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On the algebraic properties of the $H_{\frac{n}{2},\frac{1}{2}}$ spaces

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Abstract

We investigate the multiplicative properties of the spaces $H^{\frac{n}{2},\frac{1}{2}}$ As in the case of the classical Sobolev spaces $H^{\frac{n}{2}}$ this space does not form an algebra. We investigate instead the space $H^{\frac{n}{2}} \cap L^{\infty}$, more precisely a subspace of it formed by products of solutions of the homogeneous wave equation with data in $H^{\frac{n}{2}}$.

It is a well known fact that the classical Sobolev spaces $H_s(\mathbb{R}^n)$, $s > \frac{n}{2}$, form an algebra relative to the standard multiplication of functions. This properties fails however for the critical exponent $s = \frac{n}{2}$, unless we consider the smaller space $H_{\frac{n}{2}} \cap L^{\infty}(\mathbb{R})$ for which it is still true. If n is an even integer this fact can be easily proved with the help of the Gagliardo-Nirenberg inequality. If n is odd a simple proof of this fact can be obtained with the help of the following characterization of $H_s(\mathbb{R}^n)$ spaces for $0 < s < 1$: A function $f \in H_s$ if and only if $f \in L^2$ and,

$$\int\int \frac{|f(x+y) - f(x)|^2}{|y|^{n+2s}} dxdy < \infty$$

Another proof[1](see [C]) can be obtained with the help of Littlewood-Paley decompositions.

In recent years the spaces $H_{s,\delta}(\mathbb{R}^{n+1})$ have surfaced as reasonable hyperbolic analogues of the classical H_s spaces. See [B]. In [K-M2] it was proved that, if $\delta > \frac{n}{2}$, $\delta > \frac{1}{2}$ and $n \geq 2$, these spaces form an algebra. This fact played a fundamental role in the

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[1] We shall present yet another proof of this fact below

proof of local well posedness of the Wave Maps equations in H^s, for any $s > \frac{n}{2}$. By analogy with the case of classical Sobolev spaces one may expect that, in the case of the critical exponents $s = \frac{n}{2}$, $\delta = \frac{1}{2}$ the space $H_{\frac{n}{2},\frac{1}{2}} \cap L^\infty(\mathbb{R}^{n+1})$ forms also an algebra. Such a fact may play an important role to prove the well posedness of the Wave-Maps equations for the critical exponent $s = \frac{n}{2}$. There are reasons to believe that such a result is however false. In this note we shall establish a weaker version of this fact concerning products $u_1 u_2 \dots u_N$ of solutions of the wave equation $\Box u_i = 0$ with $H_{\frac{n}{2}}$ data. It is known, see [K-M1], [K-M2] and [K-S], that for $N = 2$ the product $u_1 u_2 \in H_{\frac{n}{2},\frac{1}{2}}$. This result does probably fail for $N = 3$, we believe that the following conjecture is true.

Conjecture: *If* $\Box u = 0$, $u = f$, $\partial_t u = 0$ *at* $t = 0$, *there exists* f_k *with* $\|f_k\|_{H_{\frac{n}{2}}} = 1$ *such that for the corresponding solutions* u_k,

$$\|u_k^3\|_{H_{\frac{n}{2},\frac{1}{2}}} \to \infty$$

Our main results are contained in the following theorems:

Theorem 1 *Let* $n \geq 3$. *Let* u_i, $i = 1 \dots N$ *verifying* $\Box u_i = 0$ *with data*[2] $u_i = f_i \in H_{\frac{n}{2}}$, $\partial_t u_i = g_i \in H_{\frac{n}{2}-1}$ *at* $t = 0$. *If in addition* $u_i \in L^\infty$ *then* $u_1 u_2 \dots u_N \in H_{\frac{n}{2},\frac{1}{2}}$.

Theorem 2 *Let* $n \geq 3$. *Let* F *be an analytic function of one real variable whose Fourier transform is a compactly supported measure*[3]. *Let* u *be an arbitrary solution of* $\Box u = 0$ *with* $H_{\frac{n}{2}}$ *data. Then* $F(u) \in H_{\frac{n}{2},\frac{1}{2}}$.

Remark: We expect these results to be correct also for n=2. In fact there is only one place in the proof, the estimate for $\|E_3\|_{L^2}$ that requires $n > 2$. This term could probably be handled by a direct but long calculation.

As a warm up for the proof of theorem 1 we shall start by presenting an elementary proof of the algebra property of $H_{\frac{n}{2}} \cap L^\infty$. Let $u, v \in H_{\frac{n}{2}} \cap L^\infty$; to estimate $D^{\frac{n}{2}}(uv)$ it suffices to estimate the commutator $E = D^{\frac{n}{2}}(uv) - uD^{\frac{n}{2}}v - vD^{\frac{n}{2}}u$. In fact we can show that

$$\|E\|_{L^2} \leq c \|D^{\frac{n}{2}} u\|_{L^2} \, \|D^{\frac{n}{2}} v\|_{L^2} \tag{1}$$

from which we derive,

$$\|E\|_{L^2} \leq c\Big(\|D^{\frac{n}{2}} u\|_{L^2} \, \|D^{\frac{n}{2}} v\|_{L^2} + \|D^{\frac{n}{2}} u\|_{L^2} \, |v|_{L^\infty} + \|D^{\frac{n}{2}} v\|_{L^2} \, |u|_{L^\infty} \Big) \tag{2}$$

To prove 1 we write the Fourier transform $E^\wedge$ in the form,

$$E^\wedge(\xi) = \int \sigma(\xi - \eta, \eta) u^\wedge(\xi - \eta) v^\wedge(\eta) d\eta$$

[2]In what follows we shall simply say that the u_i have $H^{\frac{n}{2}}$ data

[3]The exact reqirement is that $\int e^{c\lambda^2} |\tilde{F}| | d\lambda$ be finite.

where $\sigma(\xi-\eta,\eta)=|\xi|^{\frac{n}{2}}-|\xi-\eta|^{\frac{n}{2}}-|\eta|^{\frac{n}{2}}$. Now observe that,

$$|\sigma(\xi-\eta,\eta)|\le c\min(|\xi-\eta|,|\eta|)\max{}^{\frac{n}{2}-1}(|\xi-\eta|,|\eta|).$$

Thus , writing $|\xi|^{\frac{n}{2}}|u\hat{}(\xi)|=f(\xi), |\xi|^{\frac{n}{2}}|v\hat{}(\xi)|=g(\xi)$ with $f,g\in L^2$,

$$|E\hat{}(\xi)|\le c\int\left(\frac{1}{|\xi-\eta|^{\frac{n}{2}-1}|\eta|}+\frac{1}{|\eta|^{\frac{n}{2}-1}|\xi-\eta|}\right)f(\xi-\eta)g(\eta)d\eta$$

from which 1 is immediate.

We shall next prove Theorem 1 in the particular case of $N=3$. Define the operator $\mathcal{D}=W_+^{\frac{n}{2}}W_-^{\frac{1}{2}}$ by $W_\pm^aF(t,x)=\int\int e^{i\tau t}e^{ix\cdot\xi}\Big||\tau|\pm\xi|\Big|^a u\tilde{}(\tau,\xi)d\tau d\xi$ where $\tilde{}$ deotes the space-time Fourier transform of F. We have to prove that $\mathcal{D}(u_1u_2u_3)\in L^2$ for all solutions $\Box u_i=0$, $i=1,2,3$ with $H_{\frac{n}{2}}$ data and $u_i\in L^\infty$.

The main idea of our proof is to consider the following commutator,

$$E=\mathcal{D}(u_1u_2u_3)-u_1\mathcal{D}(u_2u_3)-u_2\mathcal{D}(u_3u_1)-u_3\mathcal{D}(u_1u_2), \tag{3}$$

for which we prove the estimate,

$$\|E\|_{L^2(\mathbb{R}^{n+1})}\le C$$

with a constant C which depends only on the size of the $H_{\frac{n}{2}}$ norm of the data for u_1, u_2u_3. Clearly, it suffices to prove this estimate for

$$u_i\tilde{}(\tau,\xi)=\delta(\tau-\epsilon_i|\xi|)\frac{f_i(\xi)}{|\xi|^{\frac{n}{2}}} \tag{4}$$

where $\epsilon_i=\pm1$ and $f_i\in L^2$. More precisely we shall prove the estimate,

$$\|E\|_{L^2(\mathbb{R}^{n+1})}\le C\|f_1\|_{L^2(\mathbb{R}^n)}\|f_2\|_{L^2(\mathbb{R}^n)}\|f_3\|_{L^2(\mathbb{R}^n)}. \tag{5}$$

The proof of he Theorem is then an immediate consequence of (5) and the result for $N=2$. To prove (5) we write the space-time Fourier transform of E in the form ,

$$E\tilde{}(\tau,\xi)=\int\int\int_{\lambda_1+\lambda_2+\lambda_3=\tau,\,\xi_1+\xi_2+\xi_3=\xi} m(\xi_1,\xi_2,\xi_3)u_1\tilde{}(\lambda_1,\xi_1)u_2\tilde{}(\lambda_2,\xi_2)u_3\tilde{}(\lambda_3,\xi_3)d\tau d\xi \tag{6}$$

where, $u_i\tilde{}$ are given by (4) and

$$\begin{aligned}m(\xi_1,\xi_2,\xi_3) &= d(\epsilon_1|\xi_1|+\epsilon_2|\xi_2|+\epsilon_3|\xi_3|,\ \xi_1+\xi_2+\xi_3)-d(\epsilon_2|\xi_2|+\epsilon_3|\xi_3|,\ \xi_2+\xi_3)\\ &- d(\epsilon_1|\xi_1|+\epsilon_3|\xi_3|,\ \xi_1+\xi_3)-d(\epsilon_1|\xi_1|+\epsilon_2|\xi_2|,\xi_1+\xi_2)\end{aligned} \tag{7}$$

and,

$$d(\tau,\xi)=(|\tau|+|\xi|)^{\frac{n}{2}}\,||\tau|-|\xi||^{\frac{1}{2}}$$

is the symbol of the operator $\mathcal{D}$. The commutator E was define with the intent that $m(\xi_1,\xi_2,\xi_3)\equiv0$ if any of the vectors ξ_1,ξ_2,ξ_3 vanish[4] . In fact we can prove the following:

$$|m(\xi_1,\xi_2,\xi_3)|\le C\min{}^{\frac{1}{2}}(|\xi_1|,\,|\xi_2|,\,|\xi_3|)\max{}^{\frac{n}{2}}(|\xi_1|,\,|\xi_2|,\,|\xi_3|) \tag{8}$$

[4]Observe that that $d(\epsilon|\xi|,\xi)=0$.

Without loss of generality we may assume that in (6) we integrate only on the region $|\xi_1| \le |\xi_2| \le |\xi_3|$. Then, $|m(\xi_1,\xi_2,\xi_3)| \le C(|\xi_1|\cdot|\xi_2|)^{\frac{1}{4}}|\xi_3|^{\frac{n}{2}}$ and,

$$|E\tilde{}(\tau,\xi)| \le C\int\int\int_{\lambda_1+\lambda_2+\lambda_3=\tau,\,\xi_1+\xi_2+\xi_3=\xi}(|\xi_1|\cdot|\xi_2|)^{\frac{1}{4}}|\xi_3|^{\frac{n}{2}}|u_1\tilde{}(\lambda_1,\xi_1)u_2\tilde{}(\lambda_2,\xi_2)u_3\tilde{}(\lambda_3,\xi_3)|.$$

Henceforth,

$$\begin{aligned}|E\tilde{}(\tau,\xi)| &\le v_1\tilde{} * v_2\tilde{} * v_3\tilde{}(\tau,\xi), \quad \text{where}\\ v_i\tilde{}(\tau,\xi) &= \delta(\tau-\epsilon_i|\xi_i|)g_i(\xi)\end{aligned}$$

with $g_1\hat{}(\xi) = \frac{1}{|\xi|^{\frac{2n-1}{4}}}|f_1(\xi)|$, $g_2\hat{}(\xi) = \frac{1}{|\xi|^{\frac{2n-1}{4}}}|f_2(\xi)|$, and $g_3\hat{}(\xi) = |f_3(\xi)|$. Thus v_1, v_2 and v_3 are solutions of $\Box v_i = 0$ with data in $H^{\frac{2n-1}{4}}$ and respectively L^2. By Plancherel formula and Holder inequality,

$$\|E\|_{L^2} \le C\|v_1v_2v_3\|_{L^2} \le C\|v_3\|_{L^\infty_t L^2_x}\|v_1\|_{L^4_t L^\infty_x}\|v_2\|_{L^4_t L^\infty_x}$$

and the proof follows from the following version of the Strichartz-Pecher inequalities. See [E-V]. It is this step than does not work for n=2.

Proposition 0.1 *Let $v\tilde{}(\tau,\xi) = \delta(\tau-|\xi|)g\hat{}(\xi)$. If $n \ge 3$ and $2 < q < \infty$, we have , with a constant $C = C_{q,n}$ depending only on q and the dimension n,*

$$\|v\|_{L^q_t L^\infty_x} \le C_{q,n}\|g\|_{\dot{H}_{\frac{n}{2}-\frac{1}{q}}}$$

Moreover, for large q,

$$C_{q,n} \le C_n\sqrt{q}$$

where C_n is a constant depending only on n and not on q. For $n=2$ we have to take $4 < q < \infty$.

We next proceed to prove the theorem in full generality. Given $u_1 \dots u_N$, as defined in (4), we form the commutator,

$$E = \mathcal{D}(u_1\dots u_N) + \sum_{k=1}^{N-2}(-1)^k\sum_{\sigma\in A_k}u_{\sigma(1)}\dots u_{\sigma(k)}\mathcal{D}(u_{\sigma(k+1)}\dots u_{\sigma(N)}) \qquad (9)$$

where A_k denotes all permutations of $\{1,\dots,N\}$ with $\sigma(1) < \sigma(2)\dots < \sigma(k)$ and $\sigma(k+1) < \sigma(k+2)\dots < \sigma(N)$. Thus A_k has $C_N^k = \frac{N!}{k!(N-k)!}$ distinct elements. We shall prove that,

$$\|E\|_{L^2(\mathbb{R}^{n+1})} \le C_N\|f_1\|_{L^2(\mathbb{R}^n)}\cdots\|f_N\|_{L^2(\mathbb{R}^n)}$$

More precisely we will prove the following,

Theorem 3 *Consider $u_i\tilde{} = \delta(\tau-\epsilon_i|\xi|)\frac{f_i(\xi)}{|\xi|^{\frac{n}{2}}}$ with $f_i \in L^2$ and the commutator E defined by (9) Then,*

$$\|E\|_{L^2(\mathbb{R}^{n+1})} \le (C\sqrt{N})^N\|f_1\|_{L^2(\mathbb{R}^n)}\cdot\|f_2\|_{L^2(\mathbb{R}^n)}\cdots\|f_N\|_{L^2(\mathbb{R}^n)}$$

Proof of Theorem 3: The Fourier transform of E can be written in the form,

$$E\tilde{}(\tau,\xi) = \int \cdots \int_{\sum_i \lambda_i=\tau,\, \sum_i \xi_i=\xi} m(\xi_1,\dots \xi_N) u_1\tilde{}(\lambda_1,\xi_1)\cdots u_N\tilde{}(\lambda_N,\xi_N) d\lambda d\xi \quad (10)$$

where

$$m(\xi_1,\dots \xi_N) = \sum_{k=0}^{N-2}(-1)^k \sum_{\sigma\in A_k} d(\sum_{i=k+1}^{N} \epsilon_{\sigma(i)}|\xi_{\sigma(i)}|, \sum_{i=k+1}^{N} \xi_{\sigma(i)}) \quad (11)$$

Observe that $m \equiv 0$ whenever any one of the vectors $\xi_1,\dots \xi_N$ vanishes. In fact we shall prove the following inequality,

$$m(\xi_1,\dots \xi_N) \le C2^N \min{}^{\frac{1}{2}}(|\xi_1|,\dots |\xi_N|) \max{}^{\frac{n}{2}}(|\xi_1|,\dots |\xi_N|) \quad (12)$$

Assuming the above inequality the theorem can be proved as follows: From (10), if we assume, without loss of generality, that $|\xi_1| \le |\xi_2| \le \dots \le |\xi_N|$ then,

$$|m| \le C|\xi_1|^{\frac{1}{2}}|\xi|^{\frac{n}{2}} \le C(|\xi_1|\cdots|\xi_{N-1}|)^{\frac{1}{2(N-1)}}|\xi_N|^{\frac{n}{2}}$$

Therefore,

$$|E\tilde{}| \le C2^N(v_1\cdot v_2\cdots v_N)\tilde{}$$

where $v_1 \dots v_N$ verify $\Box v_i = 0$ with data $g_i\hat{}(\xi) = \frac{1}{|\xi|^{\frac{n}{2}-\frac{1}{2(N-1)}}}|f_i\hat{}(\xi)|$ for $i = 1,\dots,N-1$ and $g_N\hat{}(\xi) = |f_N\hat{}(\xi)|$. Now , in view of Proposition 0.1, for all $i = 1,2\dots,N-1$

$$\|v_i\|_{L_t^{2(N-1)}L_x^\infty} \le CN^{\frac{1}{2}}\|D^{\frac{n}{2}-\frac{1}{2(N-1)}}g_i\|_{L^2} = CN^{\frac{1}{2}}\|f_i\|_{L^2}.$$

Therefore,

$$\begin{aligned}\|E\|_{L^2} &\le C2^N\|v_1\|_{L_t^{2(N-1)}L_x^\infty}\cdots|v_{N-1}\|_{L_t^{2(N-1)}L_x^\infty}\|v_N\|_{L_t^\infty L_x^2}\\ &\le C^N N^{\frac{N}{2}}\|f_1\|_{L^2}\cdots\|f_N\|_{L^2}\end{aligned}$$

and thus prove the desired inequality.

To finish the proof of Theorem 3 it remains to prove (12) and Prop. 0.1. To prove (12) first rewrite (11) in the form

$$m(\xi_1,\dots \xi_N) = \sum_{k=0}^{N-2}(-1)^k\sum_{\sigma\in A_k} d_{N-k}(\xi_{\sigma(k+1)},\, \xi_{\sigma(k+2)},\dots,\xi_{\sigma(N)})$$

where

$$d_l(\xi_1,\, \xi_2,\dots,\xi_l) = |\sum_{i=1}^k \xi_i|^{\frac{n}{2}}\Big|\,|\sum_{i=1}^k \epsilon_i|\xi_i|| - |\sum_{i=1}^k \xi_i|\Big|^{\frac{1}{2}}$$

Assume, without loss of generality, that $|\xi_1| \le \dots|\xi_N|$. Clearly $m(\xi_1,\dots\xi_N)$ can be written in the form as a sum of C_{N-1}^k terms of the type $d_{k+1}(\xi_1,\, \xi_{i_1},\dots,\xi_{i_k}) -$

$d_k(\xi_{i_1},\ldots,\xi_{i_k})$ where $1 < i_1 < i_2 \ldots i_k$, for $2 \le k \le N-1$, as well terms of the type $d_2(\xi_1,\xi_i)$. Thus the inequality (11) follows from the following,

$$\begin{aligned} d_2(\xi_1,\xi_i) &\le c|\xi_1|^{\frac{1}{2}}|\xi_N|^{\frac{n}{2}} \qquad (13)\\ |d_{k+1}(\xi_1,\xi_{i_1},\ldots,\xi_{i_k}) - d_k(\xi_{i_1},\ldots,\xi_{i_k})| &\le c|\xi_1|^{\frac{1}{2}}|\xi_N|^{\frac{n}{2}} \qquad (14)\end{aligned}$$

The inequality (13) follows easily from,

$$\Big| ||\xi| \pm |\eta|| - |\xi+\eta| \Big| \le 2\min(|\xi|,|\eta|).$$

To prove (14) we can write the left hand side L in the form

$$\begin{aligned} L &= |\xi_1 + A|^{\frac{n}{2}}\Big||\epsilon_1|\xi_1| + B| - |\xi_1 + A|\Big|^{\frac{1}{2}} - |A|^{\frac{n}{2}}\Big||B| - |A|\Big|^{\frac{1}{2}} \\ &= \left(|\xi_1 + A|^{\frac{n}{2}} - |A|^{\frac{n}{2}}\right)\Big||\epsilon_1|\xi_1| + B| - |\xi_1 + A|\Big|^{\frac{1}{2}} \\ &+ |A|^{\frac{n}{2}}\left(\Big||\epsilon_1|\xi_1| + B| - |\xi_1 + A|\Big|^{\frac{1}{2}} - \Big||B| - |A|\Big|^{\frac{1}{2}}\right)\end{aligned}$$

where $A = \xi_{i_1} + \ldots + \xi_{i_k}$, $B = \epsilon_{i_1}|\xi_{i_1}| + \ldots + \epsilon_{i_k}|\xi_{i_k}|$. Now, the result follows easily from $\Big||\xi_1 + A|^{\frac{n}{2}} - |A|^{\frac{n}{2}}\Big| \le c|\xi_1||\xi_N|^{\frac{n}{2}-\frac{1}{2}}$ and, since $\Big||u|^{\frac{1}{2}} - |v|^{\frac{1}{2}}\Big| \le |u-v|^{\frac{1}{2}}$,

$$\begin{aligned} \Big|\Big||\epsilon_1|\xi_1| + B| - |\xi_1 + A|\Big|^{\frac{1}{2}} - \Big||B| - |A|\Big|^{\frac{1}{2}}\Big| &\le \Big||\epsilon_1|\xi_1| + B| - |\xi_1 + A| - |B| + |A|\Big|^{\frac{1}{2}} \\ &\le \Big||\epsilon_1|\xi_1| + B| - |B|\Big|^{\frac{1}{2}} + \Big||\xi_1 + A| - |A|\Big|^{\frac{1}{2}} \\ &\le 2|\xi_1|^{\frac{1}{2}}\end{aligned}$$

Proof of Prop. 0.1: Let T be the operator defined from $L^2(\mathbb{R}^n)$ to functions of $(t,x) \in \mathbb{R}^{n+1}$ defined by

$$Tf(t,x) = \int e^{it|\xi| + ix\cdot\xi}\frac{1}{|\xi|^{\frac{n}{2}-\frac{1}{q}}}f\hat{}(\xi)d\xi$$

By the usual TT^* argument to show that $\|Tf\|_{L^q_tL^\infty_x} \le c\sqrt{q}\|f\|_{L^2}$ it suffices to check that $\|TT^*F\|_{L^q_tL^\infty_x} \le Cq\|F\|_{L^{q'}_tL^1_x}$. Now observe that TT^* can be written in the form,

$$TT^*F(t,x) = \int\int k(t-s,x-y)F(s,y)dsdy \qquad (15)$$

where ,

$$k(t,x) = \int e^{it|\xi| + ix\cdot\xi}\frac{1}{|\xi|^{n-\frac{2}{q}}}d\xi.$$

We shall show that, for large q,

$$|k(t,x)| \leq Cq\frac{1}{|t|^{\frac{2}{q}}} \tag{16}$$

Thus , from (15),

$$\|TT^*F(t,\cdot)\|_{L^\infty_x} \leq Cq\int \frac{1}{|t-s|^{\frac{2}{q}}}\|F(s,\cdot)\|_{L^1_x}.$$

By applying the Hardy-Littlewood inequality[5] we infer that,

$$\|TT^*F\|_{L^q_t L^\infty_x} \leq Cq\|F\|_{L^{q'}_t L^\infty_x}$$

as desired. It remains to prove (16). We shall prove it for $n = 3$, the proof for $n \geq 3$ is only slightly more involved. In that case, $k(t,x) = \int_0^\infty e^{it\lambda}\frac{sin\lambda|x|}{\lambda|x|}\frac{1}{\lambda^{1-\frac{2}{q}}}d\lambda$. Let $k = k_1 + k_2$ with

$$\begin{aligned} k_1 &= \int_0^{\frac{1}{|t|}} e^{it\lambda}\frac{sin\lambda|x|}{\lambda|x|}\frac{1}{\lambda^{1-\frac{2}{q}}}d\lambda \\ k_2 &= \int_{\frac{1}{|t|}}^\infty e^{it\lambda}\frac{sin\lambda|x|}{\lambda|x|}\frac{1}{\lambda^{1-\frac{2}{q}}}d\lambda \end{aligned}$$

Clearly

$$|k_1| \leq \int_0^{\frac{1}{|t|}} \lambda^{-1+\frac{2}{q}}d\lambda = \frac{q}{2}|t|^{-\frac{2}{q}}.$$

On the other hand , if $\frac{|t|}{2} \leq |x|$,

$$\begin{aligned} |k_2| &\leq \frac{1}{|x|}\int_{\frac{1}{|t|}}^\infty \frac{1}{\lambda^{-2+\frac{2}{q}}}d\lambda \leq \frac{1}{1-\frac{2}{q}}\frac{1}{|x|}|t|^{1-\frac{2}{q}} \\ &\leq C|t|^{-\frac{2}{q}}. \end{aligned}$$

Finally, for $|x| \leq \frac{|t|}{2}$, we make a change of variables and then integrate by parts as follows,

$$\begin{aligned} k_2(x) &= |x|^{-\frac{2}{q}}\int_{\frac{|x|}{|t|}}^\infty e^{i\frac{|t|}{|x|}\lambda}\frac{\sin\lambda}{\lambda}\lambda^{-1+\frac{2}{q}}d\lambda \\ &= |x|^{-\frac{2}{q}}\int_{\frac{|x|}{|t|}}^\infty \frac{|x|}{|t|}\frac{d}{d\lambda}(e^{i\frac{|t|}{|x|}\lambda})\frac{\sin\lambda}{\lambda}\lambda^{-1+\frac{2}{q}}d\lambda \\ &= k_{21} + k_{22} + k_{23}. \end{aligned}$$

The absolute value of the boundary term K_{21} is clearly bounded by $|x|^{-\frac{2}{q}}\frac{|x|}{|t|}(\frac{|x|}{|t|})^{-1+\frac{2}{q}} = |t|^{-\frac{2}{q}}$. Also,

[5] Which is valid for all $q > 2$, with a uniform constant.

$$
\begin{aligned}
|k_{22}| &= |x|^{-\frac{2}{q}}\frac{|x|}{|t|}\left|\int_{\frac{|x|}{|t|}}^{\infty} e^{i\frac{|t|}{|x|}\lambda}\frac{\sin\lambda}{\lambda}\frac{d}{d\lambda}\lambda^{-1+\frac{2}{q}}\right| \\
&\le |x|^{-\frac{2}{q}}\frac{|x|}{|t|}(1-\frac{2}{q})\int_{\frac{|x|}{|t|}}^{\infty}\lambda^{-2+\frac{2}{q}}d\lambda \\
&\le |t|^{-\frac{2}{q}} \\
|k_{23}| &= |x|^{-\frac{2}{q}}\frac{|x|}{|t|}\left|\int_{\frac{|x|}{|t|}}^{\infty} e^{i\frac{|t|}{|x|}\lambda}\frac{d}{d\lambda}(\frac{\sin\lambda}{\lambda})\lambda^{-1+\frac{2}{q}}d\lambda\right| \\
&\le C|x|^{-\frac{2}{q}}\frac{|x|}{|t|}\int_{\frac{|x|}{|t|}}^{\infty}\lambda^{-2+\frac{2}{q}}d\lambda \\
&\le C|t|^{-\frac{2}{q}}.
\end{aligned}
$$

Hence $|k_2| \le C|t|^{-\frac{2}{q}}$ as desired.

This ends the proof of Theorem 3. Theorem 1 is an obvious consequence of formula (9) and Theorem 3.

Proof of Theorem 2: Without loss of generality we may assume that u is a solution of $\Box u = 0$ with data $u = f \in H_{\frac{n}{2}}$, $\partial_t u = 0$ at $t = 0$. In view of Theorem 3 we have the formula,

$$
\mathcal{D}(u^N) = C_N^1 u\mathcal{D}(u^{N-1}) - C_N^2 u^2\mathcal{D}(u^{N-2}) + \ldots (-1)^{N-2}u^{N-2}\mathcal{D}(u^2) + E_N \tag{17}
$$

where E_N verifies the estimate,

$$
\|E_N\|_{L^2(\mathbb{R}^{n+1})} \le C^N N^{\frac{N}{2}} \tag{18}
$$

Next, we remark [6]

$$
\mathcal{D}(e^{i\lambda u})e^{-i\lambda u} = \sum_{k=0}^{\infty}\frac{i\lambda E_k}{k!} \tag{19}
$$

Hence,

$$
\|\mathcal{D}e^{i\lambda u}\|_{L^2} \le Ce^{C\lambda^2}\|f\|_{H^{n/2}} \tag{20}
$$

Therefore, if we write

$$
F(u) = \int e^{i\lambda u}\tilde{F}(\lambda)d\lambda \tag{21}
$$

we conclude

$$
\|\mathcal{D}F(u)\|_{L^2} \le C\|f\|_{H^{n/2}}\int e^{C\lambda^2}|\tilde{F}(\lambda)|d\lambda \tag{22}
$$

[6] Our formula is not true for general functions u, but is true for bounded ones. Out of convenience, let's assume u has Schwartz data, thus is bounded, and we prove an a priori estimate with constants independent of the L^∞ norm of u.

References

[B] J. Bourgain,Fourier transform restriction phenomena for certain lattice subsets and applications to non-linear evolution equations, I, II GAFA 3(1993), 107-156, 209-262.

[C] Jean-Yves Chemin , Fluides Parfaits Incompressible. *Asterix* 230 , 1995

[E-V] M. Escobedo and L Vega, A semilinear Dirac equation in $H^s(R^3)$ for $s > 1$, SIAM J. Math anal. vol 28, no 2, 338-362, March 1997

[K-M1] S. Klainerman and M. Machedon, Space-time estimates for null forms and the local existence theorem, *Comm. Pure Appl. Math* **46**(1993), 1221-1268

[K-M2] S. Klainerman and M. Machedon, Smoothing estimates for null forms and applications, *Duke Math J.* **81** (1995), 99-103

[K-S] S. Klainerman and S. Selberg, Remark on the optimal regularity for equations of Wave Maps type, *Comm.P.D.E* **22** (5-6), 901-918(1997).