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Abstract

We review some recent results on quantum motion analysis, and in particular lower bounds for moments in quantum dynamics. The goal of the present exposition is to stress the role played by quantities we shall call Transport Integrals and by the so called generalized dimensions of the spectral measure in the analysis of quantum motion. We start with very simple derivations that illustrate how these quantities naturally enter the game. Then, gradually, we present successive improvements, up to most recent result.

1 Introduction to the subject

Let $H$ be a self-adjoint operator acting on a Hilbert space $\mathcal{H}$, and $U(t) = e^{-iHt}$ the strongly continuous one parameter unitary group that $H$ generates. We denote by $\mathcal{D}(H) \subset \mathcal{H}$ the domain of $H$. The example to keep in mind is $H$ a Schrödinger operator acting either on $l^2(\mathbb{Z}^d)$ or $L^2(\mathbb{R}^d)$. The goal is to study the time behaviour of the solution $\psi_t$ of the Schrödinger equation $-i\partial_t \psi_t = H\psi_t$, with initial condition $\psi_{t=0} = \psi \in \mathcal{D}(H)$ of norm 1. The solution is given by $\psi_t = U(t)\psi$. Our basic tool from operator theory is the spectral theorem, which reads here: there exists a positive Borel measure $\mu_\psi$ of total mass 1, called the spectral measure associated to $\psi$, s.t. for all bounded Borel function $f$,

$$\langle \psi, f(H)\psi \rangle = \int_{\mathbb{R}} f(\lambda)\mu_\psi(d\lambda). \quad (1.1)$$

Replacing $f$ by $e^{-i\lambda t}$ leads to $\langle \psi, U(t)\psi \rangle = \widehat{\mu_\psi}(t)$, the Fourier transform of $\mu_\psi$. So that looking for the behaviour of $\langle \psi, U(t)\psi \rangle$ as $t \to \infty$ reduces to the understanding of the asymptotic behaviour of $\widehat{\mu_\psi}(t)$. It is thus quite clear that the regularity of $\mu_\psi$ plays an important role in the understanding of the solution $\psi_t$. The first remark is that, by the Riemann-Lebesgue Lemma, one has $\langle \psi, U(t)\psi \rangle \to 0$ as soon as $\mu_\psi$ is absolutely continuous with respect to Lebesgue. Note that, in some vague sense, $\langle \psi, U(t)\psi \rangle \to 0$ means that the solution $\psi_t$ dissociates itself from its initial state, which can be read as a weak notion of delocalization or quantum transport. On the opposite side one expects singular measures (say atomic) to be associated to localization or absence of quantum
transport. Here are further signs that reinforce this picture. Consider the correlation coefficient

$$C(T) = \frac{1}{T} \int_0^T |\langle \psi, U(t)\psi \rangle|^2 dt = \frac{1}{T} \int_0^T |\tilde{\mu}(t)|^2 dt. \quad (1.2)$$

Decompose $\mu_\psi$ into its atomic and continuous parts: $\mu_\psi = \sum_n a_n \delta_{\lambda_n} + \nu_\psi$, where $\delta_\lambda$ is the Dirac mass. It is a rather immediate consequence of (1.1) and the Lebesgue Dominated Convergence Theorem that $\lim_{T \to \infty} C(T) = \sum_n |a_n|^2$, a result called the Wiener Theorem. In particular solutions $\psi_t$ corresponding to continuous spectral measures exhibit some delocalization in the sense that $\lim_{T \to \infty} C(T) = 0$, while atoms prevent $\psi_t$ from being totally disconnected from $\psi$, in the sense that $\lim_{T \to \infty} C(T) \neq 0$. Next step is to understand how fast $C(T)$ vanishes if it does. Here again, better regularity of $\mu_\psi$ will yield faster delocalization. Imagine $\mu_\psi = f d\lambda$, with $f \in L^2(\mathbb{R})$, then Parseval formula implies $(C(T) + C(-T)) \sim \|f\|_2^2 T^{-1}$. A more refined statement is due to Strichartz [St]: Assume that $\mu_\psi$ is Uniformly $\alpha$ Hölder continuous $(U\alpha H)$, $\alpha \in (0, 1]$, namely,

$$\exists \varepsilon_0 > 0, \exists C_\alpha < +\infty, \forall \varepsilon \in [0, \varepsilon_0[, \text{ for } \mu \text{ a.e. } x \in \text{supp } \mu : \mu_\psi(B(x, \varepsilon)) \leq C_\alpha \varepsilon^\alpha, \quad (1.3)$$

where $B(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$. Then $C(T) \leq C'T^{-\alpha}$. In particular

$$\liminf_{T \to \infty} \frac{\log C(T)}{\log 1/T} \geq \alpha. \quad (1.4)$$

We shall recover this result Line (1.10). In other terms, in the sense of correlation coefficient, the more the spectral measure is regular, the stronger is the delocalization effect. This picture will be confirmed in the sense of time behaviour of moments $\mathcal{M}(\mu, \psi, T)$ defined in (1.17) (see (3.8) and below).

However, during the 90’s, thanks to some concrete models, it became clear that taking only into account the regularity of spectral measures cannot explain totally quantum transport. Fast delocalization (in terms of moments $\mathcal{M}(\mu, \psi, T)$) has been shown to occur even in presence of poor regularity of the spectral measure [La, BCM, DBF, Ma, DRJLS] (or more recently [JSBS, DT, DST, GKT, Tc2]).

Therefore, thinner properties of measures rather than just regularity, should play an important role in a theory of quantum transport. In this context, the goal of the present note is to stress the importance of a family of integrals that we shall call Transport Integrals, as well as their normalized growth exponents (we note that these integrals already appeared in the context of quantum dynamics in [SBB]).

**Definition 1.1 (Transport Integrals).** Let $\mu$ be a probability measure on $\mathbb{R}$. For all $q \in \mathbb{R}$ and $\varepsilon \in (0, 1)$, set

$$I_\mu(q, \varepsilon) = \int_{\text{supp } \mu} \mu(x - \varepsilon, x + \varepsilon)^{q-1} d\mu(x) \in [0, +\infty]. \quad (1.5)$$

For $q \neq 1$, we further define the lower and upper Hentschel-Proccacia dimensions

$$D^-_\mu(q) = \liminf_{\varepsilon \to 0} \frac{\log I_\mu(q, \varepsilon)}{(q - 1) \log \varepsilon}, \quad D^+_\mu(q) = \limsup_{\varepsilon \to 0} \frac{\log I_\mu(q, \varepsilon)}{(q - 1) \log \varepsilon}, \quad (1.6)$$

taking values in $[0, +\infty]$. 

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These dimensions $D_2^\nu(q)$ have many other names: generalized dimensions, generalized fractal dimensions, multifractal dimensions, generalized entropy dimensions.

A simple way to be convinced that these integrals and dimensions play a role in quantum transport is to do the following calculation:

\[
C(T) \leq c \int \int \mu_\psi(dx) \mu_\psi(dy) \int_{-\infty}^{+\infty} e^{-i(\frac{y}{T})^2} e^{-i(x-y)t} \frac{dt}{T} \tag{1.7}
\]

\[
= c' \int \mu_\psi(dx) \left( \int \mu_\psi(dy) \exp(-((x-y)T)^2) \right) \tag{1.8}
\]

\[
\sim \int \mu_\psi(dx) \left( \int \mu_\psi(dy) \chi_{[-1,1]}((x-y)T) \right) \tag{1.9}
\]

\[
= \int \mu_\psi(dx) \mu_\psi(x - \varepsilon, x + \varepsilon) = I_{\mu_\psi}(2, \varepsilon), \quad \varepsilon = \frac{1}{T}. \tag{1.10}
\]

The equivalence sign in (1.9) means: $f \sim g$ iff there exists $C > 0$ finite s.t. $C^{-1}g \leq f \leq Cg$. It is a consequence of a more general result proved in [BGT3, Tc1]. On the lower bound side, note that

\[
C(T) = c \int \int \frac{\sin((x-y)T)}{(x-y)T} \mu_\psi(dx) \mu_\psi(dy) \tag{1.11}
\]

\[
\geq c \int \int_{|x-y| \leq \frac{1}{T}} \frac{\sin((x-y)T)}{(x-y)T} \mu_\psi(dx) \mu_\psi(dy) = c' I_{\mu_\psi}(2, \frac{1}{T}), \tag{1.12}
\]

since $\frac{\sin u}{u} \geq \sin 1$ for $u \in [0, 1]$. As a consequence (a result that goes back, at least, to [SBB, BCM])

\[
\lim_{T \to \infty} \inf \frac{\log C(T)}{\log 1/T} = D_2^{-}(2), \quad \lim_{T \to \infty} \sup \frac{\log C(T)}{\log 1/T} = D_2^{+}(2). \tag{1.13}
\]

Compare to (1.4), and note that $D_2^{-}(2)$ can be significantly greater than $\alpha$. For instance consider $\mu^{(a)}(dx) = \chi_{[0,1]}(x) x^{-a} dx$, $a \in [0, 1]$. Then short calculations show that $\mu^{(a)}$ is uniformly $(1-a)$-continuous (and not better), while $D_{\mu^{(a)}}^{-}(2) = 2(1-a)$ if $a \in [\frac{1}{2}, 1]$ and $D_{\mu^{(a)}}^{-}(2) = 1$ if $a \in [0, \frac{1}{2}]$ (use the equivalence with the Rényi sums – see next section – or look at [BGT3, Section 6, Example 5]). In Proposition 2.1 we shall actually see to which dimension UoH continuity is related.

What makes the difference is that integrals $I_{\mu}(q, \varepsilon)$ not only exploit local regularities of the measure $\mu$, but they also take advantage of statistical effects of these regularities. This will be of crucial importance when we shall observe quantum transport in presence of pure point spectrum. It will be sufficient that, $\varepsilon$ being given, there are enough points of local regularity (i.e. with $\mu(B(x, \varepsilon)) = C \varepsilon^a$) to give large values to the generalized dimensions. See discussion below Proposition 2.1.

We turn to the main dynamical quantities we want to study: moments. To do that, we first need to specify a bite more what our setting is. While it is very possible to investigate moments with respect to any orthonormal basis of the Hilbert space $\mathcal{H}$, we are mostly interested in operators coming from quantum mechanics, and more precisely, in Schrödinger operators acting on $\mathcal{H} = l^2(\mathbb{Z}^d)$ or $\mathcal{H} = L^2(\mathbb{R}^d)$. Thus moments should
refer to “physical position operators” \( (X)^p \) (namely: multiplication by \( (X)^p \)), rather than to an abstract basis\(^1\), where for \( x \in \mathbb{Z}^d \) or \( \mathbb{R}^d \) we use the following notations

\[
\langle x \rangle = \sqrt{1+|x|^2} \quad \text{and} \quad \langle (X)^p \rangle(x) = \langle x \rangle^p \psi(x), \ \psi \in \mathcal{H}.
\] (1.14)

In the continuous case \( \mathcal{H} = L^2(\mathbb{R}^d) \) we assume that the potential satisfies to the following regularity property:

\[
V = V^{(1)} + V^{(2)}, \quad \text{with} \quad 0 \leq V^{(1)} \in L^1_{\text{loc}}(\mathbb{R}, dx),
\] (1.15)

and \( V^{(2)} \) is relatively \(-\Delta\) form-bounded with relative bound \( < 1 \). To fix notations we thus require the existence of two constants \( \Theta_1 < 1 \) and \( \Theta_2 \) such that

\[
\left| \langle \psi, V^{(2)} \psi \rangle \right| \leq \Theta_1 \| \nabla \psi \| + \Theta_2 \| \psi \| ^2, \quad \text{for all} \ \psi \in \mathcal{D}(\nabla).
\] (1.16)

**Definition 1.2 (Moments and transport exponents).** For \( p \in ]0, +\infty[ \), we define the (time-averaged) moment of order \( p \), with initial state \( \psi \) and at time \( T > 0 \), \( \mathbb{M}(p, \psi, T) \), and the corresponding (normalized) lower and upper transport exponent, \( \beta^{\pm}(p, \psi) \), as

\[
\mathbb{M}(p, \psi, T) = \frac{1}{T} \int_0^T \langle \psi_t, (X)^p \psi_t \rangle \, dt, \quad \beta^{\pm}(p, \psi) = \lim_{T \to \infty} \sup \frac{\log \mathbb{M}(p, \psi, T)}{p \log T}.
\] (1.17)

Note that we implicitly assume that \( \psi_t \) belongs to the domain of \( (X)^p \) for all time \( t \).

As a general result,

**Proposition 1.1 ([GK]).** Let \( f \in C^\infty_c(\mathbb{R}) \), and \( \psi = f(H)\chi_0 \). Then \( \psi_t \) belongs to the domain of \( (X)^p \) for all \( p > 0 \) and \( t \in \mathbb{R} \). And one has

(i) \( \beta^{\pm}(p, \psi) \) are increasing functions of \( p \).

(ii) \( \beta^{\pm}(p, \psi) \in [0, 1] \), for all \( p \).

The central result we want to present and to discuss here is the

**Theorem 1.1 ([BGT1]).** Let \( f \in C^\infty_c(\mathbb{R}) \), and \( \psi = f(H)\chi_0 \). For all \( p > 0 \), there exists a finite constant \( C_p > 0 \), such that

\[
\mathbb{M}(p, \psi, T) \geq \left( \frac{C_p}{\log T} I_{\mu_\psi}(q, T^{-1}) \right)^{\frac{1}{q}}, \quad q = \frac{1}{1+p/d}.
\] (1.18)

As a consequence, for all \( p > 0 \),

\[
\beta^{\pm}(p, \psi) \geq \frac{1}{d} D^{\pm}_{\mu_\psi}(q), \quad q = \frac{1}{1+p/d}.
\] (1.19)

A similar result but under stronger hypotheses is proved in [GSB2]. For previous works on lower bounds, see [G, C, La, BCM, GSB1, BT, KL] and the discussion made in Section 3. A generalized version of Theorem 1.1 that takes into account the space behaviour of generalized eigenfunctions is obtained in [Te1]. We also refer the reader

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\(^1\)In the discrete case \( \mathcal{H} = l^2(\mathbb{Z}^d) \), these two points of view coincide if one considers the orthonormal basis \( \{\delta_n\}_{n \in \mathbb{Z}^d} \). Then Position operators are given by \( (X)^p = \sum_{n \in \mathbb{Z}^d} (n)^p \langle \delta_n, \cdot \rangle \delta_n \).
to [BGSB] for another interesting extension of Theorem 1.1 in the context of self-dual operators.

The rest of the discussion is organized as follows. In Section 2 we go over some basic properties of these transport integrals, and in particular the equivalence with other definitions that will turn out to be quite crucial. We also get immediate lower bounds on these integrals, first using only regularity properties of the measure, and second, using the statistical effect of these integrals. In Section 3 we present the derivation of Theorem 1.1, following Guarneri's idea but showing how transport integrals enter the game. In Section 4 we review the analysis of wave-packets performed in [GKT], which leads to a characterization of transport exponents for both small and large values of the order \( p \). In Section 5 we present a general lower bound for moments in one dimension. This result is a continuation of Theorem 1.1 and Theorem 2.1 using transfer matrices. It will allow to get the first examples of Schrödinger operators with non trivial generalized dimensions, and thereby the first applications of Theorem 1.1 to Schrödinger operators.

## 2 Generalized dimensions: equivalent definitions and properties

We start with the

**Definition 2.1 (Generalized Rényi Dimensions.).** Let \( \mu \) be a probability measure on \( \mathbb{R} \), and \( q > 0 \). The associated generalized Rényi sums, resp. integrals, are given by

\[
S_\mu(q, \varepsilon) = \sum_{j \in \mathbb{Z}} \mu([j \varepsilon, (j + 1)\varepsilon[^q], \quad \text{ resp. } \quad L_\mu(q, \varepsilon) = \frac{1}{\varepsilon} \int \mu(B(x, \varepsilon))^q \, dx.
\]

(2.1)

The generalized lower and upper Rényi dimensions, resp. integral Rényi dimensions, are defined, for \( q \neq 1 \), as:

\[
RD^\pm_\mu(q) = \lim_{\varepsilon \downarrow 0} \sup_{\varepsilon \downarrow 0} \frac{\log S_\mu(q, \varepsilon)}{(q - 1) \log \varepsilon}, \quad IRD^\pm_\mu(q) = \lim_{\varepsilon \downarrow 0} \sup_{\varepsilon \downarrow 0} \frac{\log L_\mu(q, \varepsilon)}{(q - 1) \log \varepsilon},
\]

taking values in \([0, +\infty]\).

To our best knowledge, sums \( S_\mu(q, \varepsilon) \) have been introduced by Rényi in the 50’s [Re] when he developed Information Theory, generalizing the Shannon entropy. They have been shown to play a role in hyperbolic dynamical systems [Pe, TV].

That the above definitions are actually equivalent is a well-known fact for \( q > 1 \), and the proof is very simple. If \( q \in [0, 1[ \), it is a different story. If the measure is “doubling” then the proof for \( q \in [0, 1] \) reduces to the proof for \( q > 1 \) (see [Ol, Pe]). But in full generality, this equivalence for \( q \in [0, 1] \) is proved in [BGT3] and it is the content of Theorem 2.1 below. For \( q < 0 \), the above Rényi dimensions do not make sense. They are replaced by discrete sums over packings or coverings of \( \text{supp}{\mu} \). Their equivalence to integrals \( I_\mu(q, \varepsilon) \) is treated rigorously in [GT].
Theorem 2.1 ([BGT3]). Let \( \mu \) be a probability measure on \( \mathbb{R} \), and \( q \in [0, 1] \cup [1, +\infty] \). Transport integrals, Renyi sums and Renyi integrals, either all diverge, or are equivalent if they converge: there exists constants \( c_i(q) > 0 \), \( i = 1, 2, 3 \), s.t. for all \( \varepsilon \in (0, 1) \),

\[
I_\mu(q, \varepsilon) \leq c_1(q)S_\mu(q, \varepsilon) \leq c_2(q)L_\mu(q, \varepsilon) \leq c_3(q)I_\mu(q, \varepsilon). \tag{2.2}
\]

As a consequence for all \( q \in [0, 1] \cup [1, +\infty] \),

\[
D_\mu^-(q) = RD_\mu^-(q) = IRD_\mu^-(q) \quad \text{and} \quad D_\mu^+(q) = RD_\mu^+(q) = IRD_\mu^+(q). \tag{2.3}
\]

Following Theorem 2.1, and for \( q \in [0, 1] \cup [1, +\infty] \), we shall refer to any of these dimensions as to generalized dimensions, and use the notation \( D_\mu^\pm(q) \). One has the following basic properties.

Theorem 2.2 ([BGT3]). Let \( \mu \) be a probability measure on \( \mathbb{R} \). Set

\[
q_\mu^* = \inf \{ q \in \mathbb{R} \setminus \{1\}, \ D_\mu^+(q) < \infty \}. \tag{2.4}
\]

Then

(i) \( q_\mu^* \leq 1 \) and for all \( q > 1 \), \( D_\mu^\pm(q) \in [0, 1] \).

(ii) Dimensions \( D_\mu^\pm(q) \) are decreasing functions of \( q \) on \( \mathbb{R} \setminus \{1\} \), and continuous on \( q \in [q_\mu^*, 1] \cup [1, +\infty] \).

(iii) If \( \mu \) is compactly supported, then \( q_\mu^* \leq 0 \) and one has \( D_\mu^+(q) \in [0, 1] \), for \( q > q_\mu^* \).

(iv) For all \( q < 1 \) and \( q' > 1 \), one has

\[
D_\mu^-(q) \geq \dim_H(\mu) \geq D_\mu^-(q'), \quad \text{and} \quad D_\mu^+(q) \geq \dim_P(\mu) \geq D_\mu^+(q'),
\]

where \( \dim_H(\mu) \) and \( \dim_P(\mu) \) stand for, respectively, the Hausdorff and Packing dimensions of \( \mu \) (see e.g. [SBB, BGT3]).

Relations between Uniformly H\ölder continuous measure and the generalized dimensions are given by the following Proposition. It says that the best rate of UoH continuity is actually given by \( D_\mu^-(+\infty) \).

Proposition 2.1. Set \( D_\mu^\pm(+\infty) = \lim_{q \to +\infty} D_\mu^\pm(q) \).

(i) Suppose that \( \mu \) is UoH continuous for some \( \alpha \in [0, 1] \), then \( \forall q > 1 \), \( D_\mu^-(q) \geq \alpha \), and thus \( D_\mu^-(+\infty) \geq \alpha \).

(ii) Assume that for some \( q > 1 \), \( D_\mu^-(q) > 0 \). Then \( \mu \) is UoH continuous with \( \alpha = D_\mu^-(q)(1 - \frac{1}{q}) - \nu > 0 \), for any \( \nu > 0 \) small enough. Moreover one has \( D_\mu^-(+\infty) = \sup_{q>1}[D_\mu^-(q)(1 - \frac{1}{q})] \). As a consequence \( D_\mu^-(+\infty) > 0 \) and \( \mu \) is UoH continuous with \( \alpha = D_\mu^-(+\infty) - \nu > 0 \), for any \( \nu > 0 \).

(iii) More generally, one has

\[
D_\mu^-(+\infty) = \lim_{\varepsilon \downarrow 0} \inf \sup \frac{\log(\sup_{x \in B(x, \varepsilon)})}{\log \varepsilon}. \tag{2.5}
\]

And it is nonzero iff \( D^\pm(q) > 0 \) for some \( q > 1 \).

Remark 2.1. The case \( q = 2 \) has already been observed in [La, Theorem 3.1].
Proof: (i) is immediate. For (ii), notice that for $q > 1$ and all $x \in \text{supp} \mu$, $I_\mu(q, \varepsilon) \geq \int_{x}^{x+\varepsilon} \mu(B(y, \varepsilon)) \alpha(y) dy \geq \mu(B(x, \varepsilon))^{\alpha}$. On the other hand for all $\nu > 0$ and $\varepsilon$ small enough, $I_\mu(q, \varepsilon) \leq \varepsilon^{D_\mu(q)+\nu(q-1)}$. It follows, recalling (1.3) that $\mu$ is UoH continuous with $\alpha = D_\mu(q)(1-\frac{1}{q}) - \nu$ for all $q > 1$ and $\nu > 0$. It is a consequence of [BGT3, Proposition 3.4 (iii)], that $D_\mu^+(+\infty) = \sup_{q>1} \{D_\mu^+(q)(1-\frac{1}{q})\}$. Thus for all $q > 1$, $D_\mu^-(q) \geq D_\mu^+(+\infty) > 0$. As a consequence, $\mu$ is UoH continuous with $\alpha = \sup_{q>1} \{D_\mu^+(q)(1-\frac{1}{q})\} - \nu = D_\mu^-(+\infty) - \nu$ for all $\nu > 0$. And (iii) is a simple generalization of (i) and (ii).

We now show how to obtain lower bounds on the transport integrals $I_\mu(q, \varepsilon)$ and thus on the dimensions $D_\mu^\pm(q)$, $q \in [0, 1]$, by taking advantage of the equivalence given by Theorem 2.1.

Of course if $\mu$ is UoH continuous (recall (1.3)), then, with $q \in [0, 1]$: $I_\mu(q, \varepsilon) \geq C_\alpha \varepsilon^{a-1}$, and thus $D_\mu^-(q) \geq \alpha$. But this is of little interest for the best such $\alpha$ is related to the smallest possible dimensions, i.e., $D_\mu^-(+\infty)$ (Proposition 2.1). Let us have a look now to what the Rényi sums (or integrals) give, and how statistics can play a role. Assume that there are $\mathcal{O}(\frac{1}{\varepsilon})$ disjoint intervals $I_j = [j\varepsilon, (j+1)\varepsilon]$ such that $\mu(I_j) \approx c \varepsilon^a$. Then, as an immediate consequence $S_\mu(q, \varepsilon) \geq c \varepsilon^{a-q}$, and $D_\mu^+(q) \geq \frac{\log c}{\log \frac{1}{\varepsilon}}$, a bound that gets better as $q$ goes to zero. Note that the latter situation is only possible with $a \geq s$ (since for $q = 1$, $S_\mu(q, \varepsilon) = 1$). In other words, in the regime $q \in [0, 1]$, a sufficiently large number of intervals of very small weight gives a better contribution.

Looking at $L_\mu(q, \varepsilon)$, the natural condition to ask is that

$$\exists \varepsilon_0 > 0, \exists C_\alpha > 0, \forall \varepsilon \in [0, \varepsilon_0[, \text{ for Lebesgue a.e. } x \in \text{supp} \mu : \mu(B(x, \varepsilon)) \geq C_\alpha \varepsilon^{a}. \quad (2.6)$$

(We implicitly assume that $|\text{supp} \mu| > 0$). Compare (2.6) to (1.3). The following can then easily be derived.

Theorem 2.3. Define

$$D_\mu^\pm(-\infty) = \lim_{q \to -\infty} D_\mu^\pm(q), \text{ and } g_\mu^\pm = \lim_{q \to \infty} \sup_{\varepsilon > 0} \frac{\inf_{x \in B(x, \varepsilon)} \mu(B(x, \varepsilon))}{\log \varepsilon}. \quad (2.7)$$

Assume that $|\text{supp} \mu| > 0$ and that $g_\mu^\pm < \infty$. Then

$$D_\mu^+(q) \geq \frac{1 - qg_\mu^+}{1 - q}. \quad (2.8)$$

Remark 2.2. The quantity $g_\mu^+$ turns out to be of special importance. Its finiteness is crucial to get finite generalized dimensions for $q < 0$. It is indeed proved in [GT] for compactly supported measures:

(i) $g_\mu^+ < \infty \iff \forall q < 0 D_\mu^+(q) < \infty \iff q_\mu^* = -\infty \iff D_\mu^+(\infty) < \infty.$

(ii) If $g_\mu^+ < \infty$ then $D_\mu^+(\infty) = g_\mu^+.$

It also follows that if $g_\mu^+ < \infty$ then dimensions $D_\mu^\pm(q)$ are continuous on $\mathbb{R} \setminus \{1\}$, and $D_\mu^+(0) = \dim_\mu^+(\text{supp} \mu)$, where $\dim_\mu^+$ denotes the lower and upper box counting dimensions. Then one checks that (2.8) can actually be improved to $D_\mu^+(q) \geq \frac{D_\mu^+(0) - qg_\mu^+}{1 - q}$ (see [Te1]).

Compare Theorem 2.3 and Remark 2.2 (ii) to Proposition 2.1.

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3 Bounding moments from below: following Guarneri’s approach

Let us first slightly extend the simple calculation (1.7). Pick \( \psi \in \mathcal{H} \) and \( \varphi = f(H) \), \( \phi = g(H) \), \( f, g \in L^2(\mathbb{R}, d\mu_\psi) \). Set \( A_{\varphi, \phi} = \langle \varphi, \cdot \rangle\phi \). If \( h \in C_0^\infty(\mathbb{R}) \) is s.t. \( |\hat{h}| \) decays faster than any polynomial:

\[
\left| \int h(t/T)(U(t)\psi, A_{\varphi, \phi}U(t)\psi) \frac{dt}{T} \right| = \left| \int \int f(x)g(y)\hat{h}((x-y)T)\mu_\psi(dx)\mu_\psi(dy) \right| \leq \|f\|_{L^2(d\mu_\psi)}\|g\|_{L^2(d\mu_\psi)} \left( \int \mu_\psi(dx) \left( \int \mu_\psi(dy)|\hat{h}|^2((x-y)T) \right) \right)^{\frac{1}{2}} \leq c\|\varphi\|_\mathcal{H}\|\phi\|_\mathcal{H} \left( I_{\mu_\psi}(2, \frac{1}{T}) \right)^{\frac{1}{2}}. (3.1)
\]

Note that \( \|A_{\varphi, \phi}\|_{HS} = \|\varphi\|_\mathcal{H}\|\phi\|_\mathcal{H} \). And the above computation can indeed be extended from rank one operators to Hilbert-Schmidt ones:

\[
\left| \int h(t/T)(U(t)\psi, AU(t)\psi) \frac{dt}{T} \right| \leq \|A\|_{HS} \left( I_{\mu_\psi}(2, \frac{1}{T}) \right)^{\frac{1}{2}}. (3.5)
\]

This is enough to understand Guarneri’s way to bound from below moments of the dynamics. Assume that for some compact energy interval \( I, \psi = E_H(I)\psi \). One has, for \( N > 0 \) and \( h \in C_0^\infty([0,1]) \) as above,

\[
M(p, \psi, T) \geq N^p \left( \|\psi\|^2 - \left| \int h(t/T)(\psi_t, P_N E_H(I)\psi_t) \frac{dt}{T} \right| \right) \geq N^p \left( \|\psi\|^2 - \|P_N E_H(I)\|_{HS} \left( I_{\mu_\psi}(2, T^{-1}) \right)^{\frac{1}{2}} \right), (3.6)
\]

where we used (3.5). That \( \|P_N E_H(I)\|_{HS} \leq C N^{\frac{d}{2}} \) is immediate if \( \mathcal{H} = \ell^2(\mathbb{Z}^d) \). In the continuous case, under hypotheses (1.15)-(1.16), the latter follows from [KKK] (see [GKT]). Choosing the biggest \( N \) such that \( C N^{\frac{d}{2}} \left( I_{\mu_\psi}(2, T^{-1}) \right)^{\frac{1}{2}} \leq \frac{1}{2}\|\psi\|^2 \) leads to

\[
M(p, \psi, T) \geq C\|\psi\|^2 \left( \frac{\|\psi\|^4}{I_{\mu_\psi}(2, T^{-1})} \right)^{p/d} \geq \frac{1}{2}D_{\mu_\psi}(2)^p. (3.8)
\]

As a consequence \( \beta^+(p, \psi) \geq \frac{1}{2}D_{\mu_\psi}(2)^p \). It implies Guarneri’s initial result [G]: \( \beta^- (p, \psi) \geq \frac{1}{2}\alpha \) if \( \mu_\psi \) is UaH continuous. However it is not as good as what we announced in Theorem 1.1. It is not even sufficient to imply results of [La, BCM] and [GSB1].
which asserts, respectively, $\beta^-(p, \psi) \geq \frac{1}{d} \dim_H(\mu_\psi)$ and $\beta^+(p, \psi) \geq \frac{1}{d} \dim_P(\mu_\psi)$ (recall Proposition 2.2 (iv)). In particular (3.8) is still too rough to enable one to extract transport in presence of pure point measure; since atomic measures satisfy $\dim_H(\mu) = \dim_P(\mu) = D_H^\mu(q) = 0$, $q > 1$. However almost all the conceptual tools we shall need to reach Theorem 1.1 are already in place: bounding $M(p, \psi, T)$ by cutting out a ball of radius $N$, use transport integrals, take the best $N$. It is now mostly a matter of being more clever and more careful.

Introduce $\psi = \varphi + \chi$, $\varphi \perp \chi$, into the expression of $M(p, \psi, T)$. Rewrite $\chi$ as $\chi = \psi - \varphi$ in the crossed term. It leads to the following quantity that we bound as follows:

$$\left| \int h(t/T)\langle U(t)\varphi, AU(t)\psi \rangle \frac{dt}{T} \right| \leq \| A \|_{HS} \left( T^{h_{\varphi,\psi}(2,T^{-1})} \right)^{1/2}, \quad (3.9)$$

where we set $I_{\varphi,\psi}(q, \varepsilon) = \int \mu_\psi(dx)b_\psi^q(x, \varepsilon)^{q-1}$, with $b_\psi^q(x, \varepsilon) = \int d\mu_\psi(y)|h|^2((x-y)\varepsilon^{-1})$. Compare to (3.5). The proof is however trickier, and we refer to [BGT1, Theorem 3.2].

A derivation in the same spirit as before leads to

$$M(p, \psi, T) \geq C \sup_\varphi \| \varphi \|_2 \left( \frac{\| \varphi \|^4}{I_{\varphi,\psi}(2,T^{-1})} \right)^{p/d}. \quad (3.10)$$

As noticed in [BT], one may now choose the best $\varphi$ for each time $T$. It is a new degree of freedom that will enable us to extract from $\psi_t$, at time $T$, the best possible part of the wave-packet, i.e. the part with best statistics. The result, which clearly implies Theorem 1.1, is that

$$\left( \frac{C}{\log T} I_{\mu_\psi}(q, T^{-1}) \right)^{1/2} \leq \sup_\varphi \| \varphi \|_2 \left( \frac{\| \varphi \|^4}{I_{\varphi,\psi}(2,T^{-1})} \right)^{p/d} \leq C' I_{\mu_\psi}(q, T^{-1})^{1/2}, \quad (3.11)$$

with $q = (1+p/d)^{-1}$. The upper bound is Hölder inequality and equivalence of transport integrals (in the spirit of (1.8)-(1.9)). The lower bound is obtained by choosing $\varphi = \chi_{\Omega_r}(H)\psi$, with $\Omega_r = \{ x, b_\psi^q(x,T^{-1}) \in [T^{-r-\|x\|_\infty/T}, T^{-r}] \}$, and such that $I_{\varphi,\psi}(q, T^{-1}) \geq \frac{C}{\log T} I_{\mu_\psi}(q, T^{-1})$ for all $T > 0$.

4 Characterization of the transport exponents at low and large order

Varying the order $p$ of moments from 0 to $+\infty$ should favorize different parts of the wave-packet. Since normalized transport exponents $\beta^\pm(p, \psi)$ increase with $p$, it seems natural to expect that fastest parts of the wave-packet will play a bigger role as $p$ gets...
large, while a slow but essential part of the wave-packet should have more influence on
moments behaviour as \( p \) goes to zero. In this section, we want to give a precise contain
to this rough intuition.
Set, for \( \alpha \in [0, +\infty] \),
\[
P(\alpha, T) = \frac{1}{T} \int_0^T \| Q_{(T^\alpha - 2)} e^{-iHt} \psi \|_{L_2}^2 \, dt,
\]
where \( Q_N \) is the spatial projection outside the ball of radius \( N \) and center the origin.
Define \( S^\pm(\alpha) \) the growth exponents of \( P(\alpha, T) \):
\[
S^-_\psi(\alpha) = -\liminf_{T \to +\infty} \frac{\log P_\psi(\alpha, T)}{\log T}, \quad S^+_\psi(\alpha) = -\limsup_{T \to +\infty} \frac{\log P_\psi(\alpha, T)}{\log T}.
\]
If \( P_\psi(\alpha, T) = 0 \) for some \( \alpha > 0 \) starting from \( T \geq T_0 \), we set \( S^+_\psi(\alpha) = +\infty \). Definitions
are so that \( 0 \leq S^-_\psi(\alpha) \leq S^+_\psi(\alpha) \leq +\infty \). Then, natural quantities to define are
\[
\alpha^-_l = \sup\{ \alpha \geq 0 \mid S^-_\psi(\alpha) = 0 \}, \quad \alpha^+_u = \sup\{ \alpha \geq 0 \mid S^+_\psi(\alpha) < +\infty \}.
\]
Since \( S^+_\psi(\alpha) \) are non decreasing functions, \( 0 \leq \alpha^-_l \leq \alpha^+_u \leq +\infty \). One can interpret
\( \alpha^+_u \) as the (lower and upper) rates of propagation of the essential part of the wave
packet, and \( \alpha^-_l \) as the rates of propagation of the fastest (polynomially small) part
of the wave packet. Indeed, if \( S^-_\psi(\alpha) = 0 \) then roughly \( P_\psi(\alpha, T) \approx T^{-S^-_\psi(\alpha)} = O(1) \): most
of the wave-packet escapes the ball of radius \( T^\alpha \), meaning that the essential part of the
wave-packet travels faster than \( T^\alpha \); while if \( S^-_\psi(\alpha) = +\infty \) then \( P_\psi(\alpha, T) = O(T^{-\infty}) \):
parts of the wave-packet that may escape from the ball of radius \( T^\alpha \) are negligible.

One proves

\textbf{Theorem 4.1 ([GKT])}. Assume that \( \beta^+(\infty, \psi) \leq \xi \) (for example, under assumptions
of Proposition 1.1, \( \xi = 1 \)). Then \( 0 \leq \alpha^+_l \leq \alpha^+_u \leq \xi \), and
\[
\frac{1}{p} (S^\pm_\psi(p))^2(p) \leq \beta^+(p, \psi) \leq \inf_{\alpha \in (\alpha^-_l, \alpha^+_u)} \max\left( \alpha, \alpha^-_u - \frac{S^+_\psi(\alpha)}{p} \right).
\]
where \( g^2 \) denotes the Legendre transform of \( g \): \( g^2(p) = \sup_{\alpha} (p\alpha - g(\alpha)) \). As a conse-
quence,
\[
\beta^+(0 + 0, \psi) = \alpha^-_l, \quad \beta^+(+\infty, \psi) = \alpha^+_u.
\]

5 A lower bound in dimension 1

In this section, we assume the potential \( V \) to be polynomially bounded: there exists
\( a, b > 0 \) such that
\[
|V(x)| \leq a(x)^b,
\]
for all \( x \in \mathbb{Z}^+ \) in the discrete case, and \( x \in \mathbb{R}^+ \) in the continuous case.

For a given operator \( H \) on \( L^2([1, +\infty)) \), resp. \( L^2([0, +\infty)) \), we define the transfer
matrices \( T(E, x, y) \) between sites \( y \) and \( x \) as:
\[
T(E, x, y) = \begin{pmatrix}
  u_0(E, x + 1) & u_{x/2}(E, x + 1) \\
u_0(E, x) & u_{x/2}(E, x)
\end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix}
  u'_0(E, x) & u'_{x/2}(E, x) \\
u_0(E, x) & u'_{x/2}(E, x)
\end{pmatrix},
\]
\[
(5.2)
\]
where $u_\theta(E,x)$ denotes the solution of $Hu = Eu$, $E \in \mathbb{R}$, satisfying $u_\theta(E,y) = \sin \theta$, $u_\theta(E,y+1) = \cos \theta$, resp. $u_\theta(E,y) = \sin \theta$, $u'_\theta(E,y) = \cos \theta$ (note that $T(E,x,x) = \text{Id}$). It follows from the definitions that if $u$ is a solution of the eigenvalue equation $Hu = Eu$, $E \in \mathbb{R}$, then

$$
\begin{pmatrix}
    u(E,x+1) \\
    u(E,x)
\end{pmatrix} = T(E,x,y)
\begin{pmatrix}
    u(E,y+1) \\
    u(E,y)
\end{pmatrix},
\text{ resp. } \begin{pmatrix}
    u'(E,x) \\
    u(E,x)
\end{pmatrix} = T(E,x,y)
\begin{pmatrix}
    u'(E,y) \\
    u(E,y)
\end{pmatrix}.
$$

(5.3)

Boundedness of transfer matrices is known to be related to a.c. spectrum [Si1]. By exploiting this fact in a spirit close to [CM], we show that on a spectral scale $\varepsilon$ (i.e. on a time scale $T = \varepsilon^{-1}$), it is enough to know the behaviour of the transfer matrices up to a length scale $N = \varepsilon^{-1+\nu}$, $\nu > 0$, in order to get a useful control on the spectral measure.

**Proposition 5.1 ([GKT]).** Let $H = -\Delta + V$ where $V$ satisfies (5.1), and in addition (1.15)-(1.16) in the continuous case. Let $f \in C_c^\infty(\mathbb{R})$, $\psi = f(H)\chi_0$, and let $I$ be a compact interval. There exist a universal constant $C_1$ and for all $M > 0$ and $\sigma > 0$, a constant $C_{1,M,\sigma,a,b}$, $i = 1, 2$, such that for all $\varepsilon \in [0,1]$ and all $\lambda \in I$, one has (setting $N = [\varepsilon^{-1-\sigma}]$ is the discrete case and $N = \varepsilon^{-1-\sigma}$ in the continuum)

$$
\mu_{\psi_0}(\lambda - \varepsilon, \lambda + \varepsilon) \geq C_1 \int \frac{k(E) \, dE}{T(E,0)} - C_{1,M,\sigma,a,b} \varepsilon^M ;
$$

(5.4)

$k(E)$ is a finite constant, positive for Lebesgue a.e. $E$, given by

$$
\begin{cases}
    k(E) = 1 & \text{on } \mathcal{H} = L^2([1, +\infty)) \\
    k(E) = K_{\Theta_1,\Theta_2} \frac{||u_0(E,\chi_0)||}{1 + |E|} & \text{on } \mathcal{H} = L^2([0, +\infty)),
\end{cases}
$$

(5.5)

where the constant $K_{\Theta_1,\Theta_2} > 0$ depends only on $\Theta_1, \Theta_2$.

**Corollary 5.1.** Assume that $|\text{supp} \mu_\psi| > 0$ and that for some $\gamma < \infty$, and for all $E \in \text{supp} \mu_\psi$: $||T(E,N,0)|| \leq C(E)N^\gamma$, where $C(E)$ is positive and finite for Lebesgue almost all $E \in \text{supp} \mu_\psi$. Then $g^+_\mu_\psi \leq 1 + 2\gamma$. As a consequence,

(i) $M(p,f,T) \geq C_p \, T^{p-2\gamma}$ for all $T$, $\beta^-(p,\psi) \geq 1 - \frac{2\gamma}{p}$.

(ii) $\alpha^- = \beta^-(\infty,\psi) = 1$

Application of this corollary to sparse potential models and random decaying potentials are given in [GKT]. Examples include models with pure point spectrum, models with a spectral transition pp / sc spectrum, models for which the nature of the spectrum is not known. However it is not the application we would like to put the accent on in this introductive review. While Corollary 5.1 requires some strong and uniform statement on the polynomial behaviour of the $||T(E,N,0)||$’s, one can see from Proposition 5.1 that, at a given scale $\varepsilon$, it is enough to get a polynomial behaviour of $||T(E,N,0)||$ on a scale of order $N \approx \frac{1}{\varepsilon}$, and not for all $N$. Moreover, it is enough to get such informations on $||T(E,N,0)||$ on sets of positive Lebesgue measure that may tend to zero as $\varepsilon$ does. Both these possibilities are contained in the following general statement. And both these possibilities will be useful to re-visit a now well-known “pathological” example that is due to Last [La] and Del Río, Jitomirskaya, Last, Simon [DRJLS].

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Theorem 5.1 ([GKT]). Let $H$ be as in Proposition 5.1. Let $S$ be a set of positive Lebesgue measure. Pick $f \in C_c^\infty(\mathbb{R})$, $f \geq 0$ and $f = 1$ on $S$. For any $q \in (0, 1)$ and $\sigma > 0$ there exists constants $C_q > 0$ and $C_{q,f,\sigma,a,b} < \infty$ (depending only on the indicated parameters) such that for all $\varepsilon \in (0, 1)$,

$$I_{\mu_{f(u)\phi_0}}(q, \varepsilon) \geq C_q \varepsilon^{q-1} \int_S \frac{k(E) \, dE}{\|T(E, N, 0)\|^2} - C_{q,f,\sigma,a,b} \varepsilon,$$  

(5.6)

where $N = [\varepsilon^{-(1+\sigma)}]$ in the discrete case and $N = \varepsilon^{-(1+\sigma)}$ in the continuous case, and where $k(E)$ is given in Theorem 5.1 Eq. (5.5). As a consequence, for any $p > 0$ and $T > 0$,

$$\mathcal{M}(p, f, T) \geq C_p T^p \left( \frac{1}{\log T} \int_S \frac{k(E) \, dE}{\|T(E, N, 0)\|^2} \right)^{p+1} - C_{p,f,\sigma,a,b},$$  

(5.7)

with $N = [T^{1+\sigma}]$ in the discrete case and $N = T^{1+\sigma}$ in the continuous case. The constants $C_p > 0$ and $C_{p,f,\sigma,a,b} < \infty$ in (5.7) depend also on $\Theta_1, \Theta_2$ in the continuous case.

We turn to the following application:

$$H_{\theta,\alpha,\lambda} = -\Delta + \Lambda \cos(\pi \alpha n + \theta) + \lambda(\delta_1, \cdot) \delta_1.$$  

(5.8)

Here we take $\alpha$ irrational and $\Lambda > 2$ so that the Lyapunov exponent is positive everywhere: as a consequence the spectrum $H_{\theta,\alpha,0}$ is purely singular for a.e. $\theta$ [He, CFKS], and so is the one of $H_{\theta,\alpha,\lambda}$ for all given $\lambda$ [RS].

The result is the following

Theorem 5.2 ([GKT]). There exists a dense $G_\delta$ set of irrationals $\Omega$ such that for any $\alpha \in \Omega$, for all $\theta \in [0, 2\pi]$ and $\lambda \in [0, 1]$, for any $q \in (0, 1)$, there exist a constant $C_q$ and a sequence $\varepsilon_k \to 0$ such that

$$I_{\mu_{\delta_1}}(q, \varepsilon_k) \geq \frac{C_q}{\log |\varepsilon_k|} \left( \frac{1}{\varepsilon_k} \right)^{1-q}.$$  

(5.9)

As a consequence $D_{\mu_{\delta_1}}^+(q) = 1$ for $q \in (0, 1)$, and thus $\beta^+(p, \delta_1) = 1$.

To prove Theorem 5.2 we proceed by induction as in [La, DRJLS], and rely on Theorem 5.1. One constructs periodic approximants of $H_{\theta,\alpha,\lambda}$. The spectrum of these approximants operators is then a.c. and transfer matrices are bounded. By perturbation, it implies the following for $H_{\theta,\alpha,\lambda}$: Fix $\sigma > 0$, then there exists a sequence of $\varepsilon_k \in [0, 1]$ and of bounded sets $S_k$ such that

$$\sup_{E \in S_k} \sup_{N \leq (\frac{1}{\varepsilon_k})^{1+\sigma}} \|T_{\theta,\alpha,\lambda}(E, N, 0)\|^2 \leq \log |\varepsilon_k^{-1}|,$$  

(5.10)

and

$$|S_k| \geq \frac{1}{\log |\varepsilon_k^{-1}|}.$$  

(5.11)

The result follows from Theorem 5.1.

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