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NEW RESULTS IN VELOCITY AVERAGING

FRANÇOIS GOLSE

Abstract. This paper discusses two new directions in velocity averaging. One is an improvement of the known velocity averaging results for L^1 functions. The other shows how to adapt some of the ideas of velocity averaging to a situation that is essentially a new formulation of the Vlasov-Maxwell system.

1. Velocity averaging: a quick tour

Velocity averaging designates a procedure by which one can prove compactness (or smoothing) effects on the macroscopic quantities corresponding to a phase-space, microscopic density that satisfies some kinetic equation. The prototype of all such results is as follows:

Theorem 1. *Let $p \in (1, \infty)$. Let $\mathcal{F} \subset L^p_{loc}(\mathbf{R}^D \times \mathbf{R}^D; dxdv)$ be a bounded set such that*

$$\left\{ v \cdot \nabla_x f \,\middle|\, f \in \mathcal{F} \right\} \text{ is bounded in } L^p_{loc}(\mathbf{R}^D \times \mathbf{R}^D; dxdv) \,.$$

Then, for each $\phi \in C_c(\mathbf{R}^D)$, the set

$$\left\{ \int f(x,v)\phi(v)dv \,\middle|\, f \in \mathcal{F} \right\} \text{ is relatively compact in } L^p_{loc}(\mathbf{R}^D; dx) \,.$$

This result is essentially Theorem 1 in [21] — the result there is given only for $p = 2$ but its extension to all $p \in (1, \infty)$ follows from an easy interpolation argument and the Fréchet-Kolmogorov L^p compactness criterion (Theorem IV.8.20 and IV.8.21 in [13]).

In fact, one can show that, under the same assumptions as in Theorem 1, the family of averages in v is bounded in $W^{s,p}_{loc}(\mathbf{R}^D)$ for all $s \in (1, \inf(1 - \frac{1}{p}, \frac{1}{p}))$: see [20] — this holds in fact for a class of measures in v that is more general than $\phi(v)dv$ and allows treating both the steady and the evolution problems at once, by taking $X = (t, x)$, $V = (w, v)$ and measures of the form $d\mu(V) = \delta_{w=1} \otimes \phi(v)dv$.

Later on, this result was extended by R. DiPerna and P.-L. Lions, in the following manner:

Theorem 2. *Let $m > 0$ be an integer and let $f \in L^2_{loc}(\mathbf{R}^D \times \mathbf{R}^D; dxdv)$ be such that*

$$v \cdot \nabla_x f = \sum_{|\alpha|=m} \partial_v^\alpha g_\alpha \text{ where } g_\alpha \in L^2_{loc}(\mathbf{R}^D \times \mathbf{R}^D; dxdv)$$

with the usual multi-index notation. Then, for each test function $\phi \in C_c^\infty(\mathbf{R}^D)$

$$x \mapsto \int f(x,v)\phi(v)dv \text{ belongs to the Sobolev class } H^{1/2(m+1)}_{loc}(\mathbf{R}^D)\,.$$

Extensions of Theorem 2 to L^p with $p \in (1, \infty)$ were done successively in [12] using Littlewood-Paley decompositions and Besov spaces; the analogous compactness result can be found in [32]; a slightly more convenient decomposition better adapted to Cauchy problems can be found in [3] — see chapter 1 of [7] for a survey. More precise interpolation results for the L^p case ($1 < p < \infty$) can be found in [9]. The optimality of the regularity results in the L^p case has been discussed in [28].

Analogues of Theorems 1 and 2 for partial differential operators other than the simple advection operator $v \cdot \nabla_x$ can be found in [15], [19]. Another presentation, based on microlocal defect measures (a variant of Wigner measures in [30]) can be found in [14]; this reference has the advantage of showing some analogy between compensated compactness and velocity averaging.

The main application of velocity averaging so far is obviously the mathematical theory of nonlinear kinetic models. Following the publication of [21] and [20], several open problems in this area were solved — such as the global existence of solutions to the Boltzmann equation [11], to the BGK model [31] or to the Vlasov-Maxwell system [10] without restriction on the size of the initial data — or were put in a more promising formulation. For instance, since the work of Hilbert [24], the problem of deriving the equations of fluid dynamics from the Boltzmann equation was addressed by using asymptotic expansions that could not handle the appearance of singularities in the limiting equations and thus were limited to trivial hydrodynamic regimes (near-equilibrium states, short times etc.) With the new compactness tools provided by the velocity averaging method and the notion of "renormalized solution" proposed by DiPerna and Lions, this problem was reduced to compactness statements that were inherently global (see the program outlined in [2] or the survey talk [35]).

Even though the general procedure of velocity averaging above clearly provides valuable information on all the topics listed above, it must be adapted to the problem under consideration. The purpose of this paper is to present two such extensions of the classical velocity averaging method which may be of independent interest.

Section 2 below describes a new result obtained in the course of a collaboration with Laure Saint-Raymond on the Navier-Stokes hydrodynamic limits for the Boltzmann equation, while section 3 summarizes ongoing work with François Bouchut and Christophe Pallard on the Vlasov-Maxwell system.

2. Velocity averaging in L^1

A fundamental difficulty when applying the velocity averaging method to kinetic models is that the phase-space density $f \equiv f(x,v)$ of particles that is the unknown function satisfying a kinetic equation is by construction a (nonnegative) $L^1_{x,v}$ function — indeed, $\iint f(x,v)dxdv$ is the total number of particles in the system.

Unfortunately, Theorem 1 does not hold for $p=1$. Pick any bounded sequence $g_n \equiv g_n(x,v)$ in $L^1(\mathbf{R}^D \times \mathbf{R}^D, dxdv)$ that converges weakly to $\delta_0(x) \otimes \delta_{v^*}(v)$, where $v^* \neq 0$. Let f_n be the unique $L^1_{x,v}$ solution of the equation $f_n + v \cdot \nabla_x f_n = g_n$. Both f_n and $v \cdot \nabla_x f_n$ are bounded sequences of $L^1_{x,v}$, but an elementary computation shows that

$$\int \chi(x) \left(\int f_n(x,v)\psi(v)dv \right) dx \to \psi(v^*) \int_0^{+\infty} e^{-t}\chi(tv^*)dt \tag{1}$$

for each test function $\chi \in C_c(\mathbf{R}^D)$. In particular the sequence of velocity averages is not even *weakly* relatively compact in $L^1_{loc}(\mathbf{R}^D)$ since it converges in the sense of distributions to a density carried by the half-line $\mathbf{R}_+ \cdot v^*$. (This example can be found in [20], pp. 123–124).

In retrospect, this is not very surprising: all proofs of Theorem 1 known to this date are based on the following microlocal argument. Let $\xi^* \in \mathbf{R}^D \setminus \{0\}$; if $v \cdot \nabla_x f \in L^2_{x,v}$, then $x \mapsto f(x,v)$ is microlocally H^1 in the direction ξ^* for each $v \notin (\mathbf{R}\xi^*)^\perp$; since $(\mathbf{R}\xi^*)^\perp$ is dv-negligeable, the effect of these bad directions disappears after averaging in v. This small divisor argument fails in L^1 because the size of a function in L^1 cannot be inferred from the size of its Fourier coefficients — nor from the size of its coefficients in any decomposition other than the Fourier one, because L^1 has no unconditional basis.

However, the analogue of Theorem 1 for $p=1$ holds under the additional assumption of equiintegrability.

Theorem 3. *Let $\mathcal{F} \subset L^1_{loc}(\mathbf{R}^D \times \mathbf{R}^D; dxdv)$ be an equiintegrable set such that*

$$\left\{ v \cdot \nabla_x f \,\middle|\, f \in \mathcal{F} \right\} \text{ is equiintegrable in } L^1_{loc}(\mathbf{R}^D \times \mathbf{R}^D; dxdv)\,.$$

Then, for each $\phi \in C_c(\mathbf{R}^D)$, the set

$$\left\{ \int f(x,v)\phi(v)dv \,\Big|\, f \in \mathcal{F} \right\} \text{ is relatively compact in } L^1_{loc}(\mathbf{R}^D; dx)\,.$$

Theorem 3 is essentially Proposition 3 of [20]; this formulation of velocity averaging is the one best adapted to the Boltzmann equation and is one of the main ingredients in the construction of global renormalized solutions for arbitrarily large initial data: see [11]. Indeed, because of Boltzmann's H theorem, the unknown phase-space density F in the Boltzmann equation satisfies $\iint F \ln F dxdv \le C$, which gives the equiintegrability of well-chosen approximating sequences F_n.

However, Theorem 3 can be considerably improved: in fact it is enough to rule out possible concentrations *in the variable v only.* Such a result is nearly optimal: after all, concentrations in the variable v tend to annihilate the effect of averaging in v.

We first define more precisely the notion of "equiintegrability in the variable v".

Definition 1. *A bounded set $\mathcal{F}$ in $L^1_{loc}(\mathbf{R}^D_x \times \mathbf{R}^D_v)$ is said to be locally equiintegrable in v if and only if, for each $\eta > 0$ and each compact $K \subset \mathbf{R}^D \times \mathbf{R}^D$, there exists $\alpha > 0$ such that, for each measurable family $(A_x)_{x\in\mathbf{R}^D}$ of measurable subsets of $\mathbf{R}^D$ satisfying $\sup_{x\in\mathbf{R}^D} |A_x| < \alpha$, one has*

$$\int \left(\int_{A_x} \mathbf{1}_K(x,v)|f(x,v)|dv \right) dx < \eta$$

for each $f \in \mathcal{F}$.

For instance, any set $\mathcal{F}$ that is bounded in $L^1_{loc}(dx; L^p_v)$ with $p > 1$ is locally equiintegrable in v in the sense of the definition above. An easy adaptation of the de La Vallée-Poussin equiintegrability criterion (see [29] p. 38) shows that a set $\mathcal{F}$ is locally equiintegrable in v provided that it is bounded in $L^1_{loc}(dx; L^\Phi_v)$ where L^Φ designates the Orlicz space of measurable functions g such that $\Phi(|g|) \in L^1$, with $\Phi : \mathbf{R}_+ \to \mathbf{R}_+$ a convex function such that $\Phi(z)/z \to +\infty$ as $z \to +\infty$.

Our main new result in this section is

Theorem 4. *Let $\mathcal{F}$ be a bounded set in $L^1_{loc}(\mathbf{R}^D_x \times \mathbf{R}^D_v)$ that is locally equiintegrable in v and such that the set*

$$\{v \cdot \nabla_x f \,|\, f \in \mathcal{F}\} \text{ is also bounded in } L^1_{loc}(\mathbf{R}^D_x \times \mathbf{R}^D_v)\,.$$

Then

- *the set $\mathcal{F}$ is locally equiintegrable in $\mathbf{R}^D_x \times \mathbf{R}^D_v$ (in both variables x and v);*

• *for each* $\psi \in C_c(\mathbf{R}^D)$*, the set of velocity averages*

$$\left\{ \int f(x,v)\psi(v)dv \,\middle|\, f \in \mathcal{F} \right\} \text{ is relatively compact in } L^1_{loc}(\mathbf{R}^D)\,. \tag{2}$$

A complete proof of this theorem can be found in [22]; the first statement is one of the main ingredients in the proof of the global incompressible Navier-Stokes limit of the Boltzmann equation (from DiPerna-Lions renormalized solutions of the Boltzmann equation to Leray solutions of the Navier-Stokes equations): see [23]. A first important step in this direction is the following special case of Theorem 4: if $\mathcal{F}$ is a bounded set in $L^1_x(L^\infty_v)$ such that the set $\{v \cdot \nabla_x f \,|\, f \in \mathcal{F}\}$ is bounded in $L^1_{x,v}$, then the set of velocity averages $\{\int f(x,v)\psi(v)dv \,\big|\, f \in \mathcal{F}\}$ is locally equiintegrable in $\mathbf{R}^D_x$. This observation already appeared in [34].

Let us say a few words about the proof of Theorem 4. It is based on the next two lemmas. The first one bears on the dispersion effects for the transport operator: a variant of it appears in [1] or [8]. The following statement is Proposition 1.11 in [7].

Lemma 1. *Let* $\phi^0 \equiv \phi^0(x,v) \in L^p_x(L^q_v)$ *for some* $1 \le p < q \le +\infty$*, and let* $\phi \equiv \phi(t,x,v)$ *be the solution of the Cauchy problem*

$$\partial_t \phi + v \cdot \nabla_x \phi = 0\,, \qquad \phi(0,x,v) = \phi^0(x,v)\,, \qquad x, v \in \mathbf{R}^D\,. \tag{3}$$

Then, for all $t \in \mathbf{R}^*$*,*

$$\|\phi(t,\cdot,\cdot)\|_{L^q_x(L^p_v)} \le |t|^{-D(\frac{1}{p}-\frac{1}{q})} \|\phi^0\|_{L^p_x(L^q_v)}\,. \tag{4}$$

The next lemma is an interpolation formula involving the fictitious time variable t as the interpolation parameter; it is reminiscent of the definition by J.-L. Lions of "spaces of traces" as interpolation spaces between a Hilbert space H and the domain of an operator A that is the infinitesimal generator of a semigroup on H: see [27].

Lemma 2. *For each* $f \equiv f(x,v) \in L^1(\mathbf{R}^D \times \mathbf{R}^D)$ *such that* $v \cdot \nabla_x f$ *belongs to* $L^1(\mathbf{R}^D \times \mathbf{R}^D)$ *and each* $\phi^0 \in L^\infty(\mathbf{R}^D \times \mathbf{R}^D)$*, one has*

$$\begin{aligned} \iint f(x,v)\phi^0(x,v)dxdv = &\iint f(x,v)\phi(t,x,v)dtdxdv \\ &- \int_0^t \iint \phi(s,x,v) v \cdot \nabla_x f(x,v) ds dx dv\,, \end{aligned} \tag{5}$$

for all $t \in \mathbf{R}^*$*, where* ϕ *is the solution of (3).*

These two lemmas can be applied with $\phi^0(x,v) = \mathbf{1}_A(x)$ to control

$$\int_A \left| \int f(x,v)dv \right| dx \le \iint \mathbf{1}_A(x)|f(x,v)|dxdv$$

where $A \subset \mathbf{R}^D_x$ is a set of small measure. This gives the equiintegrability of the set of velocity averages without much difficulty. With some additional work — essentially using again the equiintegrability in v and the Bienaymé-Chebyshev inequality — one arrives at the first statement in Theorem 4.

The second statement there is a straightforward consequence of the following amplification of Theorem 3:

Theorem 5. *Let $\mathcal{F} \subset L^1_{loc}(\mathbf{R}^D \times \mathbf{R}^D; dxdv)$ be an equiintegrable set such that*

$$\left\{ v \cdot \nabla_x f \,\middle|\, f \in \mathcal{F} \right\} \text{ is bounded in } L^1_{loc}(\mathbf{R}^D \times \mathbf{R}^D; dxdv)\,.$$

Then, for each $\phi \in C_c(\mathbf{R}^D)$, the set

$$\left\{ \int f(x,v)\phi(v)dv \,\middle|\, f \in \mathcal{F} \right\} \text{ is relatively compact in } L^1_{loc}(\mathbf{R}^D; dx)\,.$$

See [22] for a proof. This result is a corollary of Theorem 3 based on the elementary inequality $\|(\lambda I + v \cdot \nabla_x)^{-1}\|_{\mathcal{L}(L^1_{x,v})} = 1/\lambda$.

3. Velocity averaging and the Vlasov-Maxwell system

3.1. A quick presentation of the Vlasov-Maxwell system. The (relativistic) Vlasov-Maxwell system is the kinetic equation that models the collisionless dynamics of charged particles accelerated by their self-consistent electro-magnetic field. It reads

$$\begin{aligned} \partial_t f + v(\xi) \cdot \nabla_x f &= -(E + v(\xi) \wedge B) \cdot \nabla_\xi f\,, \\ \partial_t E - \mathrm{curl}_x B &= -j_f\,, \\ \mathrm{div}_x E &= \rho_f\,, \\ \partial_t B + \mathrm{curl}_x E &= 0\,, \\ \mathrm{div}_x B &= 0\,, \end{aligned} \tag{6}$$

with $v(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}}$ and the notations

$$\rho_f = \int f(t,x,\xi)d\xi\,, \qquad j_f = \int f(t,x,\xi)v(\xi)d\xi\,. \tag{7}$$

Here, $f(t,x,\xi)$ is the density of particles which, at time t, are located at x with momentum ξ, $E(t,x)$ and $B(t,x)$ are respectively the electric and magnetic fields at time t and position x while $v(\xi)$ is the (relativistic) velocity in terms of the momentum ξ — the speed of light and the mass of the particles are normalized to 1.

This system for the unknown $(f(t,x,\xi), E(t,x), B(t,x))$ is posed in $\mathbf{R}_+ \times \mathbf{R}^3_x \times \mathbf{R}^3_\xi$ and is completed by the initial conditions

$$f_{|t=0} = f_I\,, \quad E_{|t=0} = E_I\,, \quad B_{|t=0} = B_I\,. \tag{8}$$

The main results known to this date on (6) are

- the global existence of weak (and even renormalized) solutions, proved by R. DiPerna and P.-L. Lions [10];
- existence and uniqueness of classical solutions under the assumption that supp $f(t,x,\cdot)$ is bounded for each $t>0$, proved by R. Glassey and W. Strauss [16].

Subsequently, the global existence and uniqueness of classical solutions to (6) was established in [17] under the weaker assumption that the macroscopic energy density satisfy

$$\int \sqrt{1+|\xi|^2} f d\xi \in L^\infty_{loc}(\mathbf{R}_+; L^\infty(\mathbf{R}^3))\,. \tag{9}$$

Finally, R. Glassey and W. Strauss established the global existence and uniqueness of classical solutions to (6) for small (in some sense) initial data in [18], by proving that (9) holds for such initial data.

The main open problem on (6) is to prove (or disprove) the same result as in [16] without assuming (9) or the support condition for all $t>0$:

"Let f_I, E_I and B_I be compactly supported and C^∞. Does there exist a unique global C^∞ solution to the Cauchy problem (6)-(8)?"

The analogous problem for the Vlasov-Poisson system has been solved by Pfaffelmoser [33]:

$$\begin{aligned} \partial_t f + \xi\cdot\nabla_x f &= \nabla_x\phi\cdot\nabla_\xi f\,,\\ \Delta_x \phi &= \rho_f\,. \end{aligned} \tag{10}$$

Let us briefly compare both systems. In the case of (6), the conservation of energy is

$$\iint \sqrt{1+|\xi|^2} f(t,x,\xi)dxd\xi + \tfrac12 \int (|E|^2+|B|^2)(t,x)dx = Cst\,, \tag{11}$$

while in the case of (10) it becomes

$$\iint \tfrac12|\xi|^2 f(t,x,\xi)dxd\xi + \tfrac12\int |\nabla_x\phi|^2(t,x)dx = Cst\,. \tag{12}$$

In both cases, the Vlasov equation satisfies the Maximum Principle, meaning that $\|f\|_{L^\infty_{t,x,\xi}} = \|f_I\|_{L^\infty_{x,\xi}}$.

However, if one assumes that $f(t,x,\xi)=0$ for all t, x and $|\xi|>R$, one sees that ρ_f and $j_f\in L^\infty_t(L^2_x)$ and in the case of the Vlasov-Poisson system, the standard ellipticity estimate for the Poisson equation implies that the electric field $E=-\nabla_x\phi$ satisfies $E\in L^\infty_t(H^1_x)$. In the case of the Vlasov-Maxwell system, what plays the role of the Poisson equation is the Maxwell system of equations for (E,B) which is notoriously hyperbolic. In other words, by going from (10) to (6), it seems that one "looses a derivative" on the fields, which is not very encouraging. Yet, with the first Glassey-Strauss result [16], the state of the art on the Vlasov-Maxwell system is essentially on par with what it was on the Vlasov-Poisson system before Pfaffelmoser's breakthrough: all that is missing is an estimate preventing the ξ-support of f to spread to infinity in finite time, as was done in [33] for the simpler Vlasov-Poisson system (10).

This indicates somehow that the argument in [16] contains somehow a way to win back the derivative on the fields lost in the manner explained above. My purpose in the remaining part of this paper is to explain why this can be seen as some kind of velocity averaging result, albeit not of the type presented in section 1.

3.2. **Non resonant wave + transport systems.** Consider a coupled system consisting of a linear wave equation and a transport equation, of the form

$$
\begin{aligned}
\square_{t,x}u&=f\,,\\
(\partial_t+v(\xi)\cdot\nabla_x)f&=P(t,x,\xi,D_\xi)g\,,
\end{aligned}
\tag{13}
$$

where $\square_{t,x}=\partial_t^2-\Delta_x$. The unknowns in that system are the real-valued functions $u\equiv u(t,x,\xi)$ and $f\equiv f(t,x,\xi)$, while the source term in the right-hand side of the transport equation involves a given real-valued function $g\equiv g(t,x,\xi)$. The notation $P(t,x,\xi,D_\xi)$ designates a (smooth) linear differential operator in the variable ξ only, while $v\equiv v(\xi)$ is a smooth $\mathbf{R}^D$-valued vector field on $\mathbf{R}^M$.

The system (13) is posed for all $(t,x,\xi)\in\mathbf{R}^*_+\times\mathbf{R}^D\times\mathbf{R}^M$. Associated to this system are the initial conditions

$$
\begin{aligned}
u_{|t=0}&=u_I\,,\\
\partial_t u_{|t=0}&=u'_I\,,\\
f_{|t=0}&=f_I\,,
\end{aligned}
\tag{14}
$$

where the functions u_I, u'_I, f_I, together with g, are the initial data of the Cauchy problem (13)-(14).

The system (13) is called "non resonant" if and only if

$$(NR) \qquad \text{for each compact } K \subset \mathbf{R}^M\,, \quad v_M := \sup_{\xi \in K} |v(\xi)| < 1\,.$$

The significance of this condition comes from its physical meaning: massive particles with phase-space density f are transported at a speed $v(\xi)$ which is less than the speed of light corresponding to the propagation of singularities for the potential u. This is obviously verified in the case of the (relativistic) Vlasov-Maxwell system, as can be seen from the formula $v(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}}$.

It has been speculated for some time after the publication of [16] that the reason to believe that the Vlasov-Maxwell system does not develop singularities in finite time is that singularities of the fields are propagated by the Maxwell system of equations at the speed of light while the singularities of the phase-space density of particles f are propagated by the transport equation at a speed that is less than the speed of light: hence both kinds of singularities cannot interact in any nonlinear way, which suggests that these singularities may not exist in the first place. However, these ideas were never put on mathematical terms, and the next theorem may be a first step in this direction.

Theorem 6. *Let f and $g \in L^2_{loc}(\mathbf{R}^*_+ \times \mathbf{R}^D \times \mathbf{R}^M)$, and assume that the initial data $f_I \in L^2_{loc}(\mathbf{R}^D \times \mathbf{R}^M)$, that $u'_I \in L^2_{loc}(\mathbf{R}^M_\xi ; H^1_{loc}(\mathbf{R}^D))$ while $u_I \in L^2_{loc}(\mathbf{R}^M_\xi ; H^2_{loc}(\mathbf{R}^D))$. Let $P(t,x,\xi,D_\xi)$ be a linear differential operator of order $m \in \mathbf{N}$ on $\mathbf{R}^M_\xi$ with smooth coefficients. Pick $\chi \equiv \chi(\xi)$ a test function in $C^m_c(\mathbf{R}^M)$ and let $v \equiv v(\xi)$ be in $C^m(\mathbf{R}^M)$ and satisfy the nonresonant condition (NR).*

Then, if (13)-(14) hold, the ξ-average

$$\rho_\chi(t,x) = \int u(t,x,\xi)\chi(\xi)d\xi$$

belongs to $H^2_{loc}(\mathbf{R}^*_+ \times \mathbf{R}^D)$.

This result might seem somewhat shocking at first sight, since everything behaves as if the operator $\Box_{t,x}$ was elliptic. In fact it is, microlocally outside of the wave cone, and this is where the non-resonance condition (NR) helps, but this does not explain everything — eg. that the gain in regularity be independent of the order of the ξ-derivatives in the transport equation. To see this, compare the result in Theorem 6 with the one implied by Theorem 2, under the assumption that the map $\xi \mapsto v(\xi)$ has no critical point on the support of χ (which is easily verified in the case where $v(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}}$):

- by applying the analogue of Theorem 2 for evolution problems to the transport equation in (13)

$$(t,x) \mapsto \int f(t,x,\xi)\chi(\xi)d\xi \in H^{1/2(m+1)}_{loc}(\mathbf{R}^*_+ \times \mathbf{R}^D)\,;$$

- by averaging in ξ the wave equation in (13) and applying the standard energy estimate for the wave equation, one finds that

$$\rho_\chi(t,x) \in H^{1+1/2(m+1)}_{loc}(\mathbf{R}^*_+ \times \mathbf{R}^D)\,.$$

Even in the most favorable case $m = 0$ this result is worse than that of Theorem 6; further, the gain of regularity depends on m. On the other hand, this result, unlike Theorem 6, still holds when condition (NR) breaks down.

The key argument in the proof of theorem 6 is that some well chosen combinations of the wave operator $\Box_{t,x}$ and of the transport operator $\partial_t + v(\xi)\cdot\nabla_x$ are elliptic in the variables t and x.

Lemma 3. *For $\chi \in C^m_c(\mathbf{R}^M)$, let $v \equiv v(\xi)$ in $C^m(\mathbf{R}^M)$ satisfy the non-resonant condition (NR), and let $\lambda \in \mathbf{R}$. The two following conditions are equivalent:*

- *λ satisfies the condition*

$$v_M^2 < \lambda < 1\,, \qquad \text{where } \; v_M = \sup_{\xi\in supp\chi} |v(\xi)|\,; \tag{15}$$

- *for each $\xi \in supp\ \chi$, the second order differential operator*

$$Q^\lambda_\xi = \lambda\Box_{t,x} - (\partial_t - v(\xi)\cdot\nabla_x)(\partial_t + v(\xi)\cdot\nabla_x) \tag{16}$$

is elliptic.

When λ verifies any of these conditions, the symbol q^λ_ξ of the operator Q^λ_ξ satisfies the following uniform ellipticity estimates: for all $m \in \mathbf{N}$

$$\sup_{\xi\in\, supp\chi}\ \sup_{\omega^2+|k|^2>0} (\omega^2 + |k|^2)\left| D^m_\xi \left(\frac{1}{q^\lambda_\xi(\omega,k)}\right)\right| < +\infty\,. \tag{17}$$

Once Lemma 3 is established, the proof of theorem 6 is based upon controlling $Q^\lambda_\xi u$ by the usual energy estimate for the wave equation. Finally, the uniform ellipticity estimates (17) are used to control the various contributions to the ξ-average ρ_χ after integrating by parts to bring all ξ-derivatives to bear on either χ or $1/q^\lambda_\xi$.

Observe that the proof of Theorem 2 involves multiplying the rhs. of the transport equation — namely the functions g_α — by

$$D^m_v \left(\frac{1}{v\cdot k}\right) = O\left(\frac{|k|^m}{|v\cdot k|^{m+1}}\right)\,;$$

as in Lemma 3 one gets a function of k that is homogeneous of degree -1, except for the effect of the small divisor $|v \cdot \frac{k}{|k|}|^{m+1}$ which is taken care of by integrating in v at the expense of loosing some of the decay in the Fourier variable k. Likewise, in Lemma 3, one gets a function of k that is homogeneous of degree -1, except that, under the non-resonance condition (NR), there is no small divisor effect fighting the regularizing effect of the term $D_\xi^m\left(\frac{1}{q_\xi^\lambda(\omega,k)}\right)$. This is why the amount of regularity gained in Theorem 6 does not depend on the order of the ξ-derivative in the rhs. of the transport equation, unlike in Theorem 2.

Small divisors appear in this situation only when the non-resonance condition is not verified uniformly in $|\xi|$ or, equivalently, when averaging in ξ involves all ξ's and not a compact set: see Theorem 7 below.

The interested reader is referred to [4] for a more thorough discussion of this result and extensions to L^p with $p \neq 2$, and to [5] for complete proofs.

3.3. **Liénard-Wiechert potentials and the Vlasov-Maxwell system.** It remains to explain how Theorem 6 can be used on the Vlasov-Maxwell system.

In order to satisfy the initial conditions (8), we first choose a vector field $A_I \equiv A_I(x)$ such that

$$\mathrm{curl}_x A_I = B_I\,, \quad \mathrm{div}_x A_I = 0\,, \tag{18}$$

and define $A^{(I)} \equiv A^{(I)}(t,x)$ by

$$\begin{aligned} \Box_{t,x} A^{(I)} &= 0\,, \\ A^{(I)}_{|t=0} &= A_I\,, \\ \partial_t A^{(I)}_{|t=0} &= -E_I\,. \end{aligned} \tag{19}$$

Solve then for $u \equiv u(t,x,\xi)$ the Cauchy problem for the wave equation

$$\begin{aligned} \Box_{t,x} u &= f\,, \\ u_{|t=0} &= 0\,, \\ \partial_t u_{|t=0} &= 0\,. \end{aligned} \tag{20}$$

Elementary computations show that

$$\phi = \int u\,d\xi\,, \quad A = A^{(I)} + \int u v(\xi)\,d\xi \tag{21}$$

are respectively the scalar and vector potentials satisfying the wave equations

$$\Box_{t,x}\phi = \rho_f\,, \quad \Box_{t,x} A = j_f\,,$$

the Lorentz gauge condition

$$\partial_t\phi + \operatorname{div}_x A = 0\,, \tag{22}$$

and giving the electromagnetic field by the formulas

$$\begin{aligned} E &= -\partial_t A - \nabla_x\phi = -\partial_t A^{(I)} - \partial_t \int uv(\xi)d\xi - \nabla_x \int u d\xi\,, \\ B &= \operatorname{curl}_x A = \operatorname{curl}_x A^{(I)} + \operatorname{curl}_x \int uv(\xi)d\xi\,. \end{aligned} \tag{23}$$

The system (6) can be somewhat simplified by using this formulation of the Maxwell equations. It becomes

$$\begin{aligned} \partial_t f + v(\xi)\cdot\nabla_x f &= \nabla_\xi \cdot [-(E + v(\xi)\wedge B)f]\,, \\ \Box_{t,x} u &= f\,, \end{aligned} \tag{24}$$

where (E, B) are given in terms of u by (23). The initial conditions are

$$f_{|t=0} = f_I\,, \quad u_{|t=0} = \partial_t u_{|t=0} = 0\,. \tag{25}$$

Introducing $u(t, x, \xi)$ is not a mere mathematical trick; in fact the function u has a physical meaning: it is the distribution of Liénard-Wiechert potentials (see [26]) created by the charged particles under the initial phase-space distribution f_I.

One can check, by applying Theorem 6 assuming that the initial data (f_I, E_I, B_I) has finite total energy (11), that $f_I \in L^\infty_{x,\xi}$ and that a global solution constructed by DiPerna-Lions [10] satisfies the finite support assumption

(26) there exists $R \in \mathbf{R}^*_+$ such that $f(t, x, \xi) = 0$ whenever $|\xi| > R$,

that one gains the lost derivative back. In other words, under these assumptions, one finds that E and B are in $H^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)$. Of course, in order to have classical solutions, one needs to show that E and B are in $W^{1,\infty}_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)$, but the present discussion can be seen as a first step in a new formulation of [16], where the main ideas are hidden by highly technical explicit computations involving the elementary solution of the Maxwell system of equations.

3.4. **A conditional regularity result.** The approach described above — especially the formulation of the Vlasov-Maxwell system in terms of Liénard-Wiechert potentials — allows one to significantly simplify the conditional regularity result in [16]: see [6]. However, the question of global existence of classical solutions to the 3D Vlasov-Maxwell system remains open as of now, in spite of a promising new formulation due to Klainerman-Staffilani [25].

In fact, reasoning along the lines of [15] shows that the classical formulation of velocity averaging (as in Theorem 2) could be useful when the speed of the particle approaches the speed of light, leading to a loss of ellipticity in the operator Q_ξ^λ above. Here is a first result in that direction; however the idea of interpolating between Theorem 2 and Theorem 6 has not been pushed very far in the proof of this — still conditional — regularity result.

Theorem 7. *Consider initial data* (f_I, E_I, B_I) *such that* $f_I \in L^\infty(\mathbf{R}^3 \times \mathbf{R}^3)$, $f_I \geq 0$ *a.e.,* E_I *and* $B_I \in H^1_{loc}(\mathbf{R}^3)$ *satisfy*

$$div_x B_I = 0\,, \quad div_x E_I = \int f_I d\xi\,, \tag{27}$$

and the finite energy condition

$$\iint \sqrt{1+|\xi|^2} f_I dx d\xi + \tfrac{1}{2}\int (|E_I|^2 + |B_I|^2) dx < +\infty \tag{28}$$

holds. Let (f, E, B) *be a weak solution of the system (6)(the existence of which is predicted by* [10]*). If the macroscopic energy density satisfies*

$$\int \sqrt{1+|\xi|^2} f d\xi \in L^p_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)\,, \quad \text{with } p \in]\tfrac{3}{2}, 2] \tag{29}$$

then the electromagnetic field has regularity given by

$$E \text{ and } B \in H^s_{loc}(\mathbf{R}^*_+ \times \mathbf{R}^3)\,, \quad \text{with } s < \frac{2p-3}{2p+4}\,. \tag{30}$$

See [4] and [5] for the proof.

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