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# THE WAVE GROUP AND RADIATION FIELDS ON ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

ANTÔNIO SÁ BARRETO

## 1. INTRODUCTION

Asymptotically hyperbolic manifolds is an important class of manifolds for which microlocal analysis techniques have been successfully applied to study spectral and scattering theory, see for example [12, 13, 18, 23, 24, 28] and references cited there. They have also received considerable attention due to their connection with mathematical physics, see [2, 9, 31] and references cited there.

First we will discuss joint work with Mark Joshi [19] on the construction of the wave group on asymptotically hyperbolic manifolds. We show that it belongs to an appropriate class of Fourier integral operators and, as an application, we analyze the singularities of its (regularized) trace.

Secondly, following the work of F.G. Friedlander for asymptotically Euclidean manifolds [10, 11], we introduce the notion of radiation fields for asymptotically hyperbolic manifolds and use them to construct a translation representation of the wave group.

Let  $X$  be a smooth compact manifold with boundary  $\partial X$ . Let  $g$  be a Riemannian metric such that, for  $x$  is a defining function of  $\partial X$ ,  $x^2g$  extends to a smooth and non-degenerate metric up to  $\partial X$ . In other words, we consider  $g$  of the form

$$g = \frac{H}{x^2},$$

where  $x \in C^\infty(\overline{X})$ ,  $x^{-1}(0) = \partial X$ , and  $dx \neq 0$  at  $\partial X$ , and  $H$  is a smooth Riemannian metric on  $X$ , non-degenerate up to  $\partial X$ . Moreover we assume that  $|dx|_H = 1$  at  $\partial X$ .

It is easy to see, by taking  $x = a(y)x'$ , with  $a > 0$ , and  $y \in \partial X$ , that  $g$  only determines  $H|_{\partial X}$  up to a multiple. So  $g$  does not determine a metric on  $\partial X$ , but only a conformal structure.

The pair  $(X, g)$  is called an asymptotically hyperbolic manifold because along a smooth curve in  $X \setminus \partial X$ , approaching a point at  $\partial X$ , all sectional curvatures of  $g$  approach  $-1$ , see [24]. The simplest examples of such manifolds are the hyperbolic space,  $\mathbb{B}^{n+1}$ , and its quotients by certain group actions, see for example section 8 of [24].

It is proved in Proposition 2.1 of [18], see also [9], that fixed a choice of  $H|_{\partial X}$ , there exists a unique product decomposition  $X \sim \partial X \times [0, \epsilon)$ , for  $\epsilon$  small enough, such that

$$(1.1) \quad g = \frac{dx^2 + h(x, y, dy)}{x^2}.$$

It was proved in [18] that the scattering matrix determines the Taylor series of  $h(x, y, dy)$  at  $x = 0$ , and in particular  $h(0, y, dy)$ . Apparently one important question in mathematical physics is which invariants are independent of the choice of  $H|_{\partial X}$ , see [9].

Let  $\Delta$  denote the (positive, self-adjoint) Laplacian corresponding to the asymptotically hyperbolic metric  $g$ , acting on half-densities. The metric  $g$  induces a canonical trivialization of the 1-density bundle given by  $\theta = \sqrt{\text{vol}(g)}|dx dy|$ , the Riemannian density. The square root of this is then a natural trivialization of the half-density bundle. The Laplacian is defined by

$$\Delta \left( f\theta^{\frac{1}{2}} \right) = (\Delta f)\theta^{\frac{1}{2}},$$

where the Laplacian on the right hand side is the usual one acting on functions. It is well known, see for example [22, 24], that the continuous spectrum of  $\Delta$  is  $[\frac{n^2}{4}, \infty)$ .

The sections of the density bundle  ${}^0\Omega(X)$  are defined to be smooth multiples of the Riemannian half-density. In local coordinates where (1.1) holds it is given by

$$\theta = f(x, y) \frac{dx \, dy}{x \, x^n}, \quad f \in C^\infty(X), \quad f \neq 0.$$

The bundle  ${}^0\Omega^{\frac{1}{2}}(X)$  is the half-density bundle obtained from  ${}^0\Omega(X)$ . Similarly we define the bundle  ${}^0\Omega^{\frac{1}{2}}(X \times X)$ .

The group  $\cos\left(t\sqrt{\Delta - \frac{n^2}{4}}\right)$  is defined to be the operator whose kernel  $U(t, w, w')$  satisfies

$$(1.2) \quad \begin{aligned} \left(\frac{\partial^2}{\partial t^2} + \Delta_w - \frac{n^2}{4}\right) U(t, w, w') &= 0, \\ U(0, w, w') &= \delta(w, w'), \quad \frac{\partial}{\partial t} U(0, w, w') = 0. \end{aligned}$$

Here  $\delta(w, w')$  acts on half densities  $f\theta^{\frac{1}{2}}$  according to

$$f(w)\theta^{\frac{1}{2}} = \int \delta(w, w') f(w') \theta^{\frac{1}{2}}(w) \theta^{\frac{1}{2}}(w')$$

In the interior of  $\mathbb{R} \times X \times X$ ,  $U(t, w, w')$  is well known as a Fourier integral operator [4, 7, 15, 16]. The difficulty here is to understand its behavior up to  $\mathbb{R} \times \partial X \times \partial X$ . It is proved in [19] that  $\cos\left(t\sqrt{\Delta - \frac{n^2}{4}}\right)$  belongs to a class of Fourier integral operators and as an application of this it is proved that it has a regularized trace, i.e that there exist constants  $C_j$ ,  $j = 1, \dots, n-1$ , such that the limit

$$0\text{-tr}(U(t)) = \lim_{\epsilon \rightarrow 0} \left[ \int_{x>\epsilon} U(t, w, w) - \sum_{j=1}^{n-1} C_j \epsilon^{-j} + C_0 \log \epsilon \right],$$

exists. This is called the zero-trace of  $U(t)$ , in analogy with the  $b$ -integral of [29], see also the notion of  $b$ -trace of [5]. It is a Hadamard regularization and it obviously depends on the choice of the boundary defining function,  $x$ , but it gives a natural regularization of the trace of  $U(t)$ . In the case of Riemann surfaces, the notion of zero-trace was introduced and studied in depth by Guillopé and Zworski [12, 13].

Again using the characterization of  $U(t, w, w')$  as a zero F.I.O, the arguments of [4, 7, 15, 16] can then be used to analyze the singularities of  $0\text{-tr}(U(t))$  trace. The following are proved in [19]

**Theorem 1.1.** *The singular support of  $0\text{-tr}(U(t))$  is contained in the set of periods of closed geodesics of  $(X, g)$ .*

The justification for this result is that there exists  $\epsilon > 0$  such that no closed geodesics intersect  $\{x < \epsilon\}$ , so the result in the interior [4, 7] extends to  $0\text{-tr}(U(t))$ .

Another application of this characterization and [4, 7] is the asymptotic formula for  $0\text{-tr}(U(t))$  as  $t \rightarrow 0$ . Suppose that the set of periods of closed geodesics is contained in  $(t_0, \infty)$ . Let  $\rho \in C_0^\infty(\mathbb{R})$  be such that  $\rho(t) = 1$  for  $|t| < \frac{t_0}{2}$  and  $\rho(t) = 0$  for  $|t| > \frac{2t_0}{3}$ .

Since the arguments in [16], see also the proof of Proposition 2.1 of [7], are entirely local, we can apply them directly to prove

**Proposition 1.1.** *Let*

$$T_\epsilon = \int_{x>\epsilon} U(t, w, w).$$

There exist  $w_k \in \mathbb{R}$ ,  $k = 0, 1, \dots$ , with  $\omega_0 = \text{vol}(X_\epsilon)$ , such that

$$(1.3) \quad \int_{\mathbb{R}} e^{t\mu} \rho(t) T_\epsilon(t) dt \sim \frac{1}{(2\pi)^{n-1}} \sum_{k=0}^{\infty} \omega_k \mu^{n-2k-1},$$

for  $\mu \rightarrow \infty$  and is rapidly decreasing if  $\mu \rightarrow -\infty$ .

The analogous result for  $0\text{-tr}(U(t))$  follows directly from its definition and (1.3). We obtain

**Theorem 1.2.** *There exist  $\theta_k \in \mathbb{R}$ ,  $k = 0, 1, \dots$ , such that*

$$(1.4) \quad \int_{\mathbb{R}} e^{t\mu} \rho(t) 0\text{-tr}(U(t)) dt \sim \frac{1}{(2\pi)^{n-1}} \sum_{k=0}^{\infty} \theta_k \mu^{n-2k-1},$$

for  $\mu \rightarrow \infty$  and is rapidly decreasing if  $\mu \rightarrow -\infty$ .

Observe that,

$$\theta_0 = \lim_{\epsilon \rightarrow 0} \left( \int_{x>\epsilon} d\text{vol}_g - \sum_{j=1}^{n-1} d_j \epsilon^{-j} - d_0 \log \epsilon \right)$$

where  $d_j$ ,  $j = 0, 1, 2, \dots, n-1$  are the unique real numbers such that the limit exists. This is called the zero-volume of  $X$  and is denoted  $0\text{-vol}(X)$ .

The behavior of  $0\text{-tr}(U(t))$  as  $t \rightarrow 0$  is related to the behaviour of the scattering phase at high energies, and the possible existence of a Poisson type formula relating the wave group and the resonances in this setting. See for example [12, 13] for the case of Riemann surfaces. The wave group for hyperbolic space has been studied in [21, 14, 20].

The construction of 0-Fourier integral operators is based on that of  $b$ -Fourier integral operators introduced by Melrose's in the  $b$ -category [25], though he does not examine the specific problem of constructing wave groups there. The construction of the parametrix for  $U(t, w, w')$  is also largely based on his work with Mazzeo [24] on the construction of the resolvent for this class of manifolds.

Finally, as an application of [24], we show the existence of radiation fields for these manifolds and use them to obtain a translation representation of the wave group. The details of the construction will appear in [32].

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## 2. THE CONSTRUCTION OF THE RESOLVENT

The spectral theorem gives that resolvent  $R(\lambda)$  is well defined for  $\Im \lambda \ll 0$  by

$$R(\lambda) = \left( \Delta - \frac{n^2}{4} - \lambda^2 \right)^{-1}.$$

To follow the notation of [24] we let  $\zeta = \frac{n}{2} + i\lambda$ , so  $-\frac{n^2}{4} - \lambda^2 = \zeta(\zeta - n)$ . Mazzeo and Melrose show in [24] that  $R(\zeta)$  has a meromorphic continuation to the complex plane and we briefly recall their construction.

Locally, in the interior of  $X \times X$ , and for  $\Re \zeta \gg 0$ , the work of Seeley [33] gives that  $R(\zeta)$  is pseudodifferential operator, so its kernel is singular at the diagonal

$$D = \{(x, y, x', y') \in X \times X; x = x', y = y'\}.$$

The problem is to understand the behavior of the kernel of  $R(\zeta)$  up to

$$D_{\partial X} = D \cap (\partial X \times \partial X)$$

and for other values of  $\zeta$ .

For that Mazzeo and Melrose blow-up the intersection  $D_{\partial X}$ . This can be done in an invariant way, but in local coordinates this can be easily seen as introduction of polar coordinates around  $D_{\partial X}$ . Taking coordinates  $(x, y)$  and  $(x', y')$  in a product decomposition of each copy of  $X$  near  $\partial X$ , the “polar coordinates” are then given by

$$R = [x^2 + x'^2 + |y - y'|^2]^{\frac{1}{2}}, \quad \rho = \frac{x}{R}, \quad \rho' = \frac{x'}{R}, \quad \omega = \frac{y - y'}{R}$$

A function is smooth in the space  $X \times_0 X$  if it is smooth in “polar coordinates”  $(R, \rho, \rho', y, \omega)$  about  $D_{\partial X}$ . As a set,  $X \times_0 X$  is  $X \times X$  with  $D_{\partial X}$  replaced by the interior pointing portion of its normal bundle. Let

$$\beta : X \times_0 X \longrightarrow X \times X$$

denote the blow-down map.

The function  $R$  is a defining function for a new face, which we call the front face,  $\mathcal{F}$ . This is the lift of  $D_{\partial X} = D \cap (\partial X \times \partial X)$ . The functions  $\rho$  and  $\rho'$  are then defining functions for the other two boundary faces which we call the top face  $\mathcal{T}$ , and bottom face  $\mathcal{B}$ , respectively, i.e.

$$\mathcal{F} = \{R = 0\}, \quad \mathcal{B} = \{\rho' = 0\}, \quad \mathcal{T} = \{\rho = 0\}.$$

See Figure 1, which is taken from section 3 of [24]. In  $X \times_0 X$  the lift of the diagonal of  $X \times X$  only meets the boundary  $\mathcal{F}$  and is disjoint from the other two boundary faces.

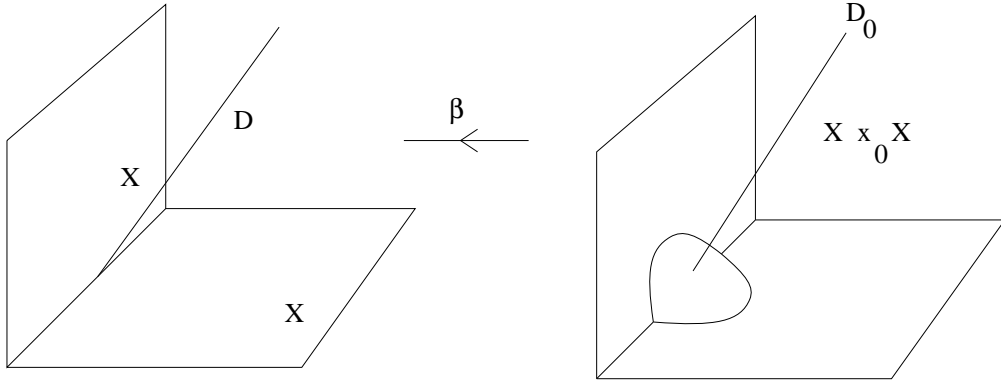


FIGURE 1.  $X \times_0 X$

It is proved in [24] that the lift of the kernel of the resolvent satisfies

$$\beta_0^* R(\zeta) = R_1(\zeta) + R_2(\zeta)$$

where  $R_1$  is conormal of order  $-2$  to the lifted diagonal,  $D_0$ , and smooth up to the front face, and vanishes to infinite order at the top and bottom faces. The second part,  $R_2$ , is of the form  $R_2 = \rho^\zeta \rho'^\zeta F(\zeta, \bullet)$ ,  $F(\zeta, \bullet) \in C^\infty(X \times_0 X, {}^0\Omega^{\frac{1}{2}}(X \times_0 X))$  and depends meromorphically on  $\zeta$ . The bundle  ${}^0\Omega^{\frac{1}{2}}(X \times_0 X)$  is defined to be the lift of  ${}^0\Omega^{\frac{1}{2}}(X \times X)$  under the blow-down map  $\beta$ .

Based on this construction, it is natural to look for the wave group to have a Schwartz kernel which is nice on the space  $\mathbb{R} \times (X \times_0 X)$ . In the interior it is a Lagrangian distribution associated with the flow of the diagonal by the Hamilton vector field of  $p = \sigma^2(\Delta)$ . The question is whether it is possible to carry on this construction uniformly up to  $\mathcal{F}$  in  $X \times_0 X$ .

As in [24], the normal operator will play a major role in the construction of the parametrrix of the wave group and we recall its definition. Let  $p \in \partial X$  and let  $T_p(X)^+$  be the inward pointing vectors in the

tangent space to  $X$  at  $p$ ,  $T_p(X)$ . This is a half space and its boundary is  $T_p(\partial X)$ . Using local coordinates in which (1.1) holds, if  $p = (0, y_0)$  we can see that  $T_p(X)^+$  has a metric

$$(2.1) \quad g_p = (x)^{-2} h(0, y_0, dy),$$

making it isometric to the hyperbolic upper half-space.

A differential operator  $P$  in  $X$  is a zero-differential operator of order  $m$ , and we denote  $P \in \text{Diff}_0^m(X)$  if it is of the form

$$P(x, y, D) = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(x, y) (xD_x)^j (xD_y)^\alpha, \quad a_{j,\alpha}(x, y) \in C^\infty(X).$$

It is shown in [24] that the operator obtained by freezing the coefficients of  $P$  at a boundary point is a well defined operator in  $T_p(X)^+$ , i.e. is independent of the choice of the coordinates  $(x, y)$  and it is defined to be the normal operator of  $P$ . If  $p = (0, y_0) \in \partial X$ , then

$$N_p(P) = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(0, y_0) (xD_x)^j (xD_y)^\alpha.$$

In coordinates (1.1) the Laplacian is given by

$$\Delta = (xD_x)^2 + inxD_x + (xD_x \log \sqrt{h}) xD_x + x^2 \Delta_h,$$

so

$$N_p(\Delta) = (xD_x)^2 + inxD_x + h(0, y_0) xD_y xD_y,$$

which is the Laplacian on  $T_p(X)^+$  with respect to the metric (2.1).

One of the key observations in Mazzeo and Melrose is that there is a natural group action on the leaf of the front face above a point  $p$  which makes it naturally isomorphic to  $X_p$ . This group action is obtained by lifting the action of the subgroup of the general linear group of the boundary of  $X_p$  to the normal bundle of  $X_p$ , as a leaf of the front face is just a quarter of the normal bundle over  $p$ . This allows the definition of normal operator to more general operators.

We say that  $B \in \Psi_0^{m,a,b}(X)$ , if the lift of its kernel under  $\beta$  can be written as  $K(B) = K(B)_1 + K(B)_2$ , with  $K(B)_1$  conormal of order  $m$  to  $D_0$ , smooth up to  $\mathcal{F}$ , and vanishing to infinite order at  $\mathcal{T}$  and  $\mathcal{B}$ , and  $K(B)_2 = \rho^a \rho'^b F$ ,  $F \in C^\infty(X \times_0 X)$ . Let  $F_p$  be the fibre of the front face lying over the point  $(p, p) \in D \cap (\partial X \times \partial X)$ . Since the kernel,  $k(B)$ , of an element  $B \in \Psi_0^{m,a,b}(X)$ , is conormal to the lift of the diagonal  $D_0$ , it can be restricted to  $F_p$  and the kernel of the normal operator,  $N_p(B)$ , is defined by

$$(2.2) \quad k(N_p(B)) = k(B)|_{F_p}.$$

Let  $(x, y)$  be local coordinates near  $p \in \partial X$ , with  $x$  a boundary defining function and also denote the natural corresponding linear coordinates on  $X_p$  by  $(x, y)$ . Let  $(x', y')$  be the same coordinates on the right factor in  $X \times X$  and let  $s = x/x'$ ,  $z = (y - y')/x$ . Then if the Schwartz kernel of a map  $B$  is  $k(x', y', s, z)\gamma$

with  $\gamma = \left| \frac{ds dz dx dy}{s x^n + 1} \right|^{\frac{1}{2}}$ , the normal operator is given at  $p = (0, \bar{y})$  by

$$(2.3) \quad [N_p(B)(f\mu)] = \int k(0, \bar{y}, s, z) f\left(\frac{x}{s}, y - \frac{x}{s}z\right) \frac{ds}{s} dz \cdot \mu,$$

where  $\mu = \left| \frac{dx}{x} \frac{dy}{x^n} \right|^{1/2}$ .

As observed in [24], each fiber  $F_p$  of the front face  $\mathcal{F}$  has a natural origin  $0_p$ , which in coordinates  $s, z$  is given by  $0_p = \{s = 1, z = 0\}$ . For example, we find that the kernel of the identity is

$$K(\text{Id}) = \delta(s - 1)\delta(z)\gamma,$$

and its normal operator is

$$(2.4) \quad N_p(\text{Id}) = \delta(s-1)\delta(z) \left| \frac{ds}{s} dz \frac{dx}{x} \frac{dy}{x^n} \right|^{\frac{1}{2}} = \delta(0_p) \left| \frac{ds}{s} dz \frac{dx}{x} \frac{dy}{x^n} \right|^{\frac{1}{2}}.$$

### 3. 0-FOURIER INTEGRAL OPERATORS

To construct the wave group we introduce the class of 0-Fourier integral operators. As in [19] we will only consider those operators whose kernels, when lifted to  $X \times_0 X$ , have support away from the top and bottom faces. The fact that makes this construction simple is the finite speed of propagation of information which guarantess that there the support of the lift of  $U(t, w, w')$  will not intersect the top and bottom faces for finite time  $t$ . So, we are able to ignore the corners formed by the intersections of the front face with the top and bottom faces. The operators considered here are closely related to the  $b$ -Fourier integral operators introduced in [25].

As observed above, the Laplacian on an asymptotically hyperbolic manifold is a second order operator which locally is the product of vector fields that vanish at  $\partial X$ . On a  $C^\infty$  manifold with boundary  $X$  the space  $\mathcal{V}_0(X)$  of smooth vector fields that vanish on the boundary is a Lie algebra. If we take local coordinates  $(x, y_1, \dots, y_n)$ , in which  $x$  is a defining function of  $\partial X$ ,  $\mathcal{V}_0(X)$  it has the local basis  $x \frac{\partial}{\partial x}, x \frac{\partial}{\partial y_j}, 1 \leq j \leq n$ , near  $\partial X$ , and so it is the space of all  $C^\infty$  sections of a vector bundle over  $X$ :

$$\mathcal{V}_0(X) = C^\infty(X, {}^0TX).$$

Restriction to the interior extends to define a smooth bundle map

$$\iota : {}^0TX \longrightarrow TX,$$

which is an isomorphism in the interior and vanishes over  $\partial X$ . Let  ${}^0T^*X$  be the dual bundle to  ${}^0TX$ . The map  $\iota$  then induces a map  $\iota^*$ , which in dual coordinates  $(x, \xi, y, \eta)$  is given by

$$\begin{aligned} \iota^* : T^*X &\longrightarrow {}^0T^*X \\ (x, \xi, y, \eta) &\longmapsto (x, x\xi, y, x\eta) = (x, \lambda, y, \mu). \end{aligned}$$

The canonical 1-form in  $T^*X$ ,

$$\alpha = \xi dx + \eta \cdot dy$$

is pulled back to

$$(3.1) \quad {}^0\alpha = \frac{\lambda}{x} dx + \frac{\mu}{x} \cdot dy \in C^\infty({}^0T^*X; {}^0T^*({}^0T^*X)).$$

For the canonical 1-form  ${}^0\alpha$ , let  ${}^0\omega = d{}^0\alpha$  be the canonical 2-form. Let  $p$  be the Hamiltonian induced by the metric  $g$ . In coordinates in which (1.1) holds

$$p = \lambda^2 + h(y, \mu) + x\tilde{h}(x, y, \mu),$$

For any  $p \in C^\infty({}^0T^*X)$  the 0-Hamiltonian vector field of  $p$ ,  ${}^0H_p$ , is defined by

$$(3.2) \quad {}^0\omega(\bullet, {}^0H_p) = dp.$$

In local coordinates where  ${}^0\alpha$  is given by (3.1),  ${}^0H_p$  is given by

$$(3.3) \quad {}^0H_p = x \frac{\partial p}{\partial \lambda} \frac{\partial}{\partial x} + x \sum_{j=1}^n \frac{\partial p}{\partial \mu_j} \frac{\partial}{\partial y_j} - \left( x \frac{\partial p}{\partial x} + 2p \right) \frac{\partial}{\partial \lambda} - \sum_{j=1}^n \left( x \frac{\partial p}{\partial y_j} - \frac{\partial p}{\partial \lambda} \mu_j \right) \frac{\partial}{\partial \mu_j}$$

We consider the bundle  $T^*\mathbb{R} \times {}^0T^*X \times {}^0T^*X$  with the one canonical 1-form

$$(3.4) \quad \alpha = \tau dt + \frac{\lambda}{x} dx + \frac{\mu}{x} \cdot dy - \frac{\lambda'}{x'} dx' - \frac{\mu'}{x'} \cdot dy'.$$

Following [7] and [8], let  $C \subset T^*\mathbb{R} \times {}^0T^*X \times {}^0T^*X$ , with the form (3.4), be defined as

$$(3.5) \quad C = \{(t, \tau, x, y, \lambda, \mu, x', y', \lambda', \mu') : \tau + \sqrt{p(x, y, \lambda, \mu)} = 0; (x', y', \lambda', \mu') = \chi_t(x, y, \lambda, \mu)\},$$

where  $\chi_t = \exp(t^0H_p)$  with  $p$  considered as a function on the first copy of  $X$ .

The key result in [19] is

**Proposition 3.1.** *The relation  $C$  lifts under the blow-down map  $\beta$  to a Lagrangian submanifold  $\Lambda_C$  of  $T^*\mathbb{R} \times T^*(X \times_0 X)$  given by*

$$(3.6) \quad \Lambda_C = \{(t, \tau, Y, H) : \tau + \sqrt{\tilde{p}(Y, H)} = 0, (Y, H) \in \Lambda_t\},$$

where  $\tilde{p}$  denotes the lift of  $p$  defined in the first copy of  ${}^0T^*X$  and  $\Lambda_t$  is the lift of the graph of  $\chi_t$ . Moreover,  $\Lambda_C$  intersects the boundary only over  $\mathcal{F}$  and

$$(3.7) \quad \Lambda_C^0 = \Lambda_C \pitchfork (T^*\mathbb{R} \times T^*(X \times_0 X))$$

is a Lagrangian submanifold of  $T^*\mathbb{R} \times T^*\mathcal{F}$  given by

$$(3.8) \quad \Lambda_C^0 = \{(t, \tau, Y_0, H_0) : \tau + \sqrt{p_0(Y_0, H_0)} = 0, (Y_0, H_0) \in \Lambda_{\mathcal{F}, t}\},$$

where  $p_0$  is the restriction of  $\tilde{p}$  to  $\mathcal{F}$  and  $\Lambda_{\mathcal{F}, t} = \exp(tH_{p_0})(N^*O_p)$ .

From (3.7), the Lagrangian  $\Lambda_C$  can clearly be extended across the front face  $\mathcal{F}$  and we define the corresponding Lagrangian distributions as the restriction to  $X \times_0 X$  of a distribution which is Lagrangian with respect to an extension  $\Lambda_e$  of  $\Lambda_C$  across  $\mathcal{F}$ .

Our class of distributions now has a pair of natural symbols. The first is the ordinary symbol of a Lagrangian distribution in the interior, which will be a smooth section of the Maslov bundle tensored with the half-density bundle over  $\Lambda_C$ , which is smooth up to the boundary of  $\Lambda_C$ . It follows from (3.7) that the restriction of a Lagrangian distribution to the front face is in fact Lagrangian with respect to  $\Lambda_C^0$ . The symbol of this restriction will give the second natural symbol. Finally, we want to define a filtration which corresponds to the order of vanishing at the front face. Let  $R$  be a boundary defining function for the front face in  $X \times_0 X$ . We define  $I^{m,s}(\Lambda_C)$  to be equal to  $R^s I^m(\Lambda_C)$ . The symbol at the front face,  $\sigma_s^f(u)$ , then defined to be the restriction of  $R^{-s}u$  to the front face. This is of course dependent on the choice of  $R$  but is invariant as a section of the normal bundle raised to the power  $s$ . In what follows we will fix a product decomposition in which (1.1) holds. This will give a defining function  $R$  of the front face, so we will ignore this coordinate dependence. The class of ordinary Lagrangian symbols will just be a pair of elements  $\sigma_{m,s}(u) = (R^s \sigma_m(u), \sigma_s^f(u))$  of the usual symbol class with the restriction that  $\sigma_m(u)$  restricted to the front face equals  $\sigma_s^f(u)$ .

It is not difficult to see the independence of the class from the choice of extension of  $\Lambda_C$ .

**Definition 3.1.** *If  $C$  is given by (3.5) and  $\Lambda_C$  is the Lagrangian defined in Proposition 3.1, then we define*

$$I_0^{m,s}(\mathbb{R} \times X, X; C, {}^0\Omega^{\frac{1}{2}}) = \{K \in I^{m,s}(\mathbb{R} \times X \times_0 X, \Lambda_C, {}^0\Omega^{\frac{1}{2}}); K \text{ vanishes in a neighbourhood of } \partial(\mathbb{R} \times X \times_0 X) \setminus \mathbb{R} \times \mathcal{F}\}.$$

Since  $\Lambda_C$  intersect the corresponding fibers over the front face transversally, we can define the normal operators of elements  $F \in I_0^{m,s}(\mathbb{R} \times X, X, C, {}^0\Omega^{\frac{1}{2}})$  as in (2.2). Moreover we find that  $N_p(F)$  is a Lagrangian distribution with respect to  $\Lambda_C^0 = \Lambda_C \cap T^*\mathbb{R} \times T^*(X \times_0 X)$ .

We want to understand the mapping properties of these operators under the action of zero differential operators - particularly the Laplacian and the wave operator. A line by line inspection of the proof of Proposition 5.19 of [24] gives its analogue for 0-Fourier integral operators. We have

**Proposition 3.2.** *The normal operator (2.2) defines an exact sequence*

$$(3.9) \quad 0 \longrightarrow I_0^{m,1}(\mathbb{R} \times X, X; C, {}^0\Omega^{\frac{1}{2}}) \longrightarrow I_0^{m,0}(\mathbb{R} \times X, X; C, {}^0\Omega^{\frac{1}{2}}) \longrightarrow I^m(\mathcal{F}, \Lambda_C^0, \Omega^{\frac{1}{2}})$$



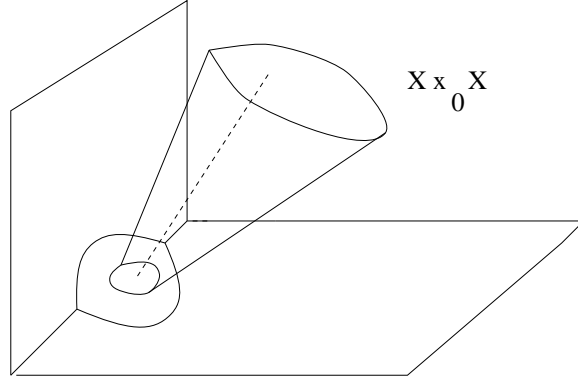


FIGURE 2. The support of the wave kernel at time  $t$ .

such that for any differential operator  $P \in \text{Diff}_0^m(X)$  and any  $F \in I_0^{m,0}(\mathbb{R} \times X, X; C, {}^0\Omega^{\frac{1}{2}})$

$$(3.10) \quad N_p((D_t^2 - P) \cdot F) = (D_t^2 - N_p(P)) \cdot N_p(F).$$

We can now prove

**Theorem 3.1.** For  $t \in \mathbb{R}$ , let  $C$  be the relation defined by (3.5). The wave group  $U(t)$  satisfies

$$(3.11) \quad U(t) = \cos\left(t\sqrt{\Delta - \frac{n^2}{4}}\right) \in I_0^{-\frac{1}{4},0}\left(\mathbb{R} \times X, X; C, {}^0\Omega^{\frac{1}{2}}\right).$$

*Proof.* This is very similar in nature to the construction in section 7 of [24]. The first step is to use the normal operator to remove the Taylor series of the lift of  $U(t)$  at  $\mathcal{F}$ . Using Proposition 3.2 and equation (2.4) we find that the normal operator  $U_0(p, t) = N_p(U(t))$  satisfies

$$(3.12) \quad \begin{aligned} \left(D_t^2 - \Delta_0 - \frac{n^2}{4}\right) U_0(p, t) &= 0, \\ U_0(p, 0) &= \delta(0_p), \quad D_t U_0(p, 0) = 0 \end{aligned}$$

where  $0_p$  is the center of  $F_p$  and  $\Delta_0$  is the normal operator of  $\Delta$ . Since  $0_p$  is away from the boundaries of  $\mathcal{F}$  and  $\Lambda_C^0$  does not intersect the boundaries for finite  $t$ , it follows from the usual theory of Fourier integral operators that  $U_0(t) \in I^{-\frac{1}{4}}(\mathbb{R} \times \mathcal{F}, \Lambda_C^0, \Omega^{\frac{1}{2}})$ , see for example [7].

Since the map (3.9) is surjective, we can choose an element,  $u_0$ , of  $I_0^{-1/4,0}(\mathbb{R} \times X, X; C, {}^0\Omega^{\frac{1}{2}})$ , with  $N_p(u_0) = U_0(t)$ , so that

$$v_0 = U(t) - u_0 \in I_0^{-1/4,1}(\mathbb{R} \times X, X; C, {}^0\Omega^{\frac{1}{2}}).$$

Intead of dividing by  $R$ , we divide by  $x'$  be, which is a defining function of the second copy of  $X$ . The advantage is that this commutes with the wave operator. Now  $(x')^{-1}v_0 \in I^{-1/4,0}$ , and, as  $v_0$  is supported away from the bottom face, no difficulties are introduced. We now solve on the front face to get  $w_1 \in I_0^{-1/4,0}$  satisfying

$$(3.13) \quad \begin{aligned} \left(D_t^2 - \Delta_0 - \frac{n^2}{4}\right) N_p(w_1) &= N_p((x')^{-1}v_0), \\ N_p(w_1)(0) &= N_p((x')^{-1}v_0), \quad D_t N_p(w_1)(0) = D_t(N_p((x')^{-1}v_0))(0). \end{aligned}$$

We let  $u_1 = x'w_1$ . We then have, by the uniqueness of the solution to (3.13) that

$$U(t) - u_0 - u_1 \in I_0^{-1/4,2}(\mathbb{R} \times X, X; C, {}^0\Omega^{\frac{1}{2}})$$

We can now iterate at each level by considering  $(x')^{-k}$  times the error achieved. Since  $u_j$  is supported away from the top and bottom faces,  $\tilde{u}_j = \frac{x'}{x}u_j$  is a Fourier integral operator in the same class. Thus we have found

$$(3.14) \quad U(t) - \sum_{j=0}^k x^j \tilde{u}_j \in I_0^{-1/4,k+1}(\mathbb{R} \times X, X; C, {}^0\Omega^{\frac{1}{2}}).$$

Asymptotically summing, we achieve an error in  $I_0^{-1/4,\infty}(\mathbb{R} \times X, X; C, {}^0\Omega^{\frac{1}{2}})$  and an error in the Cauchy data which vanishes to infinite order at the front face, and which is pseudo-differential operator of order zero.

We can now extend this error term to be identically zero across the front face and remove it in the usual way using Hörmander's Lagrangian calculus. See for example Theorem 1.1 of [7].  $\square$

#### 4. RADIATION FIELDS

The Lorentzian metric corresponding to the wave operator in  $(X, g)$ , in coordinates (1.1) is given by

$$dt^2 - \frac{dx^2}{x^2} - \frac{h(x, y, dy)}{x^2} = dt^2 - (d \log x)^2 - \frac{h(x, y, dy)}{x^2}.$$

So the surfaces  $t \pm \log x = s$  are characteristic with respect to the wave operator. Following [10, 11], we are interested in understanding the limits of solutions to the Cauchy problem

$$(4.1) \quad \begin{aligned} \left( \frac{\partial^2}{\partial t^2} + \Delta_w - \frac{n^2}{4} \right) u(t, w) &= 0, \\ u(0, w) &= f_1(w), \quad \frac{\partial}{\partial t} u(0, w) = f_2(w), \end{aligned}$$

along rays determined by these surfaces. Let  $H(t)$  denote the Heaviside function, and let  $v(t, w) = H(t)u(t, w)$ , then

$$(4.2) \quad \begin{aligned} \left( \frac{\partial^2}{\partial t^2} + \Delta - \frac{n^2}{4} \right) v(t, w) &= f_2 \delta_t(0) - f_1 \delta_t'(0), \\ v &= 0, \text{ for } t < 0. \end{aligned}$$

Let us denote  $\mathcal{S}(X)$  as the space of smooth functions in  $X$  which vanish to infinite order at  $\partial X$ .

**Proposition 4.1.** *Let  $v$  be a solution to (4.2) with  $f_j \in \mathcal{S}(X)$ . Let  $f = (f_1, f_2)$  and let  $w = (x, y)$ ,  $y \in \partial X$ . Then the limit*

$$\lim_{x \rightarrow 0} v(s - \log x, x, y) = \mathcal{R}^+(f)(s, y)$$

*gives a map*

$$\mathcal{R}^+ : \mathcal{S}(X) \times \mathcal{S}(X) \longrightarrow C^\infty(\partial X \times \mathbb{R})$$

The proof of this is based on the Mazzeo-Melrose characterization of the lift of the kernel of the resolvent by the blow-down map  $\beta$  described in Section 2. We outline the main idea here.

For simplicity consider the case where  $f_1 = 0$  and we observe that the kernel of the map  $\mathcal{R}^+$  is given by

$$\lim_{x \rightarrow 0} W(s - \log x, x, y, w') = \mathcal{R}^+(s, y, w')$$

where  $W$  is the forward fundamental solution of the wave operator, i.e

$$(4.3) \quad \left( \frac{\partial^2}{\partial t^2} + \Delta - \frac{n^2}{4} \right) W(t, w, w') = \delta(w, w') \delta_t(0),$$

$$W = 0, \text{ for } t < 0.$$

We know that the resolvent  $R(\frac{n}{2} + i\lambda)$  can be expressed in terms of  $W(t)$ , for  $\Im\lambda \ll 0$  by taking the Fourier transform of  $W$  in the  $t$  variable. We obtain the resolvent, i.e.

$$\widehat{W}(\lambda, w, w') = R(\frac{n}{2} + i\lambda)(w, w'), \text{ for } \Im\lambda \ll 0$$

Let  $\widetilde{W}(s, x, y, w') = W(s - \log x, x, y, w')$  and take  $t = s - \log x$ . Then

$$\widehat{\widetilde{W}}(\lambda, w, w') = x^{-i\lambda} R(\frac{n}{2} + i\lambda)(w, w').$$

Now observe that  $-i\lambda = \frac{n}{2} - \zeta$  and recall that the lift of the resolvent by  $\beta$  is, modulo terms that vanish to infinite order on the top and bottom faces, given by  $\rho^\zeta \rho'^\zeta F(\zeta, \bullet)$ , with  $F(\zeta, \bullet)$  a smooth half-density. Recalling that  $x = R\rho$ , we see that the factors  $\rho^\zeta$  in  $\beta^*(x^{\frac{n}{2}-\zeta} R(\zeta))$  cancel out and it can be restricted to  $\{\rho = 0\}$ . The factor in  $x^{\frac{n}{2}}$  is used to cancel out the negative power in the half-density factor. One can use the push-forward theorems proved in section 4 of [19] to prove that this is in fact the kernel of a Pseudo-differential operator. This shows that if  $F$  is the Fourier transform, then  $F \circ \mathcal{R}^+$  is well defined. One can then show that in fact the map  $\mathcal{R}^+$  is well defined as well.

One can use the methods of [10, 11] to prove that it can be extended to a map on the space defined with the energy norm and that it gives a translation representation of the wave group in a subspace of

$$\{u \in L^2(\mathbb{R} \times \partial X) : \partial_s u \in L^2(\mathbb{R} \times \partial X)\}.$$

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