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**THE CAUCHY PROBLEM FOR SYSTEMS
 -- THROUGH THE NORMAL FORM OF SYSTEMS
 AND THEORY OF WEIGHTED DETERMINANT --**

WAICHIRO MATSUMOTO

Dedicated to Professor Kiyoshi Mochizuki on his 60-th anniversary

ABSTRACT. The author propose what is the principal part of linear systems of partial differential equations in the Cauchy problem through the normal form of systems in the meromorphic formal symbol class and the theory of weighted determinant. As applications, he choose the necessary and sufficient conditions for the analytic well-posedness (Cauchy-Kowalevskaya theorem) and C^∞ well-posedness (Levi condition).

1. INTRODUCTION

The well-posedness of Cauchy problem in various classes for higher order linear scalar equations is well characterized. On the other hand, the results on the well-posedness for systems are rather poor. The reason is that the principal part of system has not been well caught. In this note, the author proposes the definition of the principal part on the Cauchy problem. In order to understand the structure of an usual matrix, the Jordan normal form and the determinant are very useful. The former includes all informations on a matrix and the latter is very convenient. Our aim is to establish the corresponding theory for the matrices of differential operators and to give some of many applications -- the necessary and sufficient conditions for the analytic well-posedness and C^∞ well-posedness --.

Let us consider the following Cauchy problem:

$$(1.1) \quad \begin{cases} \left(\frac{\partial}{\partial t} \right)^{m_k} u_k + \sum_{i=1}^{N_o} \sum_{j=1}^{m_i} \sum_{|\alpha| \leq m(j,i)} a_{\alpha j k i}(t, x) \left(\frac{\partial}{\partial x} \right)^\alpha \left(\frac{\partial}{\partial t} \right)^{m_i - j} u_i = f_k(t, x) \quad , \\ \left(\frac{\partial}{\partial t} \right)^j u_k |_{t=t_o} = u_{o j k}(x), \quad (0 \leq j \leq m_k - 1) \quad , \end{cases}$$

where k runs from 1 to N_o . Adding some unknown functions, for example $\left(\frac{\partial}{\partial x} \right)^\alpha \left(\frac{\partial}{\partial t} \right)^{m_k - j} u_k$ ($0 \leq |\alpha| \leq p j$, $1 \leq j \leq m_k$ and $1 \leq k \leq N_o$) for suitable non-negative p (1.1) is reduced to

$$(1.2) \quad \begin{cases} D_t u - \sum_{|\alpha| \leq m} A_\alpha(t, x) D_x^\alpha u = f(t, x) \quad , \\ u|_{t=t_o} = u_o(x) \quad , \end{cases}$$

Key words and phrases. normal form of systems, p-determinant of matrix of pseudo-differential operators, p-evolution, the Cauchy-Kowalevskaya theorem for systems, C^∞ well-posedness for systems.

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where, A_α is a $\mathbf{N} \times \mathbf{N}$ matrix of smooth functions ($|\alpha| \leq m$), u , u_\circ and f are vectors of dimension \mathbf{N} , $D_t = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial t}$ and $D_x = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x}$.

First, we consider some examples. Let ∂t and ∂x be $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$, respectively.

Example 1.

$$P_1(t, x, \partial t, \partial x) = I_2 \partial t - A^1(\partial x),$$

$$A^1 = \begin{pmatrix} (\partial x_1)^m (\partial x_2)^m & -(\partial x_2)^{2m} \\ (\partial x_1)^{2m} & -(\partial x_1)^m (\partial x_2)^m \end{pmatrix}, \quad (\text{of order } 2m, m \in \mathbf{N})$$

For the above operator, the Cauchy problem (1.2) has the solution

$$u = u_\circ + (t - t_\circ) A^1(\partial x) u_\circ + \int_{t_\circ}^t f(s, x) ds + \int_{t_\circ}^t (t - s) A^1(\partial x) f(s, x) ds.$$

The highest order part A^1 of P_1 has no influence for the well-posedness in any class.

Let us see this roughly. We represent P_1 using D_t and D_x .

$$P'_1 = P_1 / \sqrt{-1} = I_2 D_t - (-1)^{m-1} \sqrt{-1} \begin{pmatrix} D_{x_1}^m D_{x_2}^m & -D_{x_2}^{2m} \\ D_{x_1}^{2m} & -D_{x_1}^m D_{x_2}^m \end{pmatrix}.$$

$$\text{We transform } P_1 \text{ formally by } N_1 = \begin{pmatrix} -(-1)^{m-1} \sqrt{-1} D_{x_1}^{-2m} D_{x_2}^{2m} & 0 \\ -(-1)^{m-1} \sqrt{-1} D_{x_1}^{-m} D_{x_2}^m & 1 \end{pmatrix};$$

$$N_1^{-1} \circ P_1 \circ N_1 = I_2 D_t - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} D_{x_1}^{2m},$$

where $A \circ B$ is the operator product. Further, by $N'_1 = \text{diag}(D_{x_1}^r, 1)$ for an arbitrary real r ,

$$N_1'^{-1} \circ N_1^{-1} \circ P_1 \circ N_1 \circ N_1' = I_2 D_t - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} D_{x_1}^{2m-r}.$$

This means the true principal part of P_1 is $I_2 \partial t$ and the influence of the highest order part A^1 is negligible.

In Example 1, the highest order part is nilpotent and the transforming matrix N_1 or its inverse must have a pole set.

Example 2.

$$P_2(t, x, \partial t, \partial x) = I_2 \partial t - A^2(t, x, \partial x),$$

$$A^2 = \begin{pmatrix} tx & -x^2 \\ t^2 & -tx \end{pmatrix} (\partial x)^m, \quad (m \in \mathbf{N}, \ell = 1)$$

In this case, also A^2 is nilpotent. Let us represent P_2 using D_t and D_x ;

$$P'_2 = P_2/\sqrt{-1} = I_2 D_t - (\sqrt{-1})^{m-1} \begin{pmatrix} tx & -x^2 \\ t^2 & -tx \end{pmatrix} D_x^m .$$

By $N_2(t, x) = \begin{pmatrix} -(\sqrt{-1})^{m-1}x^2 & 0 \\ -(\sqrt{-1})^{m-1}tx & 1 \end{pmatrix}$, $N_2^{-1} \circ P'_2 \circ N_2$ becomes

$$\begin{aligned} P''_2 = I_2 D_t - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} D_x^m - (\sqrt{-1})^{m-2} m \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} D_x^{m-1} \\ - (\sqrt{-1})^{m-3} m(m-1) \begin{pmatrix} t/x & 0 \\ 0 & 0 \end{pmatrix} D_x^{m-2} - (\sqrt{-1})^{m-2} \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} . \end{aligned}$$

By the existence of the commutator, the Cauchy problem for P_2 is not analytically well-posed if $m \geq 3$. (See Section 4.) On Example 2, we shall consider again in Subsection 3.2. In Example 2, the highest order part is also nilpotent and the transforming matrix N_2 or its inverse must have a pole set.

Examples 1 and 2 suggest that in order to consider the normal form, we need accept the meromorphy or some singularity and need consider it under non-commutative product: the operator product. Of course, for the theory of the determinant, this requirement is same.

In Section 2, we explain the normal form of systems in the *formal symbol class*. In Section 3, we do the theory of the weighted determinant, so called *p-determinant* and introduce the notion of *p-evolution*. In Section 4, we give the necessary and sufficient condition for the analytic well-posedness (the Cauchy-Kowalevskaya theorem). We give a remark and a conjecture also on the C-K theorem of Nagumo type, relaxation of the regularity of coefficients. In Section 5, we give the necessary and sufficient condition for the C^∞ well-posedness assuming the constant multiplicity of characteristic roots and the real analyticity of coefficients (Levi condition). We give some remarks when the coefficients are not real analytic. The situations on the analytic well-posedness and C^∞ well-posedness in case of the constant multiplicity are very similar if coefficients are real analytic. However, the phenomena are very different when coefficients are non-quasianalytic.

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2. NORMAL FORM OF SYSTEMS

We follow the results in W.Matsumoto[25] and [27]. From an arbitrary asymptotic expansion of ultradifferentiable class, a true symbol of the same class can be constructed and the ambiguity is of class $S^{-\infty}$. (See L.Boutet de Monvel and P.Krée[7], L.Boutet de Monvel[6] and W.Matsumoto[24].) Therefore, in order to consider many problems on partial differential equations in a ultradifferentiable class, it is sufficient to consider asymptotic expansions, which we call here *formal symbols*. Let \mathbf{Z}_+ be $\mathbf{N} \cup \{0\}$. We

use the followings for α and β in $\mathbf{Z}_+^{1+\ell}$: $|\alpha| = \alpha_0 + \cdots + \alpha_\ell$, $\alpha! = \alpha_0! \alpha_1! \cdots \alpha_\ell!$, $\alpha + \beta = (\alpha_0 + \beta_0, \cdots, \alpha_\ell + \beta_\ell)$ and we denote $\beta \leq \alpha$ when $\beta_i \leq \alpha_i$ for $0 \leq i \leq \ell$. Let us set $a(t, x, \xi)_{(\alpha)}^{(\beta)} = D_t^{\alpha_0} D_{x_1}^{\alpha_1} \cdots D_{x_\ell}^{\alpha_\ell} (\frac{\partial}{\partial \xi})^\beta a(t, x, \xi)$ for $\alpha \in \mathbf{Z}_+^{1+\ell}$ and $\beta \in \mathbf{Z}_+^\ell$.

We introduce a holomorphic formal symbol and a meromorphic one. We say that a set O in $\mathbf{C}_t \times \mathbf{C}_x^\ell \times \mathbf{C}_\xi^\ell$ is conic when $\xi \in O$ implies $\lambda \xi \in O$ for arbitrary positive λ and that a subset Γ in O is conically compact in O when Γ is conic and $\Gamma \cap \{|\xi| = 1\}$ is compact in $O \cap \{|\xi| = 1\}$, where $|\xi| = \sqrt{\sum_{i=1}^\ell |\operatorname{Re} \xi_i|^2 + |\operatorname{Im} \xi_i|^2}$. We say that Σ is a subvariety of O if it is a zero set of a holomorphic function in O .

Definition 1. (*Meromorphic and holomorphic formal symbol*, [25])

I. We say that the formal sum $a(t, x, \xi) = \sum_{i=0}^\infty a_i(t, x, \xi)$ is a meromorphic formal symbol (= *m.f.s.*) on O when there exist a conic subvariety Σ in O and a real number κ such that

- 1) $a_i(t, x, \xi)$ is meromorphic in O , holomorphic in $O \setminus \Sigma$ and positively homogeneous of degree $\kappa - i$ on ξ , ($i \in \mathbf{Z}_+$).
- 2) For arbitrary conically compact set Γ in O , there are positive constants C , R and R' and we have

$$(2.1) \quad |a_i^{(\beta)}_{(\alpha)}(t, x, \xi)| \leq C R^i R^{|\alpha|+|\beta|} i! |\alpha|! |\beta|! |\xi_1|^{\kappa-i} \quad \text{on } \Gamma, \\ (i \in \mathbf{Z}_+, \alpha \in \mathbf{Z}_+^{1+\ell}, \beta \in \mathbf{Z}_+^\ell).$$

II. The formal sum $\sum_{i=0}^\infty a_i$ is called a holomorphic formal symbol (= *h.f.s.*) when it is a meromorphic formal symbol with $\Sigma = \emptyset$.

Remark 2.1. We use ξ_1 as a holomorphic scale of order in case of a complex domain and Σ includes $\{\xi_1 = 0\}$. Of course, ξ_1 can be replaced by another ξ_i and Σ includes $\{\xi_i = 0\}$.

Remark 2.2. It is important that Σ is independent of i .

Now, we define a formal symbol of class $\{M_n, L_n\}$ on a real domain. Let $\{M_n\}_{n=0}^\infty$ and $\{L_n\}_{n=0}^\infty$ be sequences of positive numbers. We assume that $\log M_n = O(n^2)$ (Differentiability condition) and $\{M_n/n!\}_{n=0}^\infty$ and $\{L_n/n!\}_{n=0}^\infty$ are logarithmically convex and non-decreasing. We say that a set O in $\mathbf{R}_t \times \mathbf{R}_x^\ell \times \mathbf{R}_\xi^\ell$ is conic when $\xi \in O$ implies $\lambda \xi \in O$ for arbitrary positive λ and that a subset Γ in O is conically compact in O when Γ is conic and $\Gamma \cap \{|\xi| = 1\}$ is compact in $O \cap \{|\xi| = 1\}$, where $|\xi| = \sqrt{\sum_{i=1}^\ell \xi_i^2}$.

Definition 2. (*Formal symbol of class $\{M_n, L_n\}$* , [25])

We say that the formal sum $a(t, x, \xi) = \sum_{i=0}^\infty a_i(t, x, \xi)$ is a formal symbol of class $\{M_n, L_n\}$ (= *f.s. of class $\{M_n, L_n\}$*) on O when there exists a real number κ such that

- 1) $a_i(t, x, \xi)$ belongs to $C^\infty(O)$ and positively homogeneous of degree $\kappa - i$ on ξ , ($i \in \mathbf{Z}_+$).
- 2) For arbitrary conically compact subset Γ in O , there are positive constants C , R and R' and we have

$$(2.2) \quad |a_i^{(\beta)}_{(\alpha)}(t, x, \xi)| \leq C R^i R^{|\alpha|+|\beta|} M_{i+|\alpha|} L_{i+|\beta|} i!^{-1} |\xi|^{\kappa-i-|\beta|} \quad \text{on } \Gamma, \\ (i \in \mathbf{Z}_+, \alpha \in \mathbf{Z}_+^{1+\ell}, \beta \in \mathbf{Z}_+^\ell).$$

The number κ is called the order of the formal symbol a and denoted by $ord a$. When $a_i = 0$ for $0 \leq i \leq i_0 - 1$ and $a_{i_0} \neq 0$, $\kappa - i_0$ is called the true order of a and denoted by *true ord a*. The order of 0 is posed $-\infty$. We set $S_M^\kappa(O) = \{ \text{the m.f.s.'s on } O \text{ of order } \kappa \}$, $S_H^\kappa(O) = \{ \text{the h.f.s.'s on } O \text{ of order } \kappa \}$, $S^\kappa\{M_n, L_n\}(O) = \{ \text{the f.s.'s of class } \{M_n, L_n\} \text{ on } O \text{ of order } \kappa \}$, and $S_M(O) = \cup_{\kappa \in \mathbf{R}} S_M^\kappa(O)$, etc. We denote one of these simply by $S(O)$.

Corresponding to the asymptotic expansion of the symbol of the product of pseudo-differential operators, we introduce the operator product of formal symbols.

Definition 3. (*Operator product*)

Let $a = \sum_{i=0}^{\infty} a_i$ and $b = \sum_{i=0}^{\infty} b_i$ be formal symbols. We set

$$(2.3) \quad a \circ b = \sum_{i=0}^{\infty} c_i \quad , \quad c_i(t, x, \xi) = \sum_{i_1+i_2+|\gamma|=i} \frac{1}{\gamma!} a_{i_1}^{(\gamma)}(t, x, \xi) b_{i_2(\gamma)}(t, x, \xi)$$

and call it the operator product of a and b .

By the operator product, S_H and $S\{M_n, L_n\}$ become non-commutative rings and S_M does a non-commutative field. S_H is a subring of S_M .

Let us consider a matrix $P = I_{\mathbf{N}} D_t - A(t, x, \xi)$, $A \in M_{\mathbf{N}}(S^m)$, ($m \in \mathbf{N}$). In [25] and [27], we obtained the following theorem.

Theorem 1. (*Normal form of system (1), [25]*)

We assume that every entry of A satisfies (2.2) ((2.1) in case of m.f.s.) with $\kappa = m$ and that the each eigenvalue $\lambda_k(t, x, \xi)$ ($1 \leq k \leq d$) of A_0 has the constant multiplicity m_k . Then, there exist finite disjoint open conical sets $\{O_h\}_h$ such that $\cup_h O_h$ is dense in O . On each O_h , there exist natural numbers d_k and $\{n_{kj}\}_{j=1}^{d_k}$ ($\sum_{j=1}^{d_k} n_{kj} = m_k$). For every point (t_0, x_0, ξ_0) in O_h , there exist a conically compact neighborhood Γ , $N(t, x, \xi) = \sum_{i=0}^{\infty} N_i$ in $GL(\mathbf{N}; S^0(\Gamma))$, and $D_{kj}(t, x, \xi) = \sum_{i=0}^{\infty} D_{kji}$ in $M_{n_{kj}}(S^m(\Gamma))$, such that

$$(2.4) \quad \begin{aligned} N^{-1}(t, x, \xi) \circ P(t, x, D_t, \xi) \circ N(t, x, \xi) &= \oplus_{1 \leq k \leq d} \oplus_{1 \leq j \leq d_k} P_{kj} , \\ P_{kj}(t, x, D_t, \xi) &= I_{n_{kj}}(D_t - \lambda_k(t, x, \xi)) - D_{kj}(t, x, \xi) \\ D_{kj0} &= J(n_{kj})|\xi|^m, \\ D_{kji} &= \begin{pmatrix} 0 \\ * \cdots * \end{pmatrix} \quad \text{homogeneous of degree } m - i \quad (i \geq 1), \end{aligned}$$

where $J(n) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix} : n \times n$. We set $\sum_{i=1}^{\infty} D_{kji} = \begin{pmatrix} O \\ d_{kj}(1) \cdots d_{kj}(n_{kj}) \end{pmatrix}$.

In case of meromorphic formal symbol, $\{O_h\}_h$ is composed by only one element and $O_1 = O \setminus \Sigma'$ for a subvariety Σ' . N and D_{kj} belong to $GL(\mathbb{N}; S_M^0(O))$ and $M_{n_{kj}}(S_M^m(O))$, respectively. In (2.4), we replace $|\xi|^m$ by ξ_1^m .

One may think that the assumption of the constant multiplicity is too strong. However, if we regard P as an operator of order $m + 1$ on D_x , the highest order part is the zero-matrix and has an unique eigenvalue zero of constant multiplicity \mathbb{N} . Thus, under no condition of the structure, we can reduce P to the normal form.

Corollary 2. (Normal form of systems (2))

We assume that every entry of A satisfies (2.2) ((2.1) in case of m.f.s.) with $\kappa = m$. There exist finite disjoint open conical sets $\{O_h\}_h$ such that $\cup_h O_h$ is dense in O . On each O_h , there exist natural numbers d and $\{n_k\}_{k=1}^d$ ($\sum_{k=1}^d n_k = \mathbb{N}$) and $N_o(t, x, \xi)$ in $GL(\mathbb{N}; S)$ such that

$$(2.5) \quad \begin{aligned} N_o^{-1}(t, x, \xi) \circ P(t, x, D_t, \xi) \circ N_o(t, x, \xi) &= Q = \oplus_{1 \leq k \leq d} Q_k, \\ Q_k(t, x, D_t, \xi) &= I_{n_k} D_t - \sum_{i=0}^{\infty} B_{ki}(t, x, \xi), \\ B_{k0} &= J(n_k) |\xi|^{m+1}, \\ B_{ki} &= \begin{pmatrix} O \\ * \dots * \end{pmatrix} : \text{homogeneous of order } m + 1 - i, \quad (i \geq 1) \end{aligned}$$

$$\text{We set } \sum_{i=1}^{\infty} B_{ki} = \begin{pmatrix} O \\ b_k(1) \dots b_k(n_k) \end{pmatrix}.$$

In case of meromorphic formal symbol, $\{O_h\}_h$ is composed by only one element and $O_1 = O \setminus \Sigma'$ for a subvariety Σ' . N_o and B_k belong to $GL(\mathbb{N}; S_M^0(O))$ and $M_{n_k}(S_M^{m+1}(O))$, respectively. In (2.5), we replace $|\xi|^{m+1}$ by ξ_1^{m+1} .

Remark 2.3. In Theorems 1 and 2, $\{O_h\}_h$ has finite elements but can have countably many connected components in case of non-quasianalytic case. This cause a difficulty on the Cauchy problem. (See Example 6 in Subsection 4.5 and Example 7 in Subsection 5.1.)

The idea of the proof is the following. First, we consider the case of m.f.s. As we accept the meromorphy, we can transform the operator to Arnold-Petkov's normal form. (See V.I.Arnold[4] and V.M.Petkov[47].) Let us consider a simple model.

Example 3. Let us take the integers $0 < k < m$ and constants a and b . We consider

$$P_3 = I_3 D_t - \begin{pmatrix} 0 & D_x^m & 0 \\ 0 & 0 & a D_x^k \\ b & 0 & 0 \end{pmatrix}.$$

The highest order part has the Jordan structure $J(2) \oplus J(1)$ and the lower order term has the form $\begin{pmatrix} 0 & 0 & 0 \\ * & * & * \\ * & 0 & * \end{pmatrix}$, then P_3 is of Arnold-Petkov's normal form.

If $a \neq 0$, by $N_3 = \text{diag}(1, 1, D_x^{m-k})$, $N_3^{-1} \circ P_3 \circ N_3$ becomes

$$P_3^1 = I_3 D_t - \begin{pmatrix} 0 & 1 & \\ & 0 & a \\ & & 0 \end{pmatrix} D_x^m + \begin{pmatrix} 0 & & \\ 0 & 0 & \\ b & 0 & 0 \end{pmatrix} D_x^{-(m-k)}$$

and its highest order part has the Jordan structure $J(3)$.

If $a = 0$ and $b \neq 0$, by $N_3' = \text{diag}(1, 1, D_x^{-m})$, $N_3'^{-1} \circ P_3 \circ N_3'$ becomes

$$P_3^2 = I_3 D_t - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} D_x^m$$

and its highest order part also has the structure $J(3)$.

Transforming again P_3^j ($j = 1, 2$) to the Arnold-Petkov's normal form, we arrive at the final normal form.

If $a = b = 0$, P_3 is splitted perfectly to $P_3(2) \oplus P_3(1)$ and P_3 itself is of the final normal form.

Thus, in finite procedures, we arrive at the above theorem. (See the detailed proof in [25].)

In case of non-quasianalytic classes, we stand on the following simple property;

For a continuous function $f(x)$ on an open set O , the set $\{x \mid f(x) \neq 0\} \cup \{x \mid f(x) = 0\}^o$ is open and dense in O , where A^o is the open kernel of A .

By this property, we can also obtain the normal form in case of non-quasianalytic classes on an open dense set.

A higher order scalar equation

$$(\partial t)^m u + \sum_{j=1}^m \sum_{|\alpha| \leq m(j)} a_{\alpha j}(t, x) (\partial x)^\alpha (\partial t)^{m-j} u = f(t, x) \quad ,$$

is reduce to a first order system on D_t for a suitable positive number p :

$$D_t u - J(N) D_x^p u - B(t, x, D_x) u = f(t, x) \quad ,$$

where the lower order term B has the form $\begin{pmatrix} 0 \\ * \dots * \end{pmatrix}$. Therefore, Corollary 2 say that a system is reduced to a direct sum of some higher order scalar equations in an open dense set in $\Omega \times \mathbf{C}^\ell \setminus \Sigma$ modulo $S^{-\infty}$. Thus, if we can obtain a result microlocally and modulo $S^{-\infty}$ and if such result on a dense open set implies the global one, we can apply the proof on scalar equations also to systems. Many necessary conditions of the well-posedness have these properties, the continuity of the conditions for the latter. On the other hand, for the

sufficiency, if we assume the real analyticity of coefficients, we can apply the maximum principle for the latter in some cases, for example, the results in Sections 4 and 5.

In Example 3, each order m , k and 0 are not essential. the product abD_x^{m+k} is essential. This leads us the theory of determinant.

3. p -DETERMINANT OF MATRIX OF DIFFERENTIAL OPERATORS AND p -EVOLUTION

3.1. Definition of p -determinant.

On the matrix of partial differential operators, G.Hufford[14] first introduced the determinant applying the theory of J.Dieudonné[12], which is a determinant theory on a non-commutative field. M.Sato and M.Kashiwara[50] obtained the regularity property of the determinant. The algebraic structure of the determinant on the ring with Ore's property is well characterized by K.Adjamagbo[2] and [3]. The determinant by G.Hufford and M.Sato and M.Kashiwara is homogeneous. However, in order to consider, for example the parabolic equations and Schrödinger type equations, we encounter inhomogeneous principal parts and need an inhomogeneous determinant. In order to describe the Levi condition for C^∞ well-posedness, we also need an inhomogeneous determinant. Recently, the author has received a preliminary version of a paper by A.D'Agnolo and G.Tagliatela[9], where they define independently the same weighted determinant as mine. Their definition and consideration are more algebraically and systematic than mine.

First we consider $S_M[D_t]$. This is a non-commutative integral domain with Ore's property: for non-zero elements a and b , we can find non-zero c and d such that $ac = bd$. Ore's property is the necessary and sufficient condition for the existence of the quotient field. (See O.Ore[44].)

We fix a positive rational number p . Let us take $a(t, x, \xi, D_t) = \sum_{j=0}^m a^{<j>}(t, x, \xi) D_t^{m-j}$, $a^{<j>} = \sum_{i=0}^\infty a_i^{<j>} \in S_M$. We reset the order of $a^{<j>}$ to its true order. let us set

$$p\text{-ord } a^{<j>}(t, x, \xi) D_t^{m-j} = \text{ord } a^{<j>} + p(m-j)$$

$$p\text{-ord } a = \max_{0 \leq j \leq m} p\text{-ord } a^{<j>}(t, x, \xi) D_t^{m-j}$$

and call them the p -order. By p -order, $S_M[D_t]$ becomes a filtered ring. We set further

$$R^{(p)}(a) = \{j : p\text{-ord } a^{<j>} D_t^{m-j} = p\text{-ord } a\}$$

$$a_{p\text{-pr}}(t, x, \xi, \tau) = \sum_{j \in R^{(p)}(a)} a_0^{<j>}(t, x, \xi) \tau^{m-j}$$

and call the latter the p -principal symbol of a . The set $\cup_{p>0} \{a_0^{<j>}(t, x, \xi) \tau^{m-j}\}_{j \in R^{(p)}(a)}$ has finite elements and composes the Newton polygon of a .

Let us take $c(t, x, \xi, \tau) = \sum_{j=0}^m c^{<j>}(t, x, \xi) \tau^{m-j}$ a polynomial on τ whose coefficients are homogeneous on ξ respectively. We say that $c(t, x, \xi, \tau)$ is a p -homogeneous polynomial on τ when all $\text{deg } c^{<j>} + p(m-j)$ coincide each other for $0 \leq j \leq m$. For p -homogeneous c , we call common $\text{deg } c^{<j>} + p(m-j)$ the p -degree of c and denote it by $p\text{-deg } c$. Let us set

$$Y = \{p\text{-homogeneous polynomials on } \tau\}.$$

Y is a commutative ring. The map σ^p from $S_M[D_t] \setminus \{0\}$ to $Y \setminus \{0\}$ defined by $\sigma^p(a) = a_{p-pr}$ is a homomorphism of the productive semigroup. This is naturally extended to the map from $S_M[D_t]^Q \setminus \{0\}$ to $Y^Q \setminus \{0\}$ by $\sigma^p(ab^{-1}) = a_{p-pr}/b_{p-pr}$ as a homomorphism of the productive group, where R^Q is the quotient field of a ring R with Ore's property. (By virtue of Ore's property, if $ab^{-1} = a'b'^{-1}$, it holds that $a_{p-pr}/b_{p-pr} = a'_{p-pr}/b'_{p-pr}$ and the map σ^p is well defined on $S_M[D_t]^Q \setminus \{0\}$.) We put $\sigma^p(0) = 0$. Thus, we can obtain the weighted determinant theory by σ^p following J.Dieudonné[12]. (See also E.Artin[5] and K.Adjamagbo[2] and [3].)

In case of non-quasianalytic classes, we stand on the following simple property mentioned in Section 2;

For a continuous function $f(x)$ on an open set O , the set $\{x \mid f(x) \neq 0\} \cup \{x \mid f(x) = 0\}^o$ is open and dense in O , where A^o is the open kernel of A .

By this property, for continuous $\{f_j(x)\}_{1 \leq j \leq d}$, we can find finite disjoint open sets $\{O_h\}_h$ such that the union is dense in O and that $f_j(x) \neq 0$ or else $\equiv 0$ on each O_h . Using this property, we can define p -determinant for matrices with entries in $S\{M_n, L_n\}[D_t]$ on an open dense set. Of course, we can also take the space of the formal symbols of C^∞ -class instead of $S\{M_n, L_n\}$. The existence of the limit of p -determinant at the boundary of the open dense set is not clear.

Definition 4. (*p-determinant*)

We call the determinant by σ^p of a matrix A with entries in $S[D_t]$ *p-determinant* of A and denote it by $p\text{-det } A$.

Remark 3.1. 1-determinant is just Hufford and Sato-Kashiwara's determinant.

K.Adjamagbo[1] pointed out that the determinant theory is not almighty.

Example 4.

$$\begin{aligned} P_4 &= I_2 D_t - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ P'_4 &= I_2 D_t. \end{aligned}$$

On the Cauchy problem for P_4 , we can pose the Cauchy data not only $(u_1(t_o, x), u_2(t_o, x))$ but also $(u_2(t_o, x), D_t u_2(t_o, x))$ to obtain a unique solution. On the other hand, for P'_4 , we can pose only $(u_1(t_o, x), u_2(t_o, x))$. The p -determinant of a triangular matrix is the product of the images of the diagonal elements by σ^p and then $p\text{-det } P_4 = p\text{-det } P'_4 = \tau^2$ for all positive p . As the off-diagonal elements of a triangular matrix has no influence on p -determinant, the number of data to be posed is decided by the determinant but the acceptable distributions of the initial data cannot be decided by the determinant theory. In case of the ordinary determinant theory, the determinant can give the dimension of a generalized eigen-space but cannot decide its structure. The situation is similar.

3.2. properties of p -determinant.

By J.Dieudonné[12], the following elementary property hold on p -determinant.

Theorem 3. (Elementary property of p -determinant)

We take $A = (a^{ij})_{1 \leq i, j \leq N}$ and B in $M_{\mathbb{N}}(S[D_t])$.

- (1) $p\text{-det } AB = p\text{-det } A \cdot p\text{-det } B$.
- (2) $p\text{-det } A \oplus p\text{-det } B = p\text{-det } A \cdot p\text{-det } B$. (In this case, the sizes of A and B can be different.)
- (3) p -determinant is invariant under the similar transformation.
- (4) If there are real numbers m_i and n_j such that $p\text{-ord } a^{ij} \leq m_i + n_j$ and the usual determinant $\det(\sigma_{m_i+n_j}^p(a^{ij}))_{1 \leq i, j \leq N}$ does not vanish, then $p\text{-det } A = \det(\sigma_{m_i+n_j}^p(a^{ij}))$, where $\sigma_{m_i+n_j}^p(a^{ij})$ is a_{p-pr}^{ij} if $p\text{-ord } a^{ij} = m_i + n_j$ and is 0 if $p\text{-ord } a^{ij} < m_i + n_j$.

Here, on the matrix of the form $P = I_{\mathbb{N}}D_t - A$, $A \in M_{\mathbb{N}}(S^m)$, we give the representation of p -determinant using the element of the normal form in Theorem 2.

Let us set

$$(3.1) \quad \begin{aligned} \text{true order } b_k(h) &= r_h^k, \\ M_k^p &= \max_{1 \leq h \leq n_k} \{r_h^k + (m+1)(n_k - h) + p(h-1)\}, \\ R_k^p &= \{h : r_h^k + (m+1)(n_k - h) + p(h-1) = M_k^p\} \end{aligned}$$

Applying the property (4) in Theorem 3, we have the following.

Proposition 3.1. (Relation between the normal form and p -determinant)

$$(3.2) \quad \begin{aligned} p\text{-det } P &= \prod_{k=1}^d p\text{-det } Q_k, \\ p\text{-det } Q_k &= \begin{cases} \tau^{n_k}, & (pn_k > M_k), \\ \tau^{n_k} - \sum_{h \in R_k^p} b_k(h)_0(t, x, \xi) |\xi|^{(m+1)(n_k-h)} \tau^{h-1}, & (pn_k = M_k), \\ - \sum_{h \in R_k^p} b_k(h)_0(t, x, \xi) |\xi|^{(m+1)(n_k-h)} \tau^{h-1}, & (pn_k < M_k), \end{cases} \\ &= \text{the highest } p\text{-degree part of the ordinary determinant of } Q_k \end{aligned}$$

In case of m.f.s., $|\xi|^{(m+1)(n_k-h)}$ is replaced by $\xi_1^{(m+1)(n_k-h)}$.

Thus, $p\text{-det } P$ is a polynomial of τ . On the determinant theory, the regularity property is important. In case of $S = S_H$, as the above P is a polynomial of τ , the meromorphy can be occur in (t, x, ξ) space and the proof of Sato-Kashiwara is directly applicable.

(We need not transform the pole set to $\xi_1 = 0$.)

Theorem 4. (Regularity of p -determinant)

- (1) For $P = I_{\mathbb{N}}D_t - A$, $A \in M_{\mathbb{N}}(S_H)$, $p\text{-det } P$ is a polynomial of τ with holomorphic coefficients on (t, x, ξ) .
- (2) For a matrix of partial differential operators with holomorphic coefficients on t and x , $p\text{-det } P$ is a polynomial of τ and ξ with holomorphic coefficients on t and x .

A.D'Agnolo and G.Tagliatela[9] algebraically showed the regularity of p -determinant without use of the normal form.

The regularity of p -determinant is delicate. To see it, we give an example, which is a little generalized one of Example 2.

Example 5. Let m be a natural number.

$$P_5(t, x, \partial t, \partial x) = I_2 \partial t - A^5(t, x, \partial x),$$

$$A^5 = \begin{pmatrix} tx & -x^2 \\ t^2 & -tx \end{pmatrix} (\partial x)^m + \begin{pmatrix} b_{11}(t, x) & b_{12}(t, x) \\ b_{21}(t, x) & b_{22}(t, x) \end{pmatrix} (\partial x)^{m-1}, \quad (m \in \mathbb{N}, \ell = 1)$$

Let us represent P_5 using D_t and D_x ;

$$P'_5 = P_5 / \sqrt{-1}$$

$$= I_2 D_t - (\sqrt{-1})^{m-1} \begin{pmatrix} tx & -x^2 \\ t^2 & -tx \end{pmatrix} D_x^m - (\sqrt{-1})^{m-2} \begin{pmatrix} b_{11}(t, x) & b_{12}(t, x) \\ b_{21}(t, x) & b_{22}(t, x) \end{pmatrix} D_x^{m-1}.$$

By the same $N_2(t, x)$ in Example 2, $N_2^{-1} \circ P'_5 \circ N_2$ becomes

$$P''_5 = I_2 D_t - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} D_x^m$$

$$- \begin{pmatrix} -(\sqrt{-1})^m \{mt + b_{11} + (t/x)b_{12}\} & \sqrt{-1}(1/x^2)b_{12} \\ (-1)^m \sqrt{-1} \{t^2 b_{12} + tx(b_{11} - b_{22}) - x^2 b_{21}\} & -(\sqrt{-1})^m \{-(t/x)b_{12} + b_{22}\} \end{pmatrix} D_x^{m-1}$$

$$- (m-1) \begin{pmatrix} -(\sqrt{-1})^{m-1}(1/x) \{mt + 2b_{11} + (t/x)b_{12}\} & 0 \\ (-1)^m [tb_{11} - xb_{21} + (1/x) \{t^2 b_{12} + tx(b_{11} - b_{22}) - x^2 b_{21}\}] & 0 \end{pmatrix} D_x^{m-2}$$

$$- (m-1)(m-2) \begin{pmatrix} (\sqrt{-1})^m (1/x^2) b_{11} & 0 \\ -(-1)^m \sqrt{-1} \{(t/x)b_{11} - b_{21}\} & 0 \end{pmatrix} D_x^{m-3}$$

$$- (\sqrt{-1})^m \begin{pmatrix} 0 & 0 \\ -x & 0 \end{pmatrix}.$$

Applying Theorem 3 (3) and (4), we can see the following.

For $m = 1$, P_5 is always 1/2-evolutive and

$$(3.3) \quad (1/2) - \det P'_5 = \tau^2 + \sqrt{-1}\{x + t^2b_{12} + tx(b_{11} - b_{22}) - x^2b_{21}\}\xi.$$

(The last term of the right-hand side of (3.3) cannot vanish identically for any choice of $\{b_{ij}\}$.)

For $m \geq 2$, if

$$(3.4) \quad t^2b_{12} + tx(b_{11} - b_{22}) - x^2b_{21} \neq 0,$$

P_5 is $(m - 1/2)$ -evolutive and

$$(3.5) \quad (m - 1/2) - \det P'_5 = \tau^2 - (-1)^m \sqrt{-1}\{t^2b_{12} + tx(b_{11} - b_{22}) - x^2b_{21}\}\xi^{2m-1}.$$

On the other hand, if

$$(3.6) \quad t^2b_{12} + tx(b_{11} - b_{22}) - x^2b_{21} = 0,$$

P'_5 becomes

$$\begin{aligned} & I_2 D_t - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} D_x^m \\ & - \begin{pmatrix} -(\sqrt{-1})^m \{mt + b_{11} + (t/x)b_{12}\} & \sqrt{-1}(1/x^2)b_{12} \\ 0 & -(\sqrt{-1})^m \{-(t/x)b_{12} + b_{22}\} \end{pmatrix} D_x^{m-1} \\ & - (m-1) \begin{pmatrix} -(\sqrt{-1})^{m-1}(1/x)\{mt + 2b_{11} + (t/x)b_{12}\} & 0 \\ (-1)^m(tb_{11} - xb_{21}) & 0 \end{pmatrix} D_x^{m-2} \\ & - (m-1)(m-2) \begin{pmatrix} (\sqrt{-1})^m(1/x^2)b_{11} & 0 \\ -(-1)^m\sqrt{-1}\{(t/x)b_{11} - b_{21}\} & 0 \end{pmatrix} D_x^{m-3} \\ & - (\sqrt{-1})^m \begin{pmatrix} 0 & 0 \\ -x & 0 \end{pmatrix}. \end{aligned}$$

For $m = 2$, if (3.6) is satisfied, P_5 is 1-evolutive and

$$(3.7) \quad 1 - \det P'_5 = \tau^2 - (2t + b_{11} + b_{22})\xi\tau + [-x + \{t + b_{11} + (t/x)b_{12}\}\{b_{22} - (t/x)b_{12}\}]\xi^2$$

(The right-hand side of (3.7) cannot be τ^2 for any choice of $\{b_{ij}\}$.)

For $m \geq 3$, further, if the following is satisfied

$$(3.8) \quad |mt + b_{11} + b_{22}|^2 + |(tx + xb_{11} + tb_{12})(tb_{12} - xb_{22})|^2 \neq 0,$$

P_5 is $(m - 1)$ -evolutionary and

$$(3.9) \quad (m - 1) - \det P'_5 = \tau^2 \quad +(\sqrt{-1})^m(mt + b_{11} + b_{22})\xi^{m-1}\tau \\ +(-1)^m\{t + b_{11} + (t/x)b_{12}\}\{b_{22} - (t/x)b_{12}\}\xi^{2m-2}$$

In (3.7) and (3.9), $(t/x)b_{12}$ appears and it looks singular. However, the condition (3.6) implies $x|b_{12}$ and $t|b_{21}$ and $(m - 1)$ - $\det P'_5$ is regular.

We can see that P_5 is at least $(m/2)$ -evolutionary. As the condition for the Cauchy-Kowalevskaya theorem is p -evolution for $0 \leq p \leq 1$ (Theorem 5), it holds for P_5 when $m = 2$ with Condition (3.6) or else $m = 1$.

The condition for C^∞ well-posedness on P_5 is 0-evolution (Theorem 6), then all P_5 cannot satisfy it by any m and any choice of $\{b_{ij}\}$. When $m = 1$, the Cauchy problem is well-posed in Gevrey class of index 2 in case of projective Gevrey and of index less than 2 in case of inductive Gevrey.

3.3. p -evolutionary system and Kowalevskian system.

By the relation in Proposition 3.1, we can find unique p for which p - $\det P$ has the term τ^N and another term or else p - $\det P = \tau^N$ for all $p > 0$. In the former case, we say that P is p -evolutionary and define the principal part (on the Cauchy problem) of P by p - $\det P$. In the latter case, we say that P is 0-evolutionary and define the principal part by τ^N . 0-evolutionary operator is essentially an ordinary differential operator. On the other hand, if P is p -evolutionary for $p \leq 1$, we say that P is Kowalevskian. Our definition of " Kowalevskian system " is different from that in S.Mizohata[36] and M.Miyake[33].

4. CAUCHY-KOWALEVSKAYA THEOREM FOR SYSTEM

4.1. Short history.

124 years ago, S.Kowalevskaya[16] showed that if all coefficients in (1.2) are real analytic and the condition " $m \leq 1$ " is satisfied, the Cauchy problem (1.2) has an unique real analytic local solution for every real analytic initial data – so called *the Cauchy-Kowalevskaya theorem*, that is, *the analytic well-posedness*. (She considered nonlinear systems. We call this result "the weak form of the Cauchy-Kowalevskaya theorem".) 62 years ago, I.G.Petrowsky[49] obtained a result for linear systems (1.2) with " $m \leq 1$ " that there exists positive δ decided by the operator such that for arbitrary small positive ρ , if one give the data analytic in $B_\rho((t_o, x_o))$: the ball with the radius ρ and the center (t_o, x_o) , a unique analytic solution exists in $B_{\delta\rho}((t_o, x_o))$. (We call this result "the strong form of the Cauchy-Kowalevskaya theorem". See also J.Leray[19].) 58 years ago, M.Nagumo[42] relaxed the regularity on t of coefficients to the continuity in order to obtain a solution analytic on x and of C^1 -class on t . (He considered nonlinear systems.)

In 1975, S.Mizohata [37] showed that in case of the linear *scalar* higher order equations, the condition corresponding to " $m \leq 1$ " in system (1.2) is necessary and sufficient for the analytic well-posedness. (He obtained more delicate result in [38].)

In 1979, M.Miyake[33] assumed that the coefficients are real analytic and the dimension ℓ of x -space is one and gave the necessary and sufficient condition for the analytic well-posedness on systems introducing the *meromorphic* formal solutions. H.Yamahara and the author[31] and [32] obtained the necessary and sufficient condition for systems in case of general ℓ . They considered the formal fundamental solution and estimate it standing on the normal form of systems in the *meromorphic* formal symbol class.

M.Miyake[34] further showed that one can reduce the analytically well-posed system to a first order one with real analytic coefficients enlarging the size of system assuming $\ell = 1$.

On the other hand, as the algebraic analysis, M.Kashiwara[15] considered the Cauchy-Kowalevskaya theorem for systems in 1971. He decided the structure of the solution space using the *determinant* of the matrices of pseudo-differential operators introduced by M.Sato and M.Kashiwara[50].

4.2. Complexification and apriori estimate.

We set $A(t, x, D_x) = \sum_{|\alpha| \leq m} A_\alpha(t, x) D_x^\alpha$ and $P(t, x, D_t, D_x) = I_N D_t - A(t, x, D_x)$. The problem (1.2) in the real analytic space is naturally extended to the problem in the holomorphic space in a complex domain. From now on, we consider the problem (1.2) in a complex domain $\Omega \subset \mathbf{C}_{t,x}^{1+\ell}$ and assume that all coefficients of $P(t, x, D_t, D_x)$ are holomorphic there and continuous on its closure. Let Ω_{t_0} be $\{x \in \mathbf{C}^\ell : (t_0, x) \in \Omega\}$.

Definition 5. (*The Cauchy-Kowalevskaya theorem = the C-K theorem*)

We say that the Cauchy-Kowalevskaya theorem (= *the C-K theorem*) for $P(t, x, D_t, D_x)$ holds in Ω (or that the Cauchy problem (1.2) is analytically well-posed in Ω) when for each (t_0, x_0) in Ω , every initial data $u_0(x)$ holomorphic in Ω_{t_0} and every right-hand side $f(t, x)$ holomorphic in Ω , there exists a neighborhood ω of (t_0, x_0) where the Cauchy problem (1.2) has a unique holomorphic solution $u(t, x)$. (The weak form.)

In Definition 5, ω may depend on u_0 and f . However, the following was given by S.Mizohata.

Proposition 4.1. (Common existence domain (1), [36])

The existence domain ω is taken independently of u_0 and f .

We denote the ε -neighborhood of K by K_ε . We say that $v(t, x)$ is holomorphic on a compact set K when v is holomorphic in K° and continuous on K , where K° is the open kernel of K . The above proposition implies

Proposition 4.2. (Common existence domain (2))

For arbitrary compact set K in Ω and arbitrary positive ε , there exists a compact neighborhood K' of K decided by the operator and ε such that the unique holomorphic solution exists on K' for arbitrary holomorphic initial data on $\bar{K}_{\varepsilon t_0}$ and arbitrary holomorphic right-hand side on \bar{K}_ε .

When we prove the necessity for the C-K theorem, we need an apriori estimate. For a bounded domain ω in Ω , we set $H(\omega) = \{ v(t, x) = {}^t(v_1(t, x), \dots, v_N(t, x)) : v_j \text{ is holomorphic in } \omega \text{ and continuous on } \bar{\omega}, (1 \leq j \leq N) \}$. It is a Banach space by the norm $\|v\|_\omega = \max_{1 \leq j \leq N} \max_{(t,x) \in \bar{\omega}} |v_j(t, x)|$.

Remark 4.1. We also use $H(\omega)$ for the scalar functions. The readers can distinguish whether it is a space of scalar functions or one of vector functions in the current of the discussion.

The following was essentially given in S.Mizohata.

Proposition 4.3. (Apriori estimate, [36])

If the C-K theorem for P holds in Ω , for arbitrary compact set K and arbitrary positive number ε there exist a neighborhood K' of K and a positive constant C independent of u_o and f such that

$$(4.1) \quad \|u\|_{K'} \leq C(\|u_o\|_{K_{\varepsilon t_o}} + \|f\|_{K_{\varepsilon}}) ,$$

where u is the solution of (1.2).

4.3. Homogeneous problem and formal fundamental solution.

Let us consider the homogeneous Cauchy problem:

$$(4.2) \quad \begin{cases} P(t, x, D_t, D_x)u \equiv D_t u - A(t, x, D_x)u = 0 & , \\ u(t_o, x) = u_o(x) & . \end{cases}$$

If we can construct the fundamental solution which has an estimate uniform on t_o , the inhomogeneous problem (1.2) is solved by the Duhamel principle. Therefore, from now on, we consider the problem (4.2).

By the relation $D_t u = A(t, x, D_x)u$, $D_t^k u$ is represented by a linear combination of the derivatives on x of u :

$$(4.3) \quad D_t^k u = A[k](t, x, D_x)u \quad , \quad (k \geq 0) .$$

$\{A[k]\}_{k=0}^{\infty}$ satisfies the recurrence formula:

$$(4.4) \quad \begin{cases} A[0] = I_N & , \\ A[k] = A[k-1] \circ A + (A[k-1])_t & , \quad (k \geq 1) & , \end{cases}$$

where $(A)_t$ is obtained by operating D_t to the coefficients of A .

The formal fundamental solution of the problem (4.2) is given by

$$(4.5) \quad U(t, x, D_x; t_o) = \sum_{k=0}^{\infty} \frac{(\sqrt{-1}(t-t_o))^k}{k!} A[k](t_o, x, D_x) \quad .$$

As $A[k]$ is differential operator and $A[k] = \sum_{i \geq 0} A[k]_i$ is a finite sum, when it satisfies (4.8) in Proposition 4.4 below, $\sum_{k=0}^{\infty} \{(\sqrt{-1}(t-t_o))^k / k!\} A[k](t_o, x, D_x)u_o$ converges in a neighborhood ω of (t_o, x_o) for arbitrary u_o in $H(\Omega_{t_o})$ and $U(t, x)$ is the true fundamental solution in ω .

4.4. Cauchy-Kowalevskaya theorem and sketch of proofs.

Now we announce our theorem on the Cauchy-Kowalevskaya theorem for systems.

Theorem 5. (Cauchy-Kowalevskaya theorem for systems, [31] and [32])

The following conditions are equivalent.

1) *The Cauchy-Kowalevskaya theorem for $P(t, x, D_t, D_x)$ holds in Ω .*

2) *The lower order terms in the normal form (2.5) satisfy*

$$(4.6) \quad \text{ord } b_k(h) \leq 1 - m(n_k - h), \quad (1 \leq h \leq n_k, 1 \leq k \leq d).$$

3) *$P(t, x, D_t, D_x)$ is reduced to a first order system through a similar transformation by an element in $GL(\mathbf{N}; S_M)$.*

4) *1-det P is of degree \mathbf{N} : the size of P .*

5) *P is Kowalevskian in our sense, that is, p -evolutive for $0 \leq p \leq 1$.*

6) *There exists a natural number k_\circ such that*

$$(4.7) \quad \text{ord } A[k](t, x, D_x) \leq k + k_\circ, \quad (k \in \mathbf{Z}_+).$$

The equivalences between 2), 4) and 5) are obvious by virtue of Proposition 3.1. We give a sketch of the proof from 1) to 2). This is the main part of the proof of the necessity for the C-K theorem. Further, we give a sketch of the proof from 3) to 7) below: more detailed version of 6). This is the essential part of the proof of the sufficiency. By the estimate (4.8), the formal fundamental solution (4.5) converges and operates on the holomorphic functions. Thus, there exists a positive δ decided by the operator such that for u_\circ in $H(B_\rho(x_\circ))$ for arbitrary small positive ρ , the unique holomorphic solution u exists in $B_\rho(x_\circ) \times B_{\delta\rho}(t_\circ)$. This means that 7) implies 1) (the strong form). The proofs from 2) to 3), 7) to 1), 7) to 6) and 6) to 2) are easy or trivial.

Proposition 4.4. (Estimate of $A[k](t, x, \xi)$, [31] and [32])

Condition 3) implies 7):

7) *For an arbitrary compact set K in Ω , there exist a positive integer k_\circ and positive constants C , R and R_\circ independent of k , for which the following estimates hold on $K \times \mathbf{C}^\ell$;*

$$(4.8) \quad |A[k]_{i(\alpha)}^{(\beta)}(t, x, \xi)| \leq CR_\circ^k \sum_{h=0}^k R^{k-h+i+|\alpha|+|\beta|} (k-h)! i! |\alpha|! |\beta|! \|\xi\|^{k_\circ+h-i-|\beta|}$$

$$(i \in \mathbf{Z}_+, \alpha \in \mathbf{Z}_+^{1+\ell}, \beta \in \mathbf{Z}_+^\ell).$$

Proof. (Sketch of the proof from 1) to 2))

The proof is similar as that in S.Mizohata[39]. We derive a contradiction assuming that 1) holds and 2) fails. As 2) fails, $P(t, x, D_t, D_x)$ is p -evolute for $p > 1$.

Let $\{\lambda_{kj}(t, x, \xi)\}$ be the roots of p -det $Q_k = 0$. As P is p -evolute for $p > 1$, we can find $(t_o, x_o, \xi_o) \in \Omega \times \mathbf{C}^\ell \setminus \Sigma$, its conic neighborhood $\omega \times \Gamma$ ($\subset \Omega \times \mathbf{C}^\ell \setminus \Sigma$), natural numbers s_k, s_{k1} ($s_{k1} \leq s_k$) and a positive number ε such that $\lambda_{kj}(t, x, \xi)$'s are distinct and have the constant multiplicities m_{kj} 's ($1 \leq j \leq s_k, \sum_{j=1}^{s_k} m_{kj} = n_k$) and

$$(4.9) \quad \text{Im} \lambda_{kj}(t, x, \xi) \begin{cases} \leq -\varepsilon \|\xi\|^p & (1 \leq j \leq s_{k1}), \\ \geq 0 & (s_{k1} + 1 \leq j \leq s_k), \end{cases}$$

in $\omega \times \Gamma$ ($1 \leq k \leq n_{k1}$). As $p > 1$, at least one of s_{k1} is positive. We can assume $s_{11} \geq 1$.

By a complex translation and a complex rotation, (t_o, x_o) and ξ_o is reduced to $(0, O)$ and $(1, 0, \dots, 0)$, respectively. After this, we use the same notation. From now on, we restrict the variables (t, x) in ω to the real section. We construct true symbols from formal symbols. Then, on the real section, $D_t - \lambda_{11}(t, x, D_x)$ is essentially the back-word heat equation of order p (> 1) and the microlocal L_x^2 energy of the solution in the direction $(1, 0, \dots, 0)$ with the initial data $\exp(\sqrt{-1}\rho x_1)$ diverges at least of order $\exp(\varepsilon' \rho^p t)$ for a positive ε' . On the other hand, the apriori estimate (4.1) implies the divergent order at most $\exp(\delta_o \rho)$. Making ρ tend to infinity, these imply a contradiction. \square

Proof. (Sketch of the proof from 3) to 7))

By 3), there exists N in $GL(\mathbf{N}; S_M(\Omega \times \mathbf{C}^\ell \setminus \Sigma))$ and $C(t, x, \xi)$ in $M_{\mathbf{N}}(S_M^1(\Omega \times \mathbf{C}^\ell \setminus \Sigma))$ which satisfy $N^{-1} \circ P \circ N = I_{\mathbf{N}} D_t - C(t, x, \xi)$ and $\text{ord} C = 1$. On the solution space $Sol = \{u \in H : Pu = 0\}$, the element in $M_{\mathbf{N}}(H[Dx, Dt])$ operate as the element in $M_{\mathbf{N}}(H[Dx, Dt])/M_{\mathbf{N}}(H[Dx, Dt])P$. This relation can be understood as the relation in $M_{\mathbf{N}}(S_M[Dt])$. $A[k](t, x, \xi)$ is the representative element without D_t of the equivalence class $[D_t^k]$ in $M_{\mathbf{N}}(S_M[Dt])/M_{\mathbf{N}}(S_M[Dt])P$. The relation $I_{\mathbf{N}} D_t - A = 0$ is represented as $N \circ (I_{\mathbf{N}} D_t - C) \circ N^{-1} = 0$. We denote the representative element without D_t of the equivalent class $[N \circ D_t^k \circ N^{-1}]$ by $N \circ C[k](t, x, \xi) \circ N^{-1}$. $\{C[k]\}_{k=0}^\infty$ satisfies

$$(4.10) \quad \begin{cases} C[0] = I_{\mathbf{N}} \quad , \\ C[k] = C[k-1] \circ C + (C[k-1])_t \quad , \quad (k \geq 1) \quad . \end{cases}$$

As $\text{ord} C$ is one, the following lemma is rather easily obtained by the induction.

Lemma 4.5. (Estimate of $C[k](t, x, \xi)$.)

For an arbitrary conically compact set $\tilde{\Gamma}$ in $\Omega \times \mathbf{C}^\ell \setminus \Sigma$, there exist a positive constants C', R and R_o independent of k , for which the following estimates hold on $\tilde{\Gamma}$;

$$(4.11) \quad |C[k]_{i(\alpha)}^{(\beta)}(t, x, \xi)| \leq C' R_o^k \sum_{h=0}^k R^{k-h+i+|\alpha|+|\beta|} (k-h)! i! |\alpha|! |\beta|! \|\xi\|^{h-i-|\beta|}$$

$$(i \in \mathbf{Z}_+, \alpha \in \mathbf{Z}_+^{1+\ell}, \beta \in \mathbf{Z}_+^\ell).$$

We set $\text{ord } N + \text{ord } N^{-1} = k_0$. By the above lemma and the relation

$$A[k] = \sum_{j=0}^k \frac{k!}{j!(k-j)!} (D_t^{k-j} N) \circ C[j] \circ N^{-1},$$

we arrive at the estimate (4.8) on $\tilde{\Gamma}$. Because $A[k](t, x, \xi)$ is holomorphic in $\Omega \times \mathbf{C}^\ell$, the estimate (4.8) holds all over $\Omega \times \mathbf{C}^\ell$ by the maximum principle. \square

4.5. Cauchy-Kowalevskaya theorem of Nagumo Type.

M.Nagumo[42] showed that one can obtain a unique solution real analytic on x and of C^1 -class on t if $m \leq 1$ in (1.2) and the coefficients are real analytic on x and continuous on t . When $m \geq 2$, does the continuity on t of the coefficients and one of 2) to 6) in Theorem 5 assure the existence of a solution? The answer is No.

Example 6. (announced at ICM'98, [28])

$$P_6 = I_{\mathbf{N}} \partial t - \begin{pmatrix} 0 & 1 & & & & \\ & \ddots & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & 1 & & \\ \nu(t) & & & 0 & \mu(t) & \\ & & & & & 0 \end{pmatrix} (\partial x)^m, \quad \mathbf{N} \times \mathbf{N},$$

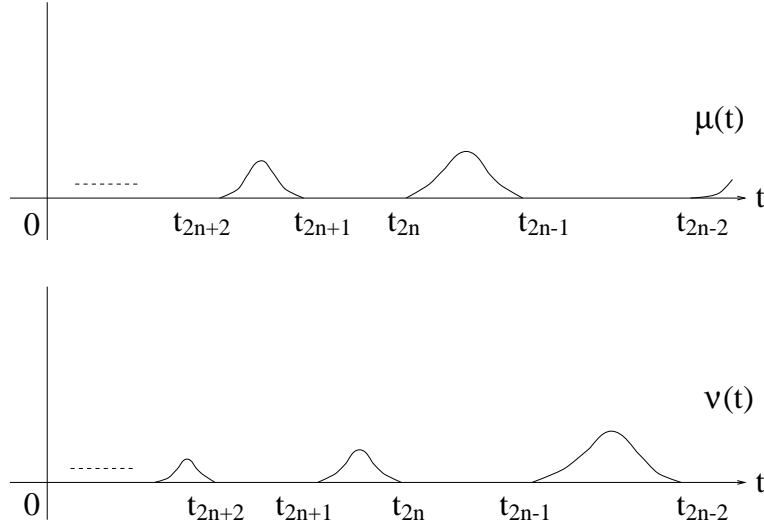
where $\mu(t)$ and $\nu(t)$ are non-negative and have the supports in $[0, \infty)$ and $\mu(t)\nu(t) \equiv 0$. More precisely, let us set $t_{2n-1} = t_{2n} = \sum_{j=n}^{\infty} j^{-1}(\log j)^{-2}$ ($\{t_n\}$ is a monotonically decreasing sequence with the limit zero) and take a natural number p ,

$$\mu(t) = \begin{cases} (t_{2n-1} - t)^p (t - t_{2n})^p & t \in (t_{2n}, t_{2n-1}), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\nu(t) = \begin{cases} (t_{2n} - t)^p (t - t_{2n+1})^p & t \in (t_{2n+1}, t_{2n}), \\ 0 & \text{otherwise.} \end{cases}$$

($n \in \mathbf{N}$)



$\mu(t)$ and $\nu(t)$ belong to $C^{p-1, 1}(\mathbf{R})$, that is, the $(p-1)$ -th derivatives are Lipschitzian. As $\mu(t)\nu(t) \equiv 0$, P_6 is 0-evolutive at every point.

For arbitrary small positive ε , we can find $t_{2q} \leq \varepsilon$. Let us take $t_o = t_{2n+2q}$. We can concretely solve the Cauchy problem for P_6 with the initial data $u_{oi} = 0$ ($0 \leq i \leq N = 1$), $u_{oN} = \varphi(x) = \exp(\rho x)$ and the right-hand side $f(t, x) = 0$ from t_o to t_{2q} and the solution u has the estimate

$$\begin{aligned}
& |u_N(t_{2q}, 0)| \\
& > \rho^{nmN} \prod_{k=1}^n \int_{t_{2n+2q-2k+1}}^{t_{2n+2q-2k}} \int_{t_{2n+2q-2k+2}}^{s_N^{(k)}} \dots \int_{t_{2n+2q-2k+2}}^{s_2^{(k)}} \nu(s_N^{(k)}) \mu(s_1^{(k)}) ds_1^{(k)} \dots ds_N^{(k)} \\
& \geq \rho^{mNn} \{K_o \prod_{j=q+1}^{n+q} j(\log j)^2\}^{-(4p+N)}.
\end{aligned}$$

On the other hand, if the Cauchy-Kowalevskaya theorem of nagumo type holds, we have the same a priori estimate as Proposition 4.3 and the following must hold

$$|u_N(t_{2q}, 0)| \leq C \exp(K_1 \rho).$$

Here, K_o and K_1 are positive constants. If $mN > 4p + N$, taking $\rho = [(n+q)\{\log(n+q)\}^2]^{4p+N}$ ($K \gg K_o K_1$) and making n tend to infinity, the both estimates are not compatible. For the detail, see W.Matsumoto[28].

Thus, in order to hold the Cauchy-Kowalevskaya theorem of Nagumo type for P_6 , we need the differentiability on t at least up to $(m-1)N/4$.

The author propose a conjecture on the Cauchy-Kowalevskaya theorem of Nagumo type for systems. Let us take $\Omega = [T_1, T_2] \times \Omega'$. We denote the space of real analytic functions in Ω' by $A(\Omega')$,

Conjecture (Conjecture on C-K theorem of Nagumo type for systems)

If all coefficients of P belong to $C^\infty([T_1, T_2]; A(\Omega'))$, the assertion in Theorem 5 also holds.

The equivalences between from 2) to 6) are rather easily seen. The assertion from 1) to 2) is also shown by the same way as the proof in Subsection 4.4, because the analyticity on x is essential but that on t is not required in the proof. Therefore, the sufficiency of 2) or 3) or \dots or 6) is open. Recently, M.Murai and the author[30] obtained an affirmative result for the most simple system but non-trivial case: $m = 2$, $N = 2$ and $\ell = 1$.

5. LEVI CONDITION FOR THE C^∞ WELL-POSEDNESS

5.1. Short history and Example.

We consider the well-posedness of the Cauchy problem for the operators with constantly multiple characteristics. 90 years ago, E.E.Levi[20] gave a sufficient condition for the C^∞ well-posedness for higher order scalar equations with constantly multiple characteristic roots in case of the dimension of x -space one. He also showed his condition becomes necessary when coefficients are constant. Through S.Mizohata and Y.Ohya[40] and [41], finally, for higher order scalar equation, the necessary and sufficient condition for the C^∞ well-posedness was obtained by H.Flaschka and G.Strang[13] (necessity) in 1971 and J.Chazarin[8] (sufficiency) in 1974.

For first order systems, in 1973, V.M.Petkov[45], [46] gave the necessary and sufficient condition assuming the constant multiplicity at most two and the constant rank. In 1977, Y.Demay[11] gave a sufficient condition for the multiplicity at most two but without the rank condition. In 1981, W.Matsumoto[21], [22] also consider a necessary condition and a sufficient condition in the same situation. He showed that when the coefficients are non-quasianalytic, it seems very difficult to give the necessary and sufficient condition through some examples. In 1994, W.Matsumoto[25] Section 4, [26] gave the necessary and sufficient condition only assuming the constant multiplicity of the characteristic roots and the real analyticity of the coefficients. As the invariance of Matsumoto's condition under the transformations was not clear, J.Vaillant continued the research on the representation of the condition in an invariant form and succeeded up to the multiplicity seven [53]. (See also J.Vaillant[54], G.Tagliatela and J.Vaillant[51] and those reference.)

The author obtained the invariant representation of his condition using determinant theory associated with a characteristic root (the condition *iv*) in Theorem 6 below). A.D'Agnolo and G.Tagliatela[9] also discussed on this type of representation.

An approach from the algebraic analysis was made by A.D'Agnolo and F.Tonin[10].

We consider the Cauchy problem of a first order system of partial differential equations ((1.2) with $m = 1$) with constantly multiple characteristic roots. If the first order part has only the zero characteristic root, the Levi condition is equivalent to 0-evolution, that is, essentially it is an ordinary differential operator of D_t . When coefficients are real analytic, this is necessary and sufficient for the C^∞ well-posedness. On the other hand, in case of non-quasianalytic coefficients, even if the first order part has constant coefficients, this condition does not rest sufficient. We give an example.

Example 7. ([25] Section 4)

$$P_7(t, \partial t, \partial x) = I_3 \partial t - \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \partial x - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu(t) \\ \nu(t) & 0 & 0 \end{pmatrix},$$

where $\mu(t)$ and $\nu(t)$ have the same forms of Figure 1 in Subsection 4.5. and belong to $C^\infty(\mathbf{R})$.

As $\mu(t)\nu(t) \equiv 0$, P_7 is 0-evolutive. For P_7 , we consider the Cauchy problem with initial time $t_o = t_{2n+2q}$. By the same calculation as on Example 6, we can see that we need n -the order derivative of initial data to obtain the solution up to $t = t_{2q}$. On the other hand, by the apriori estimate in Proposition 5.2, the loss of regularity must be finite. Making n tend to infinity, these are not compatible. On the detail, see W.Matsumoto[29].

Therefore, through this section, we assume the analyticity of all coefficients.

5.2. p -determinant associated with a characteristic root.

Let $\lambda_k(t, x, \xi)$ be the characteristic roots of constant multiplicity m_k ($1 \leq k \leq d$) of the first order part of P . By virtue of the assumption of the constant multiplicity, every characteristic roots is smooth. In order to describe the Levi condition in an invariant form, we introduce p -determinant associated with $\lambda_k(t, x, \xi)$.

Let p be a rational number such that $0 \leq p < 1$. As $S_M[D_t] = S_M[D_t - \lambda_k(t, x, \xi)]$, every $a(t, x, \xi, D_t)$ is represented as $a(t, x, \xi, D_t) = \sum_{j=0}^m a^{<j>}(t, x, \xi)(D_t - \lambda_k)^{m-j}$, $a^{<j>} = \sum_{i=0}^\infty a_i^{<j>} \in S_M$. We reset the order of $a^{<j>}$ to its true order. Let us set

$$p\text{-ord}_{\lambda_k} a^{<j>}(t, x, \xi)(D_t - \lambda_k)^{m-j} = \text{ord } a^{<j>} + p(m-j)$$

$$p\text{-ord}_{\lambda_k} a = \max_{0 \leq j \leq m} p\text{-ord}_{\lambda_k} a^{<j>}(t, x, \xi)(D_t - \lambda_k)^{m-j}$$

and call them the p -order associated with λ_k . By p -order associated with λ_k , $S_M[D_t - \lambda_k]$ becomes a filtered ring. We set further

$$R_{\lambda_k}^{(p)}(a) = \{j : p\text{-ord}_{\lambda_k} a^{<j>}(D_t - \lambda_k)^{m-j} = p\text{-ord}_{\lambda_k} a\}$$

$$a_{p-pr \lambda_k}(t, x, \xi, \tau) = \sum_{j \in R_{\lambda_k}^{(p)}(a)} a_0^{<j>}(t, x, \xi)(\tau - \lambda_k)^{m-j}$$

and call the latter the p -principal symbol of $a(t, x, \xi)$ associated with λ_k . The set $\cup_{p>0} \{a_0^{<j>}(t, x, \xi)(\tau - \lambda_k)^{m-j}\}_{j \in R_{\lambda_k}^{(p)}(a)}$ has finite elements and composes the Newton polygon of a associated with λ_k .

We define the p -homogeneous polynomial on $\tau - \lambda_k$ by the same way in Subsection 3.1. Let us set

$$Y_{\lambda_k} = \{p\text{-homogeneous polynomials on } \tau - \lambda_k\}.$$

Y_{λ_k} is a commutative ring. The map $\sigma_{\lambda_k}^p$ from $S_M[D_t - \lambda_k] \setminus \{0\}$ to $Y_{\lambda_k} \setminus \{0\}$ defined by $\sigma_{\lambda_k}^p(a) = a_{p-pr \lambda_k}$ is a homomorphism of the productive semigroup. This is naturally extended to the map from $S_M[D_t - \lambda_k]^Q \setminus \{0\}$ to $Y_{\lambda_k}^Q \setminus \{0\}$ by $\sigma_{\lambda_k}^p(ab^{-1}) = a_{p-pr \lambda_k}/b_{p-pr \lambda_k}$

as a homomorphism of the productive group. We put $\sigma_{\lambda_k}^p(0) = 0$. Thus, we can obtain the weighted determinant theory by $\sigma_{\lambda_k}^p$ following J.Dieudonné[12]. In case of non-quasianalytic classes, by the same reason as in Subsection 3.1, we can also obtain it on a open dense set.

Definition 6. (*p-determinant associated with λ_k*)

We call the determinant by $\sigma_{\lambda_k}^p$ of a matrix A with entries in $S[D_t - \lambda_k]$, *p-determinant of A associated with λ_k* and denote it by $p\text{-det}_{\lambda_k} A$.

We can obtain the corresponding properties in Theorem 3.

On the matrix of the form $P = I_{\mathbb{N}}D_t - A$, $A \in M_{\mathbb{N}}(S^1)$ ($m = 1$), we give the representation of p-determinant using the element of the normal form in Theorem 1.

Let us set

$$(5.1) \quad \begin{aligned} \text{true order } d_{kj}(h) &= r_h^{kj}, \\ M_{kj}^p &= \max_{1 \leq h \leq n_{kj}} \{r_h^{kj} + (n_{kj} - h) + p(h - 1)\}, \\ R_{kj}^p &= \{h : r_h^{kj} + (n_{kj} - h) + p(h - 1) = M_{kj}^p\} \end{aligned}$$

We have the following.

Proposition 5.1. (*Relation between the normal form and p-determinant a.w. λ_k*)

$$(5.2) \quad \begin{aligned} p\text{-det}_{\lambda_k} P &= \prod_{i=1}^d \prod_{j=1}^{d_i} p\text{-det}_{\lambda_k} P_{ij}, \\ p\text{-det} -\lambda_k P_{kj} &= \begin{cases} (\tau - \lambda_k)^{n_{kj}}, & (pn_{kj} > M_{kj}), \\ (\tau - \lambda_k)^{n_{kj}} - \sum_{h \in R_{kj}^p} d_{kj}(h)_0(t, x, \xi) |\xi|^{n_{kj}-h} (\tau - \lambda_k)^{h-1}, & (pn_{kj} = M_{kj}), \\ - \sum_{h \in R_{kj}^p} d_{kj}(h)_0(t, x, \xi) |\xi|^{n_{kj}-h} (\tau - \lambda_k)^{h-1}, & (pn_{kj} < M_{kj}), \end{cases} \\ &= \text{the highest } p\text{-degree part a.w. } \lambda_k \text{ of the ordinary determinant of } P_{kj} \\ & \hspace{15em} (1 \leq j \leq d_k) \end{aligned}$$

$$p\text{-det}_{\lambda_k} P_{ij} = (\lambda_k - \lambda_i)^{n_{ij}} \quad (1 \leq i \neq k \leq d, 1 \leq j \leq n_{ij})$$

In case of m.f.s., $|\xi|^{n_{kj}-h}$ is replaced by $\xi_1^{n_{kj}-h}$.

In case of $S = S_H$, we can obtain the regularity of p -determinant associated with λ_k corresponding to (1) in Theorem 4.

By the relation in Proposition 5.1, we can find unique p for which $p\text{-det}_{\lambda_k} P / \prod_{1 \leq i \leq d, i \neq k} (\lambda_k - \lambda_i)^{m_i}$ has the term τ^{m_k} and another term or else it is τ^{m_k} for all $0 < p < 1$. In the former case, we say that P is p -evolutionary with respect to λ_k and define the second principal part (on the Cauchy problem) of P by $p\text{-det}_{\lambda_k} P / \prod_{1 \leq i \leq d, i \neq k} (\lambda_k - \lambda_i)^{m_i} = \prod_{1 \leq j \leq d_k} p - \det_{\lambda_k} P_{kj}$ and denote it by $p\text{-det}'_{\lambda_k} P$. In the latter case, we say that P is 0-evolutionary with respect to λ_k and define the second principal part by τ^{m_k} . 0-evolutionary operator with respect to λ_k is essentially an ordinary differential operator along the bicharacteristic strip of λ_k .

5.3. Levi condition.

Let us make clear the definition of C^∞ well-posedness of the Cauchy problem. For the simplicity, we assume that Ω is bounded.

Definition 7. (C^∞ well-posedness)

We say that the Cauchy problem is C^∞ well-posed in Ω when for each (t_o, x_o) in Ω , there exists a neighborhood ω of (t_o, x_o) where every initial data $u_o(x)$ of C^∞ -class in $\bar{\Omega}_{t_o}$ and every right-hand side $f(t, x)$ of C^∞ -class in $\bar{\Omega}$, the Cauchy problem (1.2) has a unique solution $u(t, x)$ in $C^\infty(\bar{\omega})$.

We give an a priori estimate. For a bounded domain ω in Ω , we set $F(\omega) = \{ v(t, x) = {}^t(v_1(t, x), \dots, v_N(t, x)) : v_j \in C^\infty(\bar{\omega}) \text{ , } (1 \leq j \leq N) \}$. It is a Fréchet space by the semi-norms $\|v\|_{n\omega} = \max_{1 \leq j \leq N} \sum_{|\alpha| \leq n} \max_{(t,x) \in \bar{\omega}} |D_{tx}^\alpha v_j(t, x)|$.

Proposition 5.2. (A priori estimate of C^∞ well-posedness)

If the Cauchy problem for P is C^∞ well-posed in Ω , for arbitrary q in \mathbf{Z}_+ , there exist r in \mathbf{Z}_+ and a positive constant C independent of u_o and f such that

$$(5.3) \quad \|u\|_{q\bar{\omega}} \leq C(\|u_o\|_{r\bar{\omega}_{t_o}} + \|f\|_{r\bar{\omega}}) ,$$

where u is the solution of (1.2).

S.Mizohata showed that the following.

Proposition 5.3. (Hyperbolicity, [35])

In order that the Cauchy problem is C^∞ well-posed in Ω , the characteristic roots $\lambda_k(t, x, \xi)$ ($1 \leq k \leq d$) must be real.

Now we announce our theorem on the C^∞ well-posedness for systems.

Theorem 6. (C^∞ well-posedness for systems, [25] Section 4 and [29])

We assume that every characteristic root $\lambda_k(t, x, \xi)$ is real and has the constant multiplicity m_k ($1 \leq k \leq d$). The following conditions are equivalent.

i) The Cauchy problem for P is C^∞ well-posed in Ω .

ii) The lower order terms in the normal form (2.4) with $m = 1$ satisfy

$$(5.4) \quad \text{ord } d_{kj}(h) \leq -(n_k - h), \quad (1 \leq h \leq n_{kj}, 1 \leq j \leq d_k, 1 \leq k \leq d).$$

iii) P is reduced to a first order system with a diagonal first order part through a similar transformation by an element in $GL(\mathbf{N}; S_M)$.

iv) P is 0-evolutive with respect to λ_k ($1 \leq k \leq d$).

Remark 5.1. The conditions in Theorems 5 and 6 are similar each other and the proofs also similar in case of real analytic coefficients. In case of non-quasianalytic coefficients, the proofs on the necessity also hold. However, not only the proofs of the sufficiency loose the validity but also the phenomena themselves become different.

Remark 5.2. In case of non-quasianalytic coefficients, under the equivalent condition *ii)*, *iii)* or *iv)*, the greatest space for the well-posedness of the Cauchy problem was studied by W.Matsumoto[23] for 2×2 systems. It depends on the regularity of coefficients. For example, when coefficients belong to a Gevrey class, it is much bigger than the union of the Gevrey classes.

The equivalences between *ii)* and *iv)* is obvious by virtue of Proposition 5.1. The proofs from *ii)* to *iii)* is evident. We give a sketch of the proofs from *i)* to *ii)* and from *iii)* to *i)*.

Proof. (Sketch of the proof from i) to ii))

The proof is just similar as from 1) to 2) in Subsection 4.4. We derive a contradiction assuming that *i)* holds and *ii)* fails. As *ii)* fails, P is p -evolutive with respect to one of λ_k , for $0 < p < 1$, we can find (t_o, x_o, ξ_o) , its conic neighborhood $\omega \times \Gamma$, natural numbers s_{kj} , s'_{kj} ($s'_{kj} \leq s_{kj}$) and a positive number ε such that the roots $\{\mu_{kji}\}_{i=1}^{s_{kj}}$ of $p\text{-det}'_{\lambda_k} P = 0$ are distinct and have the constant multiplicities m_{kji} 's ($1 \leq i \leq s_{kj}$, $\sum_{j=i}^{s_{kj}} m_{kji} = n_{kj}$) and

$$(5.5) \quad \text{Im}\mu_{kji}(t, x, \xi) \begin{cases} \leq -\varepsilon \|\xi\|^p & (1 \leq i \leq s'_{kj}), \\ \geq 0 & (s'_{kj} + 1 \leq i \leq s_{kj}), \end{cases}$$

in $\omega \times \Gamma$ Here, at least one of s'_{kj} is positive. We can assume $s'_{11} \geq 1$ and $\xi_o = (1, 0, \dots, 0)$.

We construct true symbols from formal symbols. Then, $D_t - \lambda_1(t, x, D_x) - \mu_{111}(t, x, D_x)$ is essentially the back-word heat equation of order p ($0 < p < 1$) and the microlocal L_x^2 energy of the solution in the direction $(1, 0, \dots, 0)$ with the initial data $\exp(\sqrt{-1}\rho x_1)$ diverges at least of order $\exp(\varepsilon' \rho^p t)$ for a positive ε' but the apriori estimate (5.3) implies the divergent order on ρ is at most polynomial order. Making ρ tend to infinity, these imply a contradiction. □

Proof. (Sketch of the proof from iii) to i)
 (First step) We separate the characteristic roots.

Proposition 5.4. (Perfect Block Diagonalization, [27])

We assume the constant multiplicity of each characteristic roots λ_k . Then, for every point (t_o, x_o, ξ_o) in $\Omega \times \mathbf{R}^\ell$, there exists a conically compact neighborhood $\omega \times \Gamma$, $N^\circ(t, x, \xi) = \sum_{i=0}^\infty N_i$ in $GL(\mathbf{N}; S_H^0(\omega \times \Gamma))$, $B_k(t, x, \xi) = \sum_{i=0}^\infty B_{ki}$ in $M_{m_k}(S_H^1(\omega \times \Gamma))$, such that

$$(5.6) \quad \begin{aligned} N^{\circ-1}(t, x, \xi) \circ P(t, x, D_t, \xi) \circ N^\circ(t, x, \xi) &= \bigoplus_{1 \leq k \leq d} P^k, \\ P^k(t, x, D_t, \xi) &= I_{m_k} D_t - B_k(t, x, \xi) \end{aligned}$$

where B_{k0} has the eigenvalue $\lambda_k(t, x, \xi)$.

When we construct true symbol corresponding to each formal symbol, the error is of class $S^{-\infty}$, we can consider each P_k independently. (In our discussion, $S^{-\infty}$ is negligible.)

(Second step) By a Fourier transformation, we can transform $\lambda_k(t, x, \xi)$ to 0. (See, for example, H.Kumano-go[18] Chap 10.) From now on, we omit the index k . Thus, we need consider $\tilde{P} = I_m D_t - \tilde{A}$, $\tilde{A} \in M_m(S_H^1)$, where \tilde{A}_0 is nilpotent. (After a transformation by a Fourier integral operator, formal symbols become inhomogeneous on ξ . However, we can treat them by the same way as the homogeneous case and we use the same notation for both.)

(Third step) We define $\tilde{A}[k]$ by (4.4) replacing $A[k]$, A and \mathbf{N} by $\tilde{A}[k]$, \tilde{A} and m , respectively. As the order of \tilde{A} is one, the following estimate is rather easily seen.

Lemma 5.5. (Estimate of $\tilde{A}[k](t, x, \xi)$)

There exist positive constants C , R and R_o independent of k such that

$$(5.7) \quad \begin{aligned} |\tilde{A}[k]_{i(\alpha)}^{(\beta)}(t, x, \xi)| &\leq C R_o^k \sum_{h=0}^k R^{k-h+i+|\alpha|+|\beta|} (k-h)! i! |\alpha|! |\beta|! |\xi|^{h-i-|\beta|} \\ &(i \in \mathbf{Z}_+, \alpha \in \mathbf{Z}_+^{1+\ell}, \beta \in \mathbf{Z}_+^\ell). \end{aligned}$$

Therefore, formal fundamental solution defined by (4.5) replaced $A[k]$ by $\tilde{A}[k]$ converges in each order part.

On the other hand, by iii), there exists N in $GL(m; S_M)$ and $\tilde{C}(t, x, \xi)$ in $M_m(S_M^0)$. We denote the representative element without D_t of the equivalent class $[N \circ D_t^k \circ N^{-1}]$ by $N \circ \tilde{C}[k](t, x, \xi) \circ N^{-1}$. $\{\tilde{C}[k]\}_{k=0}^\infty$ satisfies (4.10) replacing $C[k]$, C and \mathbf{N} by $\tilde{C}[k]$, \tilde{C} and m , respectively.

As the order of \tilde{C} is one, the following lemma is rather easily obtained.

Lemma 5.6. (Estimate of $\tilde{C}[k](t, x, \xi)$)

There exist a positive constants C' , R and R_o independent of k , for which the following estimates hold.

$$(5.8) \quad |\tilde{C}[k]_{i(\alpha)}^{(\beta)}(t, x, \xi)| \leq C' R_o^k R^{k+i+|\alpha|+|\beta|} k! i! |\alpha|! |\beta|! |\xi|^{-i-|\beta|}$$

$$(i \in \mathbf{Z}_+, \alpha \in \mathbf{Z}_+^{1+\ell}, \beta \in \mathbf{Z}_+^\ell).$$

We set $\text{ord } N + \text{ord } N^{-1} = k_o$. By the above lemma and the relation

$$\tilde{A}[k] \equiv \sum_{j=0}^k \frac{k!}{j!(k-j)!} (D_t^{k-j} N) \circ \tilde{C}[j] \circ N^{-1},$$

we see that the sum in (5.7) is in reality up to $\min\{k, k_o\}$. Then, (4.5) is finite order and converges in S_M . By L.Boutet de Monvel and P.Krée[7] and L.Boutet de Monvel[6], we can construct true symbol which has the asymptotic expansion (4.5). Thus, we have obtained a parametrix as a Fourier integral operator acting on C^∞ . Following H.Kumano-go[17], we can obtain the exact fundamental solution from a parametrix. On the detailed proof, see W.Matsumoto[29]. □

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