > 0$ and where d is the dimension of M.

Proof: We consider k big enough in (5.4). The density of $q_t^k(x,y)$ is equal to $p_t(x,y)$ modulo $\exp[-C/t]$ for a very big C. We apply the techniques of [L_3] (see [BA_1] too) in order to show by using Malliavin Calculus that $q_t^k(x,y)$ has the asymptotic expansion (6.3).

$\diamondsuit$

Remark: We can apply this localization technic to the case where the two points are joined by a finite dimensional manifold of bicaracteristics (See [T.Wa]).

Let us suppose from now that $X_0(x) = 0$ at the starting point **(Hypothesis H(5))**.

Let $N(x)$ be the grad of the Lie algebra spanned by the $X_i, i > 0$ and $\tilde{X}_0$, $\tilde{X}_0$ alone excluded where we count 2 the weight of $\tilde{X}_0$.

We get:

Theorem 6.2:

i)Let us suppose that $X_0 = 0$ identically. Then there exists an asymptotic expansion

$$(6.3) \qquad p_t(x,x) = \sqrt{t}^{-N(x)} \sum_{i=0}^{N} c_i(x)\sqrt{t}^i + O(\sqrt{t}^N)$$

where $t \to 0$ and where $c_0(x) > 0$.

ii)Let us suppose only $H(5)$. Then the asymptotic expansion (6.3) is still true, but we don't know if $c_0(x) > 0$. Otherwise, all the terms of the asymptotic expansion are 0.

Proof: We consider $q_t^k(x,x)$ the density of μ_k at x. We can apply the tools of Malliavin Calculus to $q_t^k(x,x)$. In particular, it checks (6.3) by the technics of [BA_2], [T] and [L_7]. Moreover, $q_t^k(x,x)$ differs from $p_t(x,x)$ by an exponentially small term. Therefore i). ii) comes from the same considerations by using the analoguous result of [BA.L] in the bounded case.

$\diamondsuit$

Remark: Léandre has introduced in [L_5] the correct "semi-distance" associated to the hypoelliptic heat-kernel with drift, and had shown that the volume of the balls in short time associated to it is related to the theoretical integer $N(x)$.

7 Non exponential decay without boundedness assumption

We work now over R^d, and we suppose that we are in the situation ii) of Theorem IV.2 where the asymptotic expansion is trivial.

We assume the following hypothesis **(Hypothesis H(6))**: there exist two real numbers strictly positive C and C', an integer n and a real strictly positive constant K such that for all integers r, for all $i, 0 \leq i \leq d$

$$\sup_{|x-y|\leq K} |D^r X_i(y)| \leq C' r!^{Cr^n} \tag{7.1}$$

This hypothesis shows that in some sense, the vector fields belong to some generalized Gevrey class, such that the heat kernel when $t \to 0$ belongs to a generalized Gevrey class, which allows us to get an estimate of the decay of the heat kernel. This hypothesis is checked when the vector fields are polynomial near the starting point.

We get the following theorem:

Theorem 7.1: There exists a real $\alpha_0 > 1$ such that for all $t \leq 1$

$$p_t(x,x) \leq C \exp[-|\log t|^{\alpha_0}] \tag{7.2}$$

Proof: We choose k enough small. $q_t^k(x,x)$ and $p_t(x,x)$ differ by a term exponentially small. We can apply Malliavin Calculus to $q_t^k(x,x)$ and the technics of [F.L] in order to show that

$$q_t^k(x,x) \leq C \exp[-|\log t|^{\alpha_0}] \tag{7.3}$$

$\diamondsuit$

References

[A.K.S] Aida S. Kusuoka S. Stroock D.W.: On the support of Wiener functionals. In "Asymptotics problems in probability theory: Wiener functionals and asymptotics". Elworthy K.D. Kusuoka S. Ikeda N. edit. Pitman Res. Math. Series. 284 (1993), 3-34.

[A] Alexopoulos G.: Sub-Laplacians with drift on Lie groups of polynomial volume growth. Memoirs A.M.S 739 (2002).

[Az] Azencott R.: Grandes déviations et applications. Ecole de Probabilités de Saint-Flour VIII. P. Hennequin edit. L.N.M. 774 (1980), 1-176.

[B.P] Bally V. Pardoux E.: Malliavin Calculus for white noise driven parabolic S.P.D.E.S. Potential Analysis. 9.1. (1998), 27-64.
[BA_1] Ben Arous G.: Développement asymptotique du noyau de la chaleur hors du cut-locus. Ann. Sci. Eco. Norm. Sup. 21 (1988), 307-331.
[BA_2] Ben Arous G.: Développement asymptotique du noyau de la chaleur hypoelliptique sur la diagonale. Ann. Inst. Fourier. 39 (1989), 73-99.
[BA.L] Ben Arous G. Léandre R.: Décroissance exponentielle du noyau de la chaleur sur la diagonale (II). Prob. Th. Rel. Fields. 90 (1991), 372-402.
[Bi] Bismut J.M.: Large deviations and the Malliavin Calculus. Progress in Math. 45. Birkhauser (1984).
[C.K.S] Carlen E. Kusuoka S. Stroock D.W.: Upper bounds for symmetric transition functions. Ann. Inst. Henri. Poincaré. 23 (1987), 145-187.
[D] Davies E.B.: Heat kernel and spectral theory. Cambridge Tracts in Math. 92 (1990).
[F.L] Florchinger P. Léandre R.: Décroissance non exponentielle du noyau du noyau de la chaleur. Prob. Th. Rel. Fields. 95 (1993), 237-267.
[F] Fournier N.: Strict positivity of the solution to a 2-dimensional spatially homogeneous equation with cuttoff. Ann. Inst. Henri. Poincaré. Probab. stat. 31.4. (2001), 481-502.
[F.W] Freidlin M.I. Wentzell A.D.: Random perturbations of dynamical systems. Springer. (1984).
[Ga] Gaveau B.: Principe de moindre action, propagation de la chaleur et estimées sous-elliptiques sur certains groupes nilpotents. Acta. Math. 139 (1977), 95- 153.
[H] Hoermander L.: Hypoelliptic second order equation. Acta. Math. 119 (1967), 147-171.
[I.W] Ikeda N. Watanabe S.: Stochastic differential equations and diffusion processes. North-Holland (1981).
[J.L] Jones J.D.S. Léandre R.: A stochastic approach to the Dirac operator over the free loop space. In "Loop spaces and groups of diffeomorphisms". Proc. Steklov. Inst. 217 (1997), 253-282.
[J.S] Jerison D. Sanchez A.: Subelliptic second order differential operator. In "Complex analysis. III" Berenstein E. Ed. L.N.M. 1227 (1987), 46-78.
[Ku] Kusuoka S.: More recent theory of Malliavin Calculus. Sugaku 5 (1992), 155-173.
[L_1] Léandre R.: Minoration en temps petit de la densité d'une diffusion dégénérée. J. Funct. Ana. 74 (1987), 399-414.
[L_2] Léandre R.: Majoration en temps petit de la densité d'une diffusion dégénérée. Prob. Th. Rel. Fields. 74 (1987), 289-294.
[L_3] Léandre R.: Intégration dans la fibre associée a une diffusion dégénérée. Prob .Th. Rel. Fields. 76 (1987), 341-358.
[L_4] Léandre R.: Appliquations quantitatives et qualitatives du Calcul de Malliavin. In "French-Japanese Seminar". M. Métivier S. Watanabe edt. L.N.M. 1322 (1988), 109-133. English translation: in "Geometry of random motion". R. Durrett M. Pinsky edit. Contemporary Math 75 (1988), 173-197.

[L_5] Léandre R.: Volume de boules sous-Riemanniennes et explosion du noyau de la chaleur au sens de Stein. Séminaire de Probabilités XXIII Azéma J. Meyer P.A. edit. L.N.M. 1372 (1989).
[L_6] Léandre R.: Strange behaviour of the heat kernel on the diagonal. In "Stochastic processes, physics and geometry". S. Albeverio edit. World Scientific (1990), 516-527.
[L_7] Léandre R.: Développement asymptotique de la densité d'une diffusion dégénérée. Forum Math. 4 (1992), 45-75.
[L_8] Léandre R.: A simple proof of a large deviation theorem. In "Stochastic analysis". D. Nualart, M. Sanz-Solé edit. Prog. Prob. 32 Birkhauser (1993), 72-76.
[L_9] Léandre R.: Brownian motion over a Kaehler manifold and elliptic genera of level N. In "Stochastic analysis and applications in Physics". R. Sénéor L. Streit edt. Nato Asie Series. 449 (1994), 193-219.
[L_{10}] Léandre R.: Uniform upper bounds for hypoelliptic kernels with drift. J. Math. Kyoto. Univ. 34 (1994), 263-271.
[L_{11}] Léandre R.: Positivity theorem for a general manifold. Preprint (2002).
[L_{12}] Léandre R.: Positivity theorem for a stochastic delay equation on a manifold. To be published in the proceedings of the 8-th conference of Vilnius (2002. V. Paulauskas edit). Acta. Appli. Mathe.
[L_{13}] Léandre R.: Stochastic Mollifier and Nash inequality. To be published in the proceedings of the satellite conference of stochastic Analysis of I.C.M. 2002 (Ma Z.M Roeckner M. edit).
[Ma] Malliavin P.: Stochastic Calculus of variation and hypoelliptic operators. In "Stochastic analysis" Itô K. edt. Kinokuyina (1978), 155-263.
[Me] Meyer P.A.: Le Calcul de Malliavin et un peu de pédagogie. R.C.P. 25. Volume 34. Publ. Univ. Strasbourg (1984), 17-43.
[M.S] Millet A. Sanz-Solé M.: Points of positive density for the solution to a hyperbolic S.P.D.E. Potential Analysis. 7.3. (1997), 623-659.
[Mo] Molchanov S. : Diffusion processes and Riemannian geometry. Russ. Math. Surveys. 30 (1975), 1-63.
[N] Norris J.: Simplified Malliavin Calculus. Séminaire de Probabilités XX. Azéma J., Yor M. edit. L.N.M. 1204 (1986), 101-130.
[Nu] Nualart D.: The Malliavin Calculus and related topics. Springer (1995).
[T] Takanobu S.: Diagonal short time asymptotics of heat kernels for certain second order differential operator of Hoermander type. Public. RIMS. Kyoto Univ. 24 (1988), 169-203.
[T.Wa] Takanobu S. Watanabe S.: Asymptotic expansion formulas of the Schilder type for a class of conditional Wiener functional integration. In "Asymptotics problems in probability theory: Wiener functionals and asymptotics". K.D. Elworthy N. Ikeda edit. Pitman. Res. Notes. Math. Series. 284 (1993), 194-241.
[V.S.C] Varopoulos N. Saloff-Coste L. Coulhon T.: Analysis and geometry on groups. Cambridge Tracts in Maths. 100 (1992).
[Wa] Watanabe S.: Stochastic analysis and its application. Sugaku 5 (1992), 51-71.