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ASYMPTOTICS OF HOLOMORPHIC SECTIONS OF POWERS OF A POSITIVE LINE BUNDLE

STEVE ZELDITCH

1. Introduction

In this lecture we will describe some recent results [Z.1][Sh.Z] (obtained partly in collaboration with B.Shiffman) on the asymptotic properties of holomorphic sections $s_N \in H^0(M, L^N)$ of the Nth tensor power L^N of a positive hermitian holomorphic line bundle $(L, h) \to (M, \omega)$ over a compact Kahler manifold of dimension m. Here, $\omega = c_1(h)$ is the curvature form of h. Using the Boutet de Monvel-Sjostrand parametrix for the Szego kernel [BS] we first obtain a rather simple and direct proof of a result due to G. Tian on the asymptotic isometry of the Kodaira embeddings [T]. We then apply this result to prove the asymptotic uniform distribution of zeros of certain sequences of sections $s_N \in H^0(M, L^N)$ as $N \to \infty$, namely

- to 'random sequences' of sections
- to eigenfunctions of an 'ergodic quantum map'.

In the simplest case where $M = \mathbb{CP}^k$, where $L = O(1)$ (the hyperplane line bundle) and where $\omega = \omega_{FS}$ (the Fubini-Study form), $H^0(\mathbb{PC}^k, O(N))$ consists of homogeneous holomorphic polynomials $p(z_0, \dots, z_k)$ of degree N on $\mathbb{C}^{k+1}$. Our subject then specializes to the asymptotics of polynomials as the degree $N \to \infty$. The distribution of zeros of random polynomials of large degree was first studied by Bloch-Polya, Littlewood, Kac and others (see [BlD] for references). More recently, mathematical physicists [BlD][BBL][Ha][NV][Z2][V] have been interested in random polynomials (and more general sections) as a model for eigenfunctions of quantum chaotic maps. The large N limit of powers L^N of a positive line bundle arises in physics as the semi-classical limit in the geometric quantization of a compact Kahler manifold.

Our first result concerns the Kodaira embeddings $\varphi_N : M \to \mathbb{P}H^0(M, L^{\otimes N})^*$ defined by $\varphi_N(z) = \{s : s(z) = 0\}$. In a standard way [GH], one may express the Kodaira embeddings in terms of an orthonormal basis of sections. Namely, for each $N \in \mathbf{N}$, h induces a hermitian metric h_N on $L^{\otimes N}$. Let $\{S_0^N, \dots, S_{d_N}^N\}$ be any orthonormal basis of $H^0(M, L^{\otimes N})$, with respect to the inner product $\langle s_1, s_2 \rangle_{h_N} = \int_M h_N(s_1(z), s_2(z)) dV_g$ where dV_g is the volume form of g of volume one. For large N, there are no common zeroes of the sections $\{S_0^N, \dots, S_{d_N}^N\}$ and the Kodaira map may be expressed as:

$$\Phi_N(z) = [S_0^N(z), \dots, S_{d_N}^N(z)] \tag{1}$$

where $[S_0^N(z), \dots, S_{d_N}^N(z)]$ denotes the line through $(S_0^N(z), \dots, S_{d_N}^N(z))$ as defined in a local holomorphic frame.

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THEOREM **1.1.** *([Z.1]) There exists a complete asymptotic expansion:*

$$\sum_{i=0}^{d_N} ||S_i^N(z)||_{h_N}^2 = a_0 N^m + a_1(z) N^{m-1} + a_2(z) N^{m-2} + \dots$$

with $a_0 = \frac{c_1(L)^m}{m!}$ *in the sense that, for any* k, R,

$$||\sum_{i=0}^{d_N} ||S_i^N(z)||_{h_N}^2 - \sum_{j<R} a_j(z) N^{m-j}||_{C^k} \leq C_{R,k} N^{n-R}.$$

It follows easily that the Kodaira embeddings are asymptotically isometric:

COROLLARY **1.2.** *Let* ω_{FS} *denote the Fubini-Study form on* $\mathbb{CP}^{d_N}$*. Then:*

$$||\frac{1}{N}\Phi_N^*(\omega_{FS}) - \omega_g||_{C^k} = O(\frac{1}{N})$$

for any k.

This result was first proved (simeltaneously and independently) by G.Kempf [K] and S.Ji [Ji] in the case of abelian varieties (with C_0 convergence) and for general projective varieties with C^4 convergence by G.Tian [T, Lemma 3.2(i)]. Heat kernel proofs of the C^0-convergence result were later found by T. Bouche [Bou.1] [Bou.2] and J.P.Demailly [D]. The author has also recently learned that D.Catlin independently proved the theorem and its corollary by a method similar to ours [C].

Corollary (1.2) is the starting point for some results on the distribution of zeros of random sequences of holomorphic sections. Namely, we consider the probability space $(\mathcal{S}, d\mu)$, where $\mathcal{S}$ equals the product $\prod_{N=1}^{\infty} SH^0(M, L^N)$ of the unit spheres $SH^0(M, L^N)$ in $H^0(M, L^N)$ and μ is the product of Haar measures on these spheres. Given a sequence $\mathbf{s} = \{s_N\} \in \mathcal{S}$, we associate the currents of integration $[Z_{s_N}]$ over the zero divisors Z_{s_N} of the sections s_N. In complex dimension 1, $[Z_{s_N}]$ is the sum of delta functions at the zeros of s_N. Our first result states that for a random (i.e., for almost all) $\mathbf{s} \in \mathcal{S}$, the sequence of zeros of the sections s_N are asymptotically uniformly distributed:

THEOREM **1.3.** *([Sh.Z]) For* μ*-almost all* $\mathbf{s} = \{s_N\} \in \mathcal{S}$, $\frac{1}{N}[Z_{s_N}] \to \omega$ *weakly in the sense of measures; in other words,*

$$\lim_{N\to\infty} \left(\frac{1}{N}[Z_{s_N}], \varphi\right) = \int_M \omega \wedge \varphi$$

for all continuous $(m-1, m-1)$ *forms* φ*. In particular,*

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{Vol}_{2m-2}\{z \in U : s_N(z) = 0\} = m\mathrm{Vol}_{2n} U \ ,$$

for U *open in* M *(where* Vol_k *denotes the Riemannian* k*-volume in* (M, ω) *).*

Tian's theorem combined with some plurisubharmonic analysis also gives a uniform distribution theorem for zeros of eigenfunctions of ergodic quantum maps. Recall that a quantum map is a sequence $U_{\chi,N}$ of unitary operators on $H^0(M, L^N)$ which 'quantizes' a symplectic map $\chi : (M, \omega) \to (M, \omega)$. For the precise definitions, we refer to [Z3]. We call $U_{\chi,N}$ a 'quantum ergodic map' if χ is an ergodic transformation of (M, ω). Since the zero set of a wave function represents the locus of points where the quantum particle is least likely to be, it is natural to try to connect the distribution of zeros to dynamical properties of the map. In the ergodic case we have:

Theorem **1.4.** *([Sh.Z]; see also [NV]) Let $(L,h) \to (M,\omega)$ be a positive Hermitian line bundle over a Kähler manifold with $c_1(h)=\omega$ and let $U_{\chi,N}: H^0(M,L^N) \mapsto H^0(M,L^N)$ be a ergodic quantum map. Further, let $\{S_0^N,\dots,S_{d_N}^N\}$ be an orthonormal basis of eigensections of $U_{\chi,N}$. Then there exists a subsequence $\Lambda \subset \{(N,j): N=1,2,3,\dots,j\in\{0,\dots,d_N\}\}$ of density one such that*

$$\lim_{N\to\infty,(N,j)\in\Lambda} \frac{1}{N}[Z_{S_j^N}] \to \omega$$

weakly in the sense of measures.

This result was conjectured by LeBoeuf-Voros and was proved independently (and prior to us) by Nonnenmacher-Voros [NV] in the case of the theta bundle over an elliptic curve $\mathbb{C}/\mathbb{Z}^2$. We would also like to mention a result in a work in progress [NZ] at the opposite extreme where the quantum map U_χ is completely integrable in the strong sense of commuting with a Hamiltonian torus action. Let $J: M \to \mathbb{R}^n$ denote the moment map of the action and let $\Delta = J(M)$ denote the moment polytope. Then for each eigenfunction $S_{N,j}$ of $U_{\chi,N}$, we have $J(Z_{S_{N,j}}) \subset \partial\Delta$, i.e. the zeros are contained in the inverse image of the boundary of the moment polytope.

2. Preliminaries

2.1. **Notation.** We begin by recalling some basic notions and establishing notation. A holomorphic line bundle $L \to M$ over an m-dimensional compact complex (projective) manifold M is called positive if it has a hermitian metric h of positive curvature. The curvature form $c_1(h)$ is the $(1,1)$ form given locally by

$$c_1(h) = -\frac{\sqrt{-1}}{\pi}\partial\bar\partial \log \|e_L\|_h\ ,$$

where e_L is a nonvanishing local holomorphic section of L, and $\|e_L\|_h = h(e_L,e_L)^{1/2}$ denotes the h-norm of e_L. The form $c_1(h)$ is a de Rham representative of the first Chern class $c_1(L)$. Positivity means that $\omega = c_1(h)$ is a Kähler form, i.e. if $\omega = i\sum_{jk}\omega_{j\bar k}dz_j \wedge d\bar z_k$ then the hermitian form $\sum_{jk}\omega_{j\bar k}dz_j d\bar z_k$ is positive definite. We give M the volume form $dV = \frac{1}{c_1(L)^m}\omega^m$, so that M has unit volume: $\int_M dV = 1$.

As above, we denote the space of global holomorphic sections of L by $H^0(M,L)$ and that of its powers by $H^0(M,L^N)$ where $L^N = L\otimes\cdots\otimes L$. The dimension of $H^0(M,L^N)$ is denoted by d_N+1. It is well known that for N sufficiently large, d_N+1 is given by the Hilbert polynomial of L, whose leading term is $\frac{c_1(L)^m}{m!}N^m$ [G.H]. The metric h induces Hermitian metrics h_N on L^N given by $\|s^{\otimes N}\|_{h_N} = \|s\|_h^N$. We give $H^0(M,L^N)$ the inner product

$$\langle s_1, s_2\rangle = \int_M h_N(s_1,s_2)dV \qquad (s_1,s_2 \in H^0(M,L^N))\ , \tag{2}$$

and we write $|s| = \langle s,s\rangle^{1/2}$.

2.2. **Kodaira maps Φ_N.** We now fix an orthonormal basis $\{S_0^N,\dots,S_{d_N}^N\}$ of $H^0(M,L^N)$. We also fix a local holomorphic section e_L of L over $U\subset M$. It induces sections e_L^N of $L^{\otimes N}|_U$ and we may write $S_i^N(z) = f_i^N(z)e_L^N(z)$ for certain holomorphic functions f_i^N on U. The Kodaira map is defined by

$$\Phi_N(z) M \to \mathbb{PC}^{d_N}, \quad \Phi_N(z) = [f_0^N(z),\dots,f_{d_N}^N(z)] \tag{3}$$

where $[\cdots]$ denotes the line thru the vector in $\mathbb{C}^{d_N+1}$. Recall that the Kahler form $\omega_{g_{FS}}$ of the Fubini-Study metric g_{FS} on $\mathbb{C}P^m$ is given in homogeneous coordinates $[w_0, \dots, w_m]$ by $\omega_{FS} = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log(\sum_{j=0}^m |w_i|^2)$. Hence

$$\Phi_N^* \omega_{FS} = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log(\sum_{j=0}^m |f_j|^2). \tag{4}$$

It is easy to see that this form is independent of the choice of orthonormal basis.

2.3. **Currents of integration** $[Z_s]$**.** For a holomorphic section $s \in H^0(M, L^N)$, we let $[Z_s]$ denote the current of integration over the zero divisor of s. Recall here the notion of current: We let $\mathcal{D}^{p,q}(M)$ denote the space of $\mathcal{C}^\infty$ (p,q)-forms on M, and we let $\mathcal{D}'^{p,q}(M) = \mathcal{D}^{m-p,m-q}(M)'$ denote the space of (p,q)-currents on M; $(T, \varphi) = T(\varphi)$ denotes the pairing of $T \in \mathcal{D}'^{p,q}(M)$ and $\varphi \in \mathcal{D}^{m-p,m-q}(M)$. Then for $\varphi \in \mathcal{D}^{m-1,m-1}(M)$, $([Z_s], \varphi) = \int_{Z_s} \varphi$. In the local frame e_L^N for L^N, we write $s = f e_L^N$, where f is a local holomorphic function. The zero current is then given by the Poincaré-Lelong formula

$$[Z_s] = \frac{\sqrt{-1}}{\pi}\partial\bar\partial \log|f| = \frac{\sqrt{-1}}{\pi}\partial\bar\partial \log\|s\|_{h_n} + N\omega\ . \tag{5}$$

We also consider the normalized zero divisor

$$[\widetilde{Z}_s^N] = \frac{1}{N}[Z_s]\ ,$$

so that the currents $[\widetilde{Z}_s^N]$ are de Rham representatives of $c_1(L)$, and thus

$$\Big([\widetilde{Z}_s^N], \omega^{m-1}\Big) = \frac{c_1(L)^m}{m!}\ . \tag{6}$$

Equation (6) says that the currents $[\widetilde{Z}_s^N]s$ all have the same mass. For background on currents of integration see [GH][LG].

2.4. **Holomorphic sections and CR holomorphic functions.** Following [B.G], we study asymptotics of high powers of line bundles by lifting the problem to the unit circle bundle X associated to L. That is, let L^* be the dual line bundle to L and let $D = \{v \in L^* : h(v,v) < 1\}$ be its unit disc bundle relative to the metric induced by h and let $X = \partial D = \{v \in L^* : h(v,v) = 1\}$. The positivity of $c_1(h)$ is equivalent to the strict pseudoconvexity of the disc bundle D in L^* (see [Gr]). We will denote the S^1 action on X by $r_\theta x$ and its infinitesimal generator by $\frac{\partial}{\partial\theta}$.

Let us also denote by $T'D, T''D \subset TD \otimes \mathbb{C}$ the holomorphic, resp. anti-holomorphic subspaces and define $d'f = df|_{T'}, d''f = df|_{T''}$ for $f \in C^\infty(D)$. Then X inherits the CR structure $TX \otimes \mathbb{C} = T' \oplus T'' \oplus \mathbb{C}\frac{\partial}{\partial\theta}$. Here $T'X$ (resp. $T''X$) denotes the holomorphic (resp. anti-holomorphic vectors) of D which are tangent to X. They are given in local coordinates by vector fields $\sum a_j \frac{\partial}{\partial \bar z_j}$ such that $\sum a_j \frac{\partial}{\partial \bar z_j}\rho = 0$. A local basis is given by the vector fields $Z_j^k = \frac{\partial}{\partial \bar z_j} - (\frac{\partial\rho}{\partial \bar z_k})^{-1}(\frac{\partial\rho}{\partial \bar z_j})\frac{\partial}{\partial \bar z_k}$ $(j \neq k.)$

The Cauchy-Riemann operator on X is defined by

$$\bar\partial_b : C^\infty(X) \to C^\infty(X, (T'')^*), \qquad \bar\partial_b f = df|_{T''}. \tag{7}$$

In terms of the local basis above, it is given by

$$\bar{\partial}_b f = \sum_{j \neq k} Z_j^k f d\bar{z}_j|_{T''}. \tag{8}$$

Also associated to X are

$$\begin{aligned}
&\bullet \ \text{the contact form } \alpha = \tfrac{1}{i}d'\rho|_X = -\tfrac{1}{i}d''\rho|_X \\
&\bullet \ \text{the volume form } d\mu = \alpha \wedge (d\alpha)^n \\
&\bullet \ \text{the Levi form } L_\rho(z) = \sum \tfrac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} z_j \bar{z}_k. \\
&\bullet \ \text{the Levi form on} X \ L_X = L_\rho|_{T' \oplus T'' \cap TX}
\end{aligned} \tag{9}$$

which are independent of the choice of ρ. The Levi form on X is related to $d\alpha = \pi^* \omega_g$ by: $L_X(V, W) = d\alpha(V, \bar{W})$.

The Hardy space $H^2(X)$ is the space of boundary values of holomorphic functions on D which are in $L^2(X)$, or equivalently $H^2 = (\ker\ \bar{\partial}_b) \cap L^2(X)$. The S^1 action commutes with $\bar{\partial}_b$, hence $H^2(X) = \oplus_{N=1}^{\infty} H_N^2(X)$ where $H_N^2(X) = \{f \in H^2(X) : f(r_\theta x) = e^{iN\theta} f(x)\}$.

A section s of L determines an equivariant function $\hat{s}$ on $L^* - 0$ by the rule: $\hat{s}(z, \lambda) = \langle \lambda, s(z) \rangle$. Here, $z \in M, \lambda \in L_z^*$. It is clear that if $\tau \in \mathbb{C}^*$ then $\hat{s}(z, \tau\lambda) = \tau \hat{s}$. We will usually restrict $\hat{s}$ to X and then the equivariance property takes the form: $\hat{s}(r_\theta x) = e^{i\theta} \hat{s}(x)$. Similarly, a section s_N of $L^{\otimes N}$ determines an equivariant function $\hat{s}_N$ on $L^* - 0$: put $\hat{s}_N(z, \lambda) = \langle \lambda^{\otimes N}, s_N(z) \rangle$ where $\lambda^{\otimes N} = \lambda \otimes \lambda \otimes \cdots \otimes \lambda$. The following proposition is well-known and easy to prove:

Proposition 2.1. *The map $s \mapsto \hat{s}$ is a unitary equivalence between $H^0(M, L^{\otimes N})$ and $H_N^2(X)$.*

2.5. **Parametrix for the Cauchy-Szego kernel.** The key analytic objects in this paper are the orthogonal (Szego) projection $\Pi : L^2(X) \to H^2(X)$ and its Fourier components the finite dimensional projections $\Pi_N : L^2(X) \to H_N^2(X)$. We define their kernels by

$$\Pi_N f(x) = \int_X \Pi_N(x, y) f(y) d\mu(y), \tag{10}$$

which differs from the definition of [B.S] in using $d\mu$ as the reference density.

The main result on the Szego kernel $\Pi(x, y)$ for a strictly pseudoconvex domain is the following:

Theorem 2.2 (BS, Theorem 1.5 and §2.c)**.** *Let $\Pi(x, y)$ be the Szego kernel of the boundary X of a bounded strictly pseudo-convex domain Ω in a complex manifold L. Then: there exists a symbol $s \in S^n(X \times X \times \mathbb{R}^+)$ of the type*

$$s(x, y, t) \sim \sum_{k=0}^{\infty} t^{m-k} s_k(x, y)$$

so that

$$\Pi(x, y) = \int_0^{\infty} e^{it\psi(x,y)} s(x, y, t) dt$$

where the phase $\psi \in C^\infty(D \times D)$ is determined by the following properties:

- *$\psi(x, x) = \frac{1}{i}\rho(x)$ where ρ is the defining function of X.*

- $d''_x\psi$ *and* $d'_y\psi$ *vanish to infinite order along the diagonal.*
- $\psi(x,y) = -\bar{\psi}(y,x)$.

The integrals are regularized by taking the principal value (see [BS]). The second condition states that $\psi(x,y)$ is almost analytic. Roughly speaking, ψ is obtained by Taylor expanding $\rho(z,\bar{z})$ and replacing all the $\bar{z}$'s by $\bar{w}$'s, i.e. the Taylor expansion of ψ near the diagonal is given by

$$\psi(x+h, x+k) = \frac{1}{i}\sum \frac{\partial^{\alpha+\beta}\rho}{\partial z^\alpha \partial \bar{z}^\beta}(x)\frac{h^\alpha}{\alpha!}\frac{\bar{k}^\beta}{\beta!}.$$

For example, in the case of the unit ball $B_{n+1} \subset \mathbb{C}^{n+1}$, the Szego kernel is given by

$$K_{\partial B}(z,w) = \frac{1}{(1-\langle z,w\rangle)^{n+1}} = \int_0^\infty e^{it\psi_{\partial B}(z,w)}t^n dt$$

with $\psi_{\partial B}(z,w) = 1 - \langle z,w\rangle$.

The principal term $s_0(x,x)$ on the diagonal was determined in [B.S (4.10)], using that Π is a projection:

$$s_0(x,x)d\mu(x) = \frac{1}{4\pi^m}(\det\ \mathrm{L_X})||\mathrm{d}\rho||\mathrm{dx} \tag{11}$$

where $L_X = L_\rho|_{T'\oplus T''\cap TX}$ is the restriction of the Levi form to the maximal complex subspace of TX.

3. Proof of Theorem (1.1) and Corollary (1.2)

The proof of Theorem 1.2 begins by lifting everything to X. As above, we fix an orthonormal basis $\{S_i^N\}$ of $H^0(M, L^N)$ and lift it to abasis $\{\hat{S}_i^N\}$ of $H^2_N(X)$. The following proposition is straightforward and we refer to [Z1] for the proof:

Proposition **3.1.** $\{\hat{S}_i^N\}$ *is an orthonormal basis of* $H^2_N(X)$. *Moreover,* $||S_j^N(z)||^2_{h_N} = |\hat{S}_i^N(x)|^2$ *for any* x *with* $\pi(x) = z$.

The following lemma links our problem to the Szego kernels:

Lemma **3.2.** $\frac{\sqrt{-1}}{2\pi N}\Phi_N^*\omega_{FS} = \omega_g + \frac{\sqrt{-1}}{2\pi N}\bar{\partial}_b\partial_b \log \Pi_N(x,x)$.

Proof We first observe that

$$\frac{\sqrt{-1}}{2\pi N}\Phi_N^*\omega_{FS} = \omega_g + \frac{\sqrt{-1}}{2\pi N}\bar{\partial}\partial\log(\sum_{j=0}^{d_N}||S_j^N(z)||^2_{h_N}) \tag{12}$$

which follows by writing $||S_j^N(z)||^2_{h_N} = a^N|f_j^N|$ and using that $\bar{\partial}\partial\log a^N = -N\omega_g$. We lift this statement to X in the form:

$$\pi^*\frac{\sqrt{-1}}{2\pi N}\Phi_N^*\omega_{FS} = \frac{\sqrt{-1}}{2\pi N}\bar{\partial}_b\partial_b\log(\sum_{j=0}^{d_N}|\hat{S}_j^N|^2). \tag{13}$$

Next we observe that

$$\Pi_N(x,y) = \sum_{i=0}^{d_N}\hat{S}_i^N(x)\hat{S}^{N*}(y) \tag{14}$$

or, in local coordinates,

$$\Pi_N(z,\theta,w,\theta') = a(z)^{\frac{N}{2}} a(w)^{\frac{N}{2}} e^{iN(\theta-\theta')} \sum_{i=0}^{d_N} f_i^N(z) \bar{f}_i^N(w). \tag{15}$$

Hence we have

$$\begin{array}{ll} (a) & \sum_{j=0}^{d_N} ||S_j^N(z)||^2_{h_N} = \Pi_N(z,0,z,0) \\ (b) & \frac{\sqrt{-1}}{2\pi N}\partial\bar\partial \log(\sum_{j=0}^{d_N} ||S_j^N||^2_{h_N}) = \frac{\sqrt{-1}}{2\pi N}\bar\partial_b\partial_b \log \Pi_N(z,0,w,0). \end{array} \tag{16}$$

This completes the proof. □

The projections Π_N are Fourier coefficients of Π and hence may be expressed as:

$$\Pi_N(x,y) = \int_0^\infty \int_{S^1} e^{-iN\theta} e^{it\psi(r_\theta x, y)} s(r_\theta x, y, t) dt d\theta \tag{17}$$

where as above r_θ denotes the S^1 action on X. Changing variables $t \mapsto Nt$ gives

$$\Pi_N(x,y) = N \int_0^\infty \int_{S^1} e^{iN(-\theta + t\psi(r_\theta x, y)} s(r_\theta x, y, tN) dt d\theta. \tag{18}$$

¿From the fact that $\text{Im}\psi(x,y) \geq C(d(x,X) + d(y,X) + |x-y|^2 + O(|x-y|^3)$ (see [B.S]) it follows that the phase

$$\Psi(t,\theta;x,y) = t\psi(r_\theta x, y) - \theta. \tag{19}$$

has positive imaginary part. Hence the integral is a complex oscillatory integral. Before analysing its asymptotics we simplify the phase. As above, we choose a local holomorphic co-frame e_L^*, put $a(z) = |e_L^*|_z^2$, and write $\nu \in L_z^*$ as $\nu = \lambda e_L^*$. In the associated coordinates $(x,y) = (z,\lambda,w,\mu)$ on $X \times X$ we have:

$$\rho(z,\lambda) = a(z)|\lambda|^2, \quad \psi(z,\lambda,w,\mu) = \frac{1}{i} a(z,w)\lambda\bar\mu \tag{20}$$

where $a(z,w)$ is the almost analytic function on $M \times M$ satisfying $a(z,z) = a(z)$. On X we have $a(z)|\lambda|^2 = 1$ so we may write $\lambda = a(z)^{-\frac{1}{2}} e^{i\varphi}$. Similarly for μ. So for $(x,y) = (z,\varphi,w,\varphi') \in X \times X$ we have

$$\psi(z,\varphi,w,\varphi') = \frac{1}{i}(1 - \frac{a(z,w)}{i\sqrt{a(z)}\sqrt{a(w)}}) e^{i(\varphi-\varphi')}. \tag{21}$$

On the diagonal $x = y$ we have $\psi(r_\theta x, x) = \frac{1}{i}(1 - \frac{a(z,z)}{a(z)} e^{i\theta}) = \frac{1}{i}(1 - e^{i\theta})$. So

$$\Psi(t,\theta;x,x) = \frac{t}{i}(1 - e^{i\theta}) - \theta. \tag{22}$$

We have

$$\begin{array}{l} d_t\Psi = \frac{1}{i}(1 - e^{i\theta}) \\ d_\theta\Psi = te^{i\theta} - 1 \end{array} \tag{23}$$

so the critical set is $\mathcal{C} = \{(x,t,\theta) : \theta = 0, t = 1\}$. The Hessian Ψ'' on the critical set equals

$$\begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$$

so the phase is non-degenerate and the Hessian operator is given by $L_\Psi = \langle (\Psi''(1,0)^{-1}D, D\rangle = 2\frac{\partial^2}{\partial t \partial \theta} - i\frac{\partial^2}{\partial t^2}$. It follows by the stationary phase method that

$$\Pi_N(x,x) \sim N \frac{1}{\sqrt{\det(\mathrm{N}\Psi''(1,0)/2\pi \mathrm{i})}} \sum_{j,k=0}^{\infty} N^{m-k-j} L_j s_k(x,x) \tag{24}$$

for various differential operators L_j of order $2j$ More precisely, for any $R \geq 0$, one has (cf. [Ho I, Theorem 7.7.5])

$$|\Pi_N(x,x) - N \frac{1}{\det(\mathrm{N}\Psi''(1,0)/2\pi \mathrm{i})} \sum_{j+k<R}^{\infty} N^{m-k-j} L_j s_k(x,x)| \leq C N^{m-R} \sum_{k<R, |\alpha| \leq 2R-2k} ||D^\alpha s_k||_\infty. \tag{25}$$

The expansion can be differentiated any number of times. After some rearrangement, the series has the form

$$\Pi_N(x,x) = N^m s_0(x,x) + N^{m-1} a_1(x,x) + \dots \tag{26}$$

where the coefficients $s_0(x,x), a_1(x,x) \dots$ depend only on the jets of the terms s_k along the diagonal. From the description above of the leading coefficient $s_0(x,x)$ we have:

$$\Pi_N(x,x) d\mu(x) = \frac{1}{(2\pi^m)} N^m \alpha \wedge \omega^n + O(N^{m-1}). \tag{27}$$

Relative to the canonical volume measure, the coefficient is a (non-zero) constant times N^m; it is determined by comparison to the leading coefficient of the Hilbert polynomial, completing the proof of the theorem. □

3.1. **Proof of Corollary (1.2).** Because Φ_N is a CR map, the asymptotics of the derivatives follow immediately from the asymptotics of $\Pi_N(x,x)$. Indeed, $\bar{\partial}_b \partial_b \log \Pi_N(x,x) = \bar{\partial}_b \partial_b \log \Pi_N(x,y)|_{y=x}$.

By (a) we have

$$\begin{aligned} &\log \Pi_N(x,x) = \log(N^m s_0(x,x)[1 + N^{-1}\tfrac{s_1}{s_0} + \dots]) \\ &= m \log N + \log s_0(x,x) + \log[1 + N^{-1}\tfrac{s_1}{s_0} + \dots]) = m \log N + \log s_0(x,x) + O(\tfrac{1}{N}). \end{aligned} \tag{28}$$

By differentiating the expansion we get

$$\bar{\partial}_b \partial_b \log \Pi_N(x,x) = \bar{\partial}_b \partial_b \log s_0(x,x) + O(\frac{1}{N}) = O(1). \tag{29}$$

□

4. Zeros of random sequences of sections

We now turn to the distribution of zeros of random sequences of sections. Our purpose is to show that if we choose a sequence $\{s_1, s_2, \dots\}$ at random from the space $(\mathcal{S}, \mu)$, then the zero sets Z_{s_N} become uniformly distributed relative to ω.

A few remarks on the theorem before we begin the proof. First, it may seem to be rather tautologous that the limit distribution reproduces ω since the measure μ was defined in terms of ω in the first place. However, the result is not circular and in related settings the problem of determining the almost certain distribution of zeros is open. For instance, let us choose a (real-valued) spherical harmonic p_N of degree N at random from the unit sphere SV_N of the space of Nth order real spherical harmonics $V_N \subset L^2(S^n, \mathbb{R})$ and look at its zero set Z_{p_N}, a real hypersurface in S^n. Does the zero (nodal) set of almost any sequence $\{p_N\}$ become uniformaly distributed relative to the volume measure as $N \to \infty$? The answer is probably 'yes' but has not (at this time) been proved; the difficulty than is that the formula for the δ function on Z_{p_N} is more complicated than in the holomorphic case.

Now let us turn to the proof. What we do is calculate the expected values and variances of the random variables $([Z_s], \varphi)$. Corollary (1.2) determines the expected value and it turns out that the variances are so small that one can immediately deduce almost everywhere convergence. Somewhat surprisingly it is not even necessary to use the strong law of large numbers.

4.1. **Expected distribution of zeros.** We first determine the expected value of the normalized zero divisor $\widetilde{Z}_s$ as s is chosen at random from the unit sphere

$$SH^0(M, L^N) := \{s \in H^0(M, L^N) : |s| = 1\}$$

(or equivalently as $[s] \in \mathbb{P}H^0(M, L^N)$ is chosen at random with respect to the Fubini-Study volume). As above, we fix one orthonormal basis $\{S_j^N\}$ of $H^0(M, L^N)$ and write $S_j^N = f_j e_L^N$ relative to a holomorphic frame (= nonvanishing section) e_L^N over an open set $U \subset M$. Any section in $SH^0(M, L^N)$ may then be written as $s = \sum_{j=1}^{d_N} a_j f_j e_L^N$ with $\sum_{j=1}^{d_N} |a_j|^2 = 1$. To simplify the notation we let $f = (f_1, \dots, f_{d_N}) : U \to \mathbb{C}^{d_N}$ (which is a local representation of Φ_N) and we put

$$\sum_{j=1}^{d_N} a_j f_j = \langle a, f \rangle.$$

Hence

$$\widetilde{Z}_s^N = \frac{\sqrt{-1}}{N\pi} \partial\bar{\partial} \log |\langle a, f \rangle| \,. \tag{30}$$

We shall frequently use the notation $E(Y)$ for the expected value of a random variable Y on a probability space $(\Omega, d\mu)$, i.e. $E(Y) = \int_\Omega Y d\mu$.

We view $\widetilde{Z}_s^N$ as a $\mathcal{D}'^{1,1}(M)$-valued random variable (which we call simply a 'random current') as s varies over $SH^0(M, L^N)$ regarded as a probability space with the standard measure, which we denote by μ_N. The expected distribution of zeros of the random section s is the current $E(\widetilde{Z}_s^N) \in \mathcal{D}'^{1,1}(M)$ given by

$$\Big(E(\widetilde{Z}_s^N), \varphi\Big) = \int_{S^{2d_N-1}} \Big(\widetilde{Z}_s^N, \varphi\Big)\, d\mu_N, \quad \varphi \in \mathcal{D}^{m-1,m-1}(M), \tag{31}$$

where we identify $SH^0(M, L^N)$ with the unit $(2d_N + 1)$-sphere $S^{2d_N-1} \subset \mathbb{C}^{d_N}$. In fact, we have the following simple formula for the expected zero-distribution in terms of the map Φ_N:

Lemma **4.1.** *For N sufficiently large so that Φ_N is defined, we have:*

$$E(\widetilde{Z}_s^N) = \frac{1}{N} \Phi_N^* \omega_{\mathrm{FS}}$$

Proof

For simplicity, write $\omega_N = \frac{1}{N}\Phi_N^*\omega_{\rm FS}$. Then we have:

$$\omega_N = \frac{\sqrt{-1}}{2\pi N}\partial\bar\partial \log \sum_{j=1}^{d_N} |f_j^N|^2 = \frac{\sqrt{-1}}{2\pi N}\partial\bar\partial \log |f|^2, \tag{32}$$

where $f = (f_0, \dots, f_{d_N})$. Let φ be a smooth $(m-1, m-1)$ form, which we shall refer to as a 'test form'. We may assume that we have a coordinate frame for L on Support φ. By (30), we must show that

$$\frac{\sqrt{-1}}{\pi N}\int_{S^{2d_N-1}}\int_M \partial\bar\partial \log|\langle a, f\rangle| \wedge \varphi d\mu_N(a) = (\omega_N, \varphi)\ . \tag{33}$$

To compute the integral, we write $f = |f|u$ where $|u| \equiv 1$. Evidently, $\log|\langle a, f\rangle| = \log|f| + \log|\langle a, u\rangle|$. The first term gives

$$\frac{\sqrt{-1}}{\pi N}\int_M \partial\bar\partial \log|f| \wedge \varphi = \int_M \omega_N \wedge \varphi. \tag{34}$$

To complete the proof it suffices to show that the second term equals zero. But it is given by

$$\frac{\sqrt{-1}}{\pi}\int_{S^{2d_N-1}}\int_M \partial\bar\partial \log|\langle a, u\rangle| \wedge \varphi d\mu_N(a) = \frac{\sqrt{-1}}{\pi}\int_M \partial\bar\partial \left[\int_{S^{2d_N-1}} \log|\langle a, u\rangle| d\mu_N(a)\right] \wedge \varphi = 0, \tag{35}$$

since the average $\int \log|\langle a, \omega\rangle| d\mu_N(a)$ is a constant independent of u for $|u| = 1$, and thus the operator $\partial\bar\partial$ kills it. □

Combining Corollary **??** and Lemma 4.1, we obtain:

Corollary **4.2.** *$E(\widetilde{Z}_s^N) = \omega + O(\frac{1}{N})$; i.e., for each smooth test form φ, we have*

$$E(\widetilde{Z}_s^N, \varphi) = \int_M \omega \wedge \varphi + O(\frac{1}{N})\ .$$

4.2. **Variance estimate.** Now we obtain the variance estimate we need to obtain Theorem **??**. Let φ be a test form. It follows from our formula for the expectation (Lemma 4.1) that the variance of $(\widetilde{Z}_s^N, \varphi)$ is given by

$$E\left((\widetilde{Z}_s^N - \omega_N, \varphi)^2\right) = E\left(|(\widetilde{Z}_s^N, \varphi) - (\omega_N, \varphi)|^2\right) = E\left((\widetilde{Z}_s^N, \varphi)^2\right) - (\omega_N, \varphi)^2\ . \tag{36}$$

We have the following estimate of the variance:

Lemma **4.3.** *Let φ be any smooth test form. Then*

$$E\left(|(\widetilde{Z}_s^N, \varphi) - (\omega_N, \varphi)|^2\right) = O(\frac{1}{N^2}).$$

Proof: By (30) we easily obtain

$$E\left((\widetilde{Z}_s^N, \varphi)^2\right) = \frac{-1}{\pi^2 N^2}\int_M\int_M (\partial\bar\partial\varphi(z))(\partial\bar\partial\varphi(w)) \int_{S^{2d_N-1}} \log|\langle f(z), a\rangle| \log|\langle f(w), a\rangle| d\mu_N(a) \tag{37}$$

As in the previous lemma we write $f = |f|u$ with $|u| \equiv 1$. Then

$$\begin{aligned}\log|\langle f(z),a\rangle|\log|\langle f(w),a\rangle| &= \log|f(z)|\log|f(w)| + \log|f(z)|\log|\langle u(w),a\rangle| \\ &\quad + \log|f(w)|\log|\langle u(z),a\rangle| + \log|\langle u(w),a\rangle|\log|\langle u(z),a\rangle|.\end{aligned}$$

The first term contributes

$$\frac{-1}{\pi^2N^2}\int_M\int_M (\partial\bar\partial\varphi(z))(\partial\bar\partial\varphi(w))\log|f(z)|\log|f(w)| = \frac{1}{N^2}(\varphi,\Phi_N^*\omega_{\rm FS})^2 = (\varphi,\omega_N)^2. \tag{38}$$

The middle two terms contribute zero to the integral by (35). The lemma at hand thus comes down to the following claim:

$$\left|\int_M\int_M (\partial\bar\partial\varphi(z))(\partial\bar\partial\varphi(w))\int_{S^{2d_N-1}}\log|\langle u(z),a\rangle|\log|\langle u(w),a\rangle| d\mu_N(a)\right| = O(1). \tag{39}$$

By an explicit calculation of the integral one can show ([SZ]) that

$$G_N(x,y) := \int_{S^{2d_N-1}}\log|\langle x,a\rangle|\log|\langle y,a\rangle| d\mu_N(a) = C_N + O(1) \quad (x,y\in S^{2d_N-1}), \tag{40}$$

where C_N is a constant and the $O(1)$ term is uniformly bounded on $S^{2d_N-1}\times S^{2d_N-1}$. It follows that

$$E\left(|(\widetilde Z_s^N,\varphi) - (\omega_N,\varphi)|^2\right) \le \frac{C}{\pi^4N^2}\sup\|\partial\bar\partial\varphi\|^2 \tag{41}$$

□

4.3. **Almost everywhere convergence.** We can now complete the proof of Theorem 1.3. An element in $\mathcal{S}$ will be denoted $\mathbf{s} = \{s_N\}$. Since

$$|(\widetilde Z_{s_N},\varphi)| \le (\widetilde Z_{s_N},\omega^{m-1})\|\varphi\|_{\mathcal{C}^0} = c_1(L)^m\|\varphi\|_{\mathcal{C}^0}\ ,$$

by considering a countable $\mathcal{C}^0$-dense family of test forms, we need only consider one test form φ. By Lemma 1.2, it suffices to show that

$$(\widetilde Z_{s_N} - \omega_N,\varphi)\to 0 \qquad \text{almost surely}\ .$$

Consider the random variables

$$Y_N(\mathbf{s}) = (\widetilde Z_{s_N} - \omega_N,\varphi)^2 \ge 0\ . \tag{42}$$

By Lemma 4.3,

$$\int_{\mathcal{S}} Y_N(\mathbf{s})d\mu(\mathbf{s}) = O(\frac{1}{N^2})\ .$$

Therefore

$$\int_{\mathcal{S}}\sum_{N=1}^\infty Y_N d\mu = \sum_{N=1}^\infty\int_{\mathcal{S}} Y_N d\mu < +\infty,$$

and hence $Y_N \to 0$ almost surely. □

5. Zeros of SU(k) polynomials

To make the preceding result more concrete, we apply Lemma 4.1 to the case of random polynomials, with $M = \mathbb{CP}^m$, $L = \mathcal{O}(1)$, where we give L the standard Hermitian metric $h_{\rm FS}$, whose curvature is the Fubini-Study Kähler form $\omega = \omega_{\rm FS}$ on $\mathbb{CP}^m$.

5.1. SU(2) **polynomials.** First consider $m = 1$. Elements of $H^0(M, L^N) = H^0(\mathbb{CP}^1, \mathcal{O}(N))$ are homogeneous polynomials in two variables of degree N, or equivalently, polynomials in one variable of degree $\leq N$. A basis is given by $\sigma_j = z^j,\ j = 0, \dots, N$. The inner product in $H^0(M, L^N)$ is given by

$$\langle \sigma_j, \sigma_k \rangle = \int_{\mathbb{C}} \frac{z^j \bar{z}^k}{(1+|z|^2)^N} \omega = \frac{1}{\pi} \int_{\mathbb{C}} \frac{z^j \bar{z}^k}{(1+|z|^2)^{N+2}} dxdy \ .$$

Writing the integral in polar coordinates, we see that the σ_j are orthogonal, and

$$|\sigma_j|^2 = 2 \int_0^\infty \frac{r^{2j+1}}{(1+r^2)^{N+2}} dr = \frac{1}{(N+1)\binom{N}{j}} \ . \tag{43}$$

We thus can choose an orthonormal basis

$$S_j^N = (N+1)^{\frac{1}{2}} \binom{N}{j}^{\frac{1}{2}} z^j, \qquad j = 0, \dots, N \ .$$

Next, we note that

$$\sum_{j=1}^N \|S_j^N\|^2 = (1+|z|^2)^{-N} \sum_{j=1}^N (N+1)\binom{N}{j} |z^{2j}| \equiv N+1 \ ,$$

and thus $\omega_N = \frac{1}{N} \Phi_N^* \omega_{\mathrm{FS}} = \omega$. We thus recover the following result of [**?**, Appendix C] on 'random SU(2) polynomials':

THEOREM **5.1.** [?] *Suppose we have a random polynomial*

$$P(z) = c_0 + c_1 z + \cdots + c_N z^N \ ,$$

where $\mathrm{Re}\, c_0,\ \mathrm{Im}\, c_0,\ \dots,\ \mathrm{Re}\, c_N,\ \mathrm{Im}\, c_N$ *are independent Gaussian random variables with mean 0 and variances*

$$E\left((\mathrm{Re}\, c_j)^2\right) = E\left((\mathrm{Im}\, c_j)^2\right) = \binom{N}{j} \ .$$

Then the expected distribution of zeros of P is uniform over $\mathbb{CP}^1 \approx S^2$.

In fact, Theorem 1.3 tells us that for a random sequence of such polynomials, the distribution of zeros approaches uniformity.

5.2. SU$(m+1)$ **polynomials.** We now turn to the case of polynomials in several variables. An 'SU$(m+1)$ polynomial of degree N' is an element of the probability space of homogeneous polynomials of degree N on $\mathbb{C}^{m+1}$ with an SU$(m+1)$-invariant Gaussian probability measure. Recall that this space can be identified with $H^0(\mathbb{CP}^m, \mathcal{O}(N))$. We give $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ the standard inner product. A basis for $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ is given by the monomials

$$\sigma_J = z_0^{j_0} \cdots z_m^{j_m}, \qquad J = (j_0, \dots, j_m),\ |J| = N \ .$$

One easily sees that the σ_J are orthogonal. We compute

$$|\sigma_J|^2 = \int_{\mathbb{CP}^m} \frac{|\sigma_J(z)|^2}{|z|^{2N}} \omega_{FS}^m = \int_{S^{2m+1}} |\sigma_J(z)|^2 d\mu^{2m+1} = \frac{m! j_0! \cdots j_m!}{(N+m)!} \tag{44}$$

(where μ^{2m+1} is Haar probability measure on S^{2m+1}), by writing

$$\int_{\mathbb{C}^{m+1}} e^{-|z|^2} |\sigma_J(z)|^2 dz = \left(\int_{\mathbb{C}} e^{-|z_0|^2} |z_0|^{2j_0} dz_0 \right) \cdots \left(\int_{\mathbb{C}} e^{-|z_m|^2} |z_m|^{2j_m} dz_m \right) \ .$$

Therefore, the sections

$$S_J^N := \left[\frac{(N+m)!}{m!j_0!\cdots j_m!}\right]^{\frac{1}{2}} z^J$$

form an orthonormal basis for $H^0(\mathbb{CP}^m, \mathcal{O}(N))$. Furthermore

$$\sum_{|J|=N} \|S_J^N\|^2 \equiv \binom{N+m}{m} , \tag{45}$$

since the sum is $\mathrm{SU}(m+1)$ invariant, hence constant, and the integral of the left side equals $\dim H^0(\mathbb{CP}^m, \mathcal{O}(N))$.

In our results on zeros, we can replace the unit sphere $SH^0(M, L^N)$ with the complex d_N-dimensional vector space $H^0(M, L^N)$ with the Gaussian probability measure $\frac{1}{\pi^{d_N}} e^{-|s|^2} ds$ (where ds means $2d_N$-dimensional Lebesgue measure). The space of $\mathrm{SU}(m+1)$ polynomials of degree N is by definition the space $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ of homogeneous polynomials of degree N in $m+1$ variables (or equivalently, polynomials in m variables of degree $\leq N$) with this Gaussian measure. We can use (44) to describe the space of $\mathrm{SU}(m+1)$ polynomials explicitly as follows. For $P \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$, we write

$$P(z_0, \dots, z_m) = \sum_{|J|=N} \frac{a_J}{\sqrt{j_0! \cdots j_m!}} z_0^{j_0} \cdots z_m^{j_m} . \tag{46}$$

The Gaussian measure on $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ is then given by

$$\frac{1}{\pi^{d_N}} e^{-|A|^2} dA , \qquad A = (a_J) \in \mathbb{C}^{d_N} ,$$

where $d_N = \binom{N+m}{m}$.

Lemma 4.1 and (45) now tell us that if P is a polynomial given by (46), with the a_J being independent Gaussian random variables with mean 0 and variance 1, then the expected zero current Z_P equals $N\omega_{FS}$. Furthermore, Theorem 1.3 yields the following:

Proposition **5.2.** *Suppose we have a sequence of polynomials*

$$P_N(z_0, \dots, z_m) = \sum_{|J|=N} \frac{a_J^N}{\sqrt{j_0! \cdots j_m!}} z_0^{j_0} \cdots z_m^{j_m} ,$$

where the a_J^N are independent Gaussian random variables with mean 0 and variance 1. Then

$$\frac{1}{N} Z_{P_N} \to \omega_{FS} \qquad \text{almost surely}$$

(weakly in the sense of measures).

6. Ergodic eigenfunctions

Now we consider eigenfunctions of ergodic quantum maps. We recall that a quantum map is the quantization $U_{\chi,N} = \Pi_N \sigma T_\chi \Pi_N$ of a symplectic map χ_o of (M, ω). To be quantizable, χ_o must lift to a contact transformation χ of (X, α). We can then define the the unitary translation operator T_χ by χ on $L^2(X)$. It commutes with the S^1 action since it is lifted from the base. Unless χ_o is a holomorphic map, T_χ will not preserve $H^2(X)$ so to get an operator on $H^2_N(X)$ we need to compress it by Π_N. This will not usually be unitary so we need to put in a symbol $\sigma \in C^\infty(M)$ to make it so (at least modulo smoothing operators).

The quantum map $U_{\chi,N}$ is then a sequence of unitary operators which forms a semiclassical Toeplitz-Fourier integral operator. We refer to [Z3] for further details.

We call the quantum map $U_{\chi,N}$ ergodic if χ_o is an ergodic transformation of (M, ω). The following result, proved in [Z3], belongs to a long line of results originating in the work of A. Shnirelman [**?**] in 1974 on eigenfunctions of the Laplacian on compact Riemannian manifolds with ergodic geodesic flow.

THEOREM **6.1.** [Z3] *Let $\{S_j^N\}$ be an orthonormal basis of eigenfunctions of an ergodic quantum map $U_{\chi,N}$ on $H^0(M, L^N)$. Then there is a subsequence $\{S_{j_k}^N\}$ of density one such that $||S_j^N(z)||^2_{h_N} \to 1$ in the weak sense that for any open set U whose boundary has measure zero, $\int_U ||S_j^N(z)||^2_{h_N} dV \to vol(U)$.*

We now prove 1.4 by showing that if $||S_j^N(z)||^2_{h_N} \to 1$ then $[Z_{S_j^N}] \to \omega$. We write

$$u_N = \frac{1}{N} \log \|s_N(z)\|_{h_N} \ .$$

The main point of the proof is:

LEMMA **6.2.** *If $||S_j^N(z)||^2_{h_N} \to 1$ then $u_N \to 0$ in $L^1(M)$.*

Granted the lemma, it follows by the Poincaré-Lelong formula (5) that for any smooth test form $\varphi \in \mathcal{D}^{m-1,m-1}(M)$,

$$\left(\frac{1}{N} Z_N - \omega, \varphi\right) = \left(u_N, \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}\varphi\right) \to 0 \ ,$$

proving the theorem for smooth forms. Since by(6),

$$\left(\frac{1}{N} Z_N, \varphi\right) \leq \frac{c_1(L)^m}{m!} \sup |\varphi| \ ,$$

the conclusion extends to all $\mathcal{C}^0$ test forms φ.

Proof of Lemma: We first note that

i) the functions u_N are uniformly bounded above on M;
ii) $\limsup_{N\to\infty} u_N \leq 0$.

Indeed, since $s_N(z) = \int_M \Pi_N(z, w) s_N(w) dV$ we have by the Shwartz inequality that

$$\|s_N(z)\|^2_{h_N} \leq \Pi_N(z, z) = \left(\frac{c_1(L)^m}{m!} + O(1/N)\right) N^m.$$

Hence $\|s_N(z)\|_{h_N} \leq C N^{m/2}$ for some $C < \infty$ and taking the logarithm gives both statements.

As always, let e_L denote a local holomorphic frame for L over $U \subset M$ and let e_L^N denote the corresponding frame for L^N. Let $a(z) = \|e_L(z)\|_h$ so that $\|e_L^N(z)\|_{h_N} = a(z)^N$, and write $s_N = f_N e_L^N$ with $f_N \in \mathcal{O}(U)$ and $\|s_N\|_{h_N} = |f_N| a^N$. Instead of u_N let us consider the function

$$v_N = \frac{1}{N} \log |f_N| = u_N - \log a \ ,$$

since it is plurisubharmonic on U. Let U' be a relatively compact, open subset of U. To show that $u_N \to 0$ it suffices to show that $v_N \to -\log a$ in $L^1(U')$. Suppose on the contrary that $u_N \not\to 0$ in $L^1(U')$. Then we can find a subsequence $\{u_{N_k}\}$ with $\|u_{N_k}\|_{L^1(U')} \geq \delta > 0$.

By a standard result on subharmonic functions (see [Ho I, Theorem 4.1.9]), we know that the sequence $\{v_{N_k}\}$ either converges uniformly to $-\infty$ on U' or else has a subsequence which is convergent in $L^1(U')$. Let us now rule out the first possibility. If it occurred, there would exist $K > 0$ such that for $k \geq K$,

$$\frac{1}{N_k} \log \|s_{N_k}(z)\|_{h_{N_k}} \leq -1. \tag{47}$$

However, (47) implies that

$$\|s_{N_k}(z)\|^2_{h_{N_k}} \leq e^{-2N_k} \quad \forall z \in U' \ ,$$

which is inconsistent with the hypothesis that $\|s_{N_k}(z)\|^2_{h_{N_k}} \to 1$ in the weak* sense.

Therefore there must exist a subsequence, which we continue to denote by $\{v_{N_k}\}$, which converges in $L^1(U')$ to some $v \in L^1(U')$. By passing if necessary to a further subsequence, we may assume that $\{v_{N_k}\}$ converges pointwise almost everywhere in U' to v, and hence

$$v(z) = \limsup_{k\to\infty} u_{N_k}(z) - \log a \leq -\log a \qquad \text{(a.e)} \ .$$

Now let

$$v^*(z) := \limsup_{w\to z} v(w) \leq -\log a$$

be the upper-semicontinuous regularization of v. Then v^* is plurisubharmonic on U' and $v^* = v$ almost everywhere. Since $\|v_{N_k} + \log a\|_{L^1(U')} = \|u_{N_k}\|_{L^1(U')} \geq \delta > 0$, we know that $v^* \not\equiv -\log a$. Hence, for some $\epsilon > 0$, the open set $U_\epsilon = \{z \in U' : v^* < -\log a - \epsilon\}$ is non-empty. Let U'' be a non-empty, relatively compact, open subset of U_ϵ; by Hartogs' Lemma, there exists a positive integer K such that $v^* \leq -\log a - \epsilon/2$ for $z \in U''$, $k \geq K$; i.e.,

$$\|s_{N_k}(z)\|^2_{h_{N_k}} \leq e^{-\epsilon N_k}, \qquad z \in U'', \ \ k \geq K, \tag{48}$$

which contradicts the weak convergence to 1. This contradiction completes the proof of the lemma and hence of the theorem. □

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