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Local energy decay of solutions to the wave equation for nontrapping metrics

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Let (M, g) be an n-dimensional complete, unbounded, connected Riemannian manifold with a Riemannian metric g of class $C^\infty(\overline{M})$ and a compact C^∞-smooth boundary ∂M (which may be empty). We suppose that M is of the form $M = X_0 \cup X$, where X_0 is a compact, connected Riemannian manifold with a metric $g_{|X_0}$ of class $C^\infty(\overline{X}_0)$ with a compact boundary $\partial X_0 = \partial M \cup \partial X$, $\partial M \cap \partial X = \emptyset$, $X = [r_0, +\infty) \times S$, $r_0 \gg 1$, with metric $g_{|X} := dr^2 + \sigma(r)$. Here $(S, \sigma(r))$ is an $n-1$ dimensional compact Riemannian manifold without boundary equipped with a family of Riemannian metrics $\sigma(r)$ depending smoothly on r which can be written in any local coordinates $\theta \in S$ in the form

$$\sigma(r) = \sum_{i,j} g_{ij}(r, \theta) d\theta_i d\theta_j, \quad g_{ij} \in C^\infty(X).$$

Denote $X_r = [r, +\infty) \times S$. Clearly, ∂X_r can be identified with the Riemannian manifold $(S, \sigma(r))$ with the Laplace-Beltrami operator $\Delta_{\partial X_r}$ written as follows

$$\Delta_{\partial X_r} = -p^{-1} \sum_{i,j} \partial_{\theta_i}(p g^{ij} \partial_{\theta_j}),$$

where (g^{ij}) is the inverse matrix to (g_{ij}) and $p = (\det(g_{ij}))^{1/2} = (\det(g^{ij}))^{-1/2}$. Let Δ_g denote the (positive) Laplace-Beltrami operator on (M, g) and let ∇_g be the corresponding gradient. We have

$$\Delta_X := \Delta_g|_X = -p^{-1}\partial_r(p\partial_r) + \Delta_{\partial X_r} = -\partial_r^2 - \frac{p'}{p}\partial_r + \Delta_{\partial X_r},$$

where $p' = \partial p / \partial r$. We have the identity

$$\Delta_X^\sharp := p^{1/2} \Delta_X p^{-1/2} = -\partial_r^2 + \Lambda_r + q(r, \theta), \tag{1}$$

where

$$\Lambda_r = -\sum_{i,j} \partial_{\theta_i}(g^{ij} \partial_{\theta_j}),$$

and q is an effective potential given by

$$q(r,\theta) = (2p)^{-2}\left(\frac{\partial p}{\partial r}\right)^2 + (2p)^{-2}\sum_{i,j}\frac{\partial p}{\partial \theta_i}\frac{\partial p}{\partial \theta_j}g^{ij} + 2^{-1}p\Delta_X(p^{-1}).$$

We make the following assumptions:

$$|q(r,\theta)| \le C, \quad \frac{\partial q}{\partial r}(r,\theta) \le Cr^{-1-\delta_0}, \tag{2}$$

with constants $C, \delta_0 > 0$. Denote

$$h(r,\theta,\xi) = \sum_{i,j} g^{ij}(r,\theta)\xi_i\xi_j, \quad (\theta,\xi)\in T^*S.$$

We suppose that

$$-\frac{\partial h}{\partial r}(r,\theta,\xi) \ge \frac{C}{r}h(r,\theta,\xi), \quad \forall(\theta,\xi)\in T^*S, \tag{3}$$

with a constant $C > 0$.

Denote by G the selfadjoint realization of Δ_g on the Hilbert space $H = L^2(M, d\mathrm{Vol}_g)$ with Dirichlet or Neumann boundary conditions, $Bu = 0$, on ∂M. Let $\chi \in C^\infty(\overline{M})$, $\chi = 1$ on X_0, $\chi = 0$ in X_{r_0+1}. Denote $\mathcal{G} := (G+1)^{1/2}$. We make the following assumption:

there exist constants $T, \sigma > 0$ so that the operators $\mathcal{G}^{\sigma_1}\chi\cos\left(T\sqrt{G}\right)\chi\mathcal{G}^{\sigma_2}$ and

$$\mathcal{G}^{\sigma_1}\chi\frac{\sin\left(T\sqrt{G}\right)}{\sqrt{G}}\chi\mathcal{G}^{\sigma_2+1} \text{ belong to } \mathcal{L}(H), \forall\sigma_1,\sigma_2\in\mathbf{R} \text{ such that } \sigma_1+\sigma_2=\sigma. \tag{4}$$

The metric g will be said *nontrapping* if there exists a constant $T_0 > 0$ such that for every generalized geodesics (see [3], [4] for the definition), γ, with $\gamma(0) \in M\setminus X_{r_0+1/2}$, $\exists 0 < t = t_\gamma \le T_0$ so that $\gamma(t) \in X_{r_0+1}$. For such a metric, it follows from the result of Melrose-Sjöstrand [3], [4] on propagation of C^∞ singularities that the distribution kernels of the operators $\chi\cos\left(T_0\sqrt{G}\right)\chi$ and $\chi\frac{\sin\left(T_0\sqrt{G}\right)}{\sqrt{G}}\chi$ are of class $C^\infty(\overline{M}\times\overline{M})$ (this is known as generalized Huyghens principle). Therefore, (4) is fulfilled for nontrapping metrics.

Given a real s, choose a real-valued function $\chi_s \in C^\infty(\overline{M})$, $\chi_s = 1$ on $M\setminus X_{r_0+1/2}$, $\chi_s|_X$ depending only on r, $\chi_s = r^{-s}$ on X_{r_0+1}, $\chi_s\chi_{-s} \equiv 1$. Denote by G_0 the Dirichlet selfadjoint realization of Δ_X on the Hilbert space $H_0 = L^2(X, d\mathrm{Vol}_g)$.

Our first result is the following

Theorem 1. *Assume(2) and (3) fulfilled. Then, for every $s > 1/2$, there exist constants $C_0, C > 0$ such that for $z \ge C_0$, $0 < \varepsilon \le 1$, we have (with $j = 0,1$)*

$$\|r^{-s}\nabla_g^j(G_0 - z \pm i\varepsilon)^{-1}r^{-s}\|_{\mathcal{L}(H_0)} \le Cz^{(j-1)/2}. \tag{5}$$

Moreover, if (4) holds, we have

$$\|\chi_s\nabla_g^j(G - z \pm i\varepsilon)^{-1}\chi_s\|_{\mathcal{L}(H)} \le Cz^{(j-1)/2}. \tag{6}$$

We use this theorem to study the local energy of the solutions of the following mixed problem

$$\begin{cases} (\partial_t^2 + \Delta_g)u(t,x) = 0 \quad \text{in} \quad \mathbf{R}\times M, \\ Bu(t,x) = 0 \quad \text{on} \quad \mathbf{R}\times \partial M, \\ u(0,x) = \varphi(G)f_1(x), \partial_t u(0,x) = \varphi(G)f_2(x), \quad x\in M, \end{cases} \tag{7}$$

where $\varphi \in C^\infty(\mathbf{R})$, $\varphi = 0$ in a neighbourhood of the interval $(-\infty, A_0]$, $\varphi = 1$ outside a larger neighbourhood, where $A_0 = \max\{C_0, C_0'\}$, C_0 being as in Theorem 1 and C_0' being as in (9) and (10) below. Recall that the solutions to (7) can be expressed by the formula

$$u = \cos\left(t\sqrt{G}\right)\varphi(G)f_1 + \frac{\sin\left(t\sqrt{G}\right)}{\sqrt{G}}\varphi(G)f_2. \tag{8}$$

Given $s > 0$ and a function $\chi \in C^\infty(\overline{M})$, $\chi = 1$ in X_0, $\chi = 0$ outside some compact, set

$$E_s(t) = \int_M \left(|\partial_t u(t,x)|^2 + |\nabla_g u(t,x)|^2\right)\chi_{2s} d\mathrm{Vol}_g,$$

$$E_{loc}(t) = \int_M \left(|\partial_t u(t,x)|^2 + |\nabla_g u(t,x)|^2\right)\chi d\mathrm{Vol}_g,$$

$$\widetilde{E}(0) = \int_M \left(|f_2|^2 + |\nabla_g f_1|^2 + |f_1|^2\right) d\mathrm{Vol}_g,$$

$$\widetilde{E}_{-s}(0) = \int_M \left(|f_2|^2 + |\nabla_g f_1|^2\right)\chi_{-2s} d\mathrm{Vol}_g + \int_M |f_1|^2 \chi_{-2s+2} d\mathrm{Vol}_g.$$

To get uniform local energy decay estimates of the solutions to (7) we need to impose additional conditions on the behaviour of the resolvent of the operator G_0. We suppose that there exist $s > 1/2$, $C_0' > 0$ and an integer $m \geq 0$ so that for $z \geq C_0'$, $0 < \varepsilon \leq 1$, the following estimates hold (with $j = 0, 1$):

$$\|r^{-s}\nabla_g^j (G_0 - z \pm i\varepsilon)^{-k} r^{-s}\|_{\mathcal{L}(H_0)} \leq C z^{(j-k)/2}, \quad k = 1, ..., m+1, \tag{9}$$

$$\|r^{-s}\nabla_g^j \left(G_0 - z \pm i\varepsilon z^{1/2}\right)^{-m-2} r^{-s}\|_{\mathcal{L}(H_0)} \leq C z^{(j-m-2)/2}\varepsilon^{-1+\mu}, \tag{10}$$

with constants $C > 0$ and $0 < \mu \leq 1$ independent of z and ε.

Our main result is the following

Theorem 2. *Assume (2), (3), (4), (9) and (10) fulfilled. Then, we have, for* $t \gg 1$,

$$E_{s-1/2}(t) \leq O\left(t^{-2m-2\mu}\right)\widetilde{E}_{-s}(0). \tag{11}$$

In particular, if f_1 *and* f_2 *are of compact support, we have*

$$E_{loc}(t) \leq O\left(t^{-2m-2\mu}\right)\widetilde{E}(0). \tag{12}$$

Remark 1. If (9) holds for every integer $k \geq 1$ with $s = s_k > 1/2$ and $C = C_k > 0$, then (12) holds with $O_N\left(t^{-N}\right)$, $\forall N \gg 1$, in place of $O\left(t^{-2m-2\mu}\right)$.

The key point in the proof of Theorem 2 is the following

Proposition 3. *The estimates (5), (6), (9) and (10) imply the following ones (with $j = 0, 1$)*

$$\|\chi_s \nabla_g^j (G - z \pm i\varepsilon)^{-k} \chi_s\|_{\mathcal{L}(H)} \leq C z^{(j-k)/2}, \quad k = 1, ..., m+1, \tag{13}$$

$$\|\chi_s \nabla_g^j \left(G - z \pm i\varepsilon z^{1/2}\right)^{-m-2} \chi_s\|_{\mathcal{L}(H)} \leq C z^{(j-m-2)/2} \varepsilon^{-1+\mu}, \tag{14}$$

for $z \geq A_0$, $0 < \varepsilon \leq 1$, with m and $0 < \mu \leq 1$ the same as in (9) and (10), and a new constant $C > 0$ independent of z and ε.

As an application of the above theorem we get uniform local energy decay of the solutions to (7) for a class of asymptotically Euclidean manifolds. To describe this class, denote

$$q^\flat := r^{-1} \frac{\partial (r^2 q)}{\partial r}, \quad \Lambda_r^\flat := -r^{-1} \sum_{i,j} \partial_{\theta_i} \left(\frac{\partial (r^2 g^{ij})}{\partial r} \partial_{\theta_j} \right).$$

Denote by $G_0^\sharp$ the Dirichlet self-adjoint realization of the operator $\Delta_X^\sharp$ on the Hilbert space $H_0^\sharp = L^2(X, dr d\theta)$. We make the following assumptions:

$$\left| q^\flat(r, \theta) \right| \leq C r^{-\delta}, \tag{15}$$

$$r^\delta \Lambda_r^\flat (G_0^\sharp - i)^{-1} \in \mathcal{L}(H_0^\sharp), \tag{16}$$

for some constants $C > 0$ and $0 < \delta \leq 1$. We have the following

Proposition 4. *Assume (2), (3), (15) and (16) fulfilled. Then, (9) and (10) hold with $m = 0$, $s = 1 + \delta/2$, for every $0 < \mu < \delta$.*

Corollary 5. *Assume (2), (3), (4), (15) and (16) fulfilled. Then, we have, for $t \gg 1$,*

$$E_{(1+\delta)/2}(t) \leq O_\epsilon\left(t^{-2\delta+\epsilon}\right) \widetilde{E}_{-(1+\delta/2)}(0), \quad \forall 0 < \epsilon \ll 1. \tag{17}$$

In particular, if f_1 and f_2 are of compact support, we have

$$E_{loc}(t) \leq O_\epsilon\left(t^{-2\delta+\epsilon}\right) \widetilde{E}(0), \quad \forall 0 < \epsilon \ll 1. \tag{18}$$

Remark 2. It is worth noticing that the above results still hold for the self-adjoint realization of $\Delta_g + V(x)$, where V is a real-valued potential, $V(x) \geq 0$, provided the assumptions (2) and (15) are satisfied with q replaced by $q + V|_X$.

It is easy to see that the assumptions (2), (3), (15) and (16) are fulfilled for long-range perturbations of the Euclidean metric on $\mathbf{R}^n$, $n \geq 2$. More precisely, let $\mathcal{O} \subset \mathbf{R}^n$ be a bounded domain with a C^∞-smooth boundary and a connected complement $\Omega = \mathbf{R}^n \setminus \mathcal{O}$. Let g be a Riemannian metric in Ω of the form

$$g = \sum_{i,j=1}^{n} g_{ij}(x) dx_i dx_j, \quad g_{ij}(x) \in C^\infty(\overline{\Omega}),$$

satisfying

$$|\partial_x^\alpha (g_{ij}(x) - \delta_{ij})| \leq C_\alpha \langle x \rangle^{-\gamma_0 - |\alpha|}, \tag{19}$$

for every multi-index α, with constants $C_\alpha, \gamma_0 > 0$, where $\langle x \rangle := (1 + |x|^2)^{1/2}$ and δ_{ij} denotes the Kronecker symbol. It follows from (19) that outside a sufficiently big compact there exists a global smooth change of variables, $(r, \theta) = (r(x), \theta(x))$, $r \in [r_0, +\infty)$, $r_0 \gg 1$, $\theta \in S = \{y \in \mathbf{R}^n : |y| = 1\}$, which transforms the metric g in the form $dr^2 + \sigma(r)$. Therefore, (Ω, g) is isometric to a Riemannian manifold of the class described above, and (17) and (18) hold with $\delta = \min\{1, \gamma_0\}$, provided the metric g is nontrapping.

In the case when $g_{ij} = \delta_{ij}$ for $|x| \geq \rho_0$ with some $\rho_0 \gg 1$, and the metric g is nontrapping, a better estimate than (18) is known to hold true with $\varphi \equiv 1$. In this case, Vainberg [5], [6] showed that the generalized Huyghens principle implies (18) with a rate of decay $O\left(e^{-ct}\right)$, $c > 0$, if $n \geq 3$ is odd, and $O\left(t^{-2n}\right)$ if $n \geq 4$ is even. The fact that the metric coincides with the Euclidean one outside some compact plays an important role in Vainberg's method. In particular, this implies that the cutoff resolvent extends analytically to some strip through the real axis. This approach, however, does not work anymore in the setting described above, and it does not allow to get estimates like (11) and (17).

To prove Theorem 1 we make use of some ideas developed in [1], [8], where uniform high frequency resolvent estimates have been obtained without assuming (4). The assumption (4), however, allows to get much better estimates than those proved in the above papers.

Given any domain $M_0 \subset M$, equipe the Sobolev space $H^1(M_0, d\mathrm{Vol}_g)$ with the semi-classical norm defined by

$$\|u\|^2_{H^1(M_0, d\mathrm{Vol}_g)} := \|u\|^2_{L^2(M_0, d\mathrm{Vol}_g)} + \|\lambda^{-1} \nabla_g u\|^2_{L^2(M_0, d\mathrm{Vol}_g)},$$

where $\lambda \gg 1$. The estimate (6) follows from combining the following two estimates

Proposition 6. *Assume (4) fulfilled. Then, given any $u \in D(G)$, the following estimate holds:*

$$\begin{aligned} \|u\|_{H^1(M \setminus X_{r_0+1}, d\mathrm{Vol}_g)} \leq C\lambda^{-1} \|(\Delta_g - \lambda^2 + i\varepsilon)u\|_{L^2(M \setminus X_{r_0+1}, d\mathrm{Vol}_g)} \\ + C\|u\|_{H^1(X_{r_0+1/2} \setminus X_{r_0+1}, d\mathrm{Vol}_g)}, \end{aligned} \tag{20}$$

for $\lambda \geq \lambda_0$, $0 < \varepsilon \leq 1$, with constants $C, \lambda_0 > 0$ independent of λ and ε.

Proposition 7. *Let $u \in H^2(X_a, d\mathrm{Vol}_g)$, $a > r_0$, be such that*

$$r^s (\Delta_g - \lambda^2 + i\varepsilon) u \in L^2(X_a, d\mathrm{Vol}_g)$$

for $1/2 < s \leq (1 + \delta_0)/2$, $0 < \varepsilon \leq 1$. Then, for every $0 < \gamma \ll 1$ there exist constants $C_1, C_2, \lambda_0 > 0$ (which may depend on γ but are independent of λ and ε) so that for $\lambda \geq \lambda_0$, $\forall a_1 > a$, we have

$$\|r^{-s} u\|^2_{H^1(X_{a_1}, d\mathrm{Vol}_g)} \leq C_1 \lambda^{-2} \|r^s (\Delta_g - \lambda^2 + i\varepsilon) u\|^2_{L^2(X_a, d\mathrm{Vol}_g)}$$

$$-C_2\lambda^{-1}\mathrm{Im}\,\langle \partial_r u, u\rangle_{L^2(\partial X_a)} + \gamma\|u\|^2_{H^1(X_a\setminus X_{a_1}, d\mathrm{Vol}_g)}. \tag{21}$$

The estimate (11) follows from combining the following estimates

Proposition 8. *a) The estimate (6) implies,* $\forall s > 1/2$,

$$\int_0^\infty E_s(\tau)d\tau \le C\widetilde{E}_{-s}(0). \tag{22}$$

b) The estimates (13) and (14) imply, for $t \ge 1$,

$$\int_t^\infty E_s(\tau)d\tau \le Ct^{-2m-2\mu}\widetilde{E}_{-s}(0). \tag{23}$$

Lemma 9. *For* $\forall s > 1/2$, $t \ge 1$, *we have*

$$E_{s-1/2}(t) \le C\int_t^\infty E_s(\tau)d\tau, \tag{24}$$

with a constant $C > 0$ *independent of* t.

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