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Wave packets techniques

Nicolas Lerner*

0. Introduction

We give here a short summary of a detailed article to appear ([L6]). We are interested in proving some energy estimates for $L = D_t + iQ(t)$, where t is one real variable, $iD_t = \partial/\partial t$ and $Q(t)$ is a self-adjoint operator on $\mathcal{H} = L^2(\mathbb{R}^n_x)$. We look for estimates of the following type:

$$C\|D_t u + iQ(t)u\|_{L^2(\mathbb{R},\mathcal{H})} \geq \|u\|_{L^2(\mathbb{R},\mathcal{H})}, \tag{0.1}$$

for $u \in C_0^\infty(\mathbb{R}, \mathcal{H})$ and a controlled constant C. The estimate (0.1) yields solvability properties for the adjoint operator L^* . When $Q(t) = q(t, x, D_x)$ is a classical pseudo-differential operator of order one with real-valued symbol $q(t, x, \xi)$, condition (ψ) for $\tau - iq(t, x, \xi)$ means that

$$q(t, x, \xi) > 0 \quad \text{and} \quad s > t \quad \Longrightarrow \quad q(s, x, \xi) \geq 0. \tag{0.2}$$

It was conjectured in the early seventies by Nirenberg and Treves that condition (0.2) is equivalent to solvability of $\frac{\partial}{\partial t} + Q(t)$. It is known that this condition is necessary for solvability to hold ([H1]). On the other hand condition (0.2) implies (0.1) for differential operators ([NT], [BF], [H1]) and also if the total dimension is two ([L1]) or in various special cases ([L2], [H2]). One can note that in all these cases, condition (ψ) implies "optimal" solvability, that is the estimate (0.1), yielding $H^{s+\text{order}L-1}$ solutions for equations $L^*u = f$ with $f \in H^s$. It was proved in [L3] that (0.2) does not imply (0.1) : one should not expect solvability in its optimal version expressed by (0.1) as a consequence of the geometric condition (0.2). Dencker ([D1]) was able to prove that the non optimally solvable examples of [L3] were solvable in H^{-1} (see also [D2]). To sum-up one could say, leaving aside the important and complete results on differential operators,

- Condition (ψ) is necessary for solvability of principal type pseudo-differential operators.
- Contrarily to various claims (published from 1971 to 1983), (ψ) does not imply optimal solvability.
- The sufficiency of (ψ) for solvability is an open problem.

Anyhow, our goal here is to prove (0.1) and it is therefore natural to assume a strengthened version of (0.2). In this situation, the ordinary differential equation $D_t + iq(t, x, \xi)$ with parameters (x, ξ) is the "wave packet" version of the pseudo-differential equation $D_t + iq(t, x, D_x)$ (see [CF], [Un]). In particular, it is easy to see that the good multiplier for the ODE is the sign $s(t, x, \xi)$ of $q(t, x, \xi)$. If properly defined, using (0.2), this sign function is non-decreasing with t: we can then study energy identities coming from the expression

$$2\,\mathrm{Re}\,\langle D_t\Phi(t, x, \xi) + iq(t, x, \xi)\Phi, i\ s(t, x, \xi)\Phi\rangle_{L^2(\mathbb{R}_t)}. \tag{0.3}$$

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Our first idea will be to quantify the previous energy identity in such a way that the operator with the very irregular symbol $s(t,x,\xi)$ is still $L^2(\mathbb{R}^n_x)$ bounded. Of course, neither the ordinary nor the Weyl quantization will do such a job, and we resort to the "Wick" quantization, which amounts to take a Gaussian regularization prior to a Weyl quantization ; this method relies on a decomposition of our operator into an integral of rank-one projections, whose range are the so-called coherent states (see [Be], [La], [L4]). We shall denote by $s^{\rm Wick}$ the Wick quantization of s. This quantization is non-negative, that is associates to a non-negative symbol a non-negative operator (this fails to be true for the Weyl or the ordinary quantization). Moreover this Wick quantization, whose precise definition is given in section 4 below, is close enough to the ordinary quantization to be useful. Namely, if q is a first order symbol and q^w its Weyl quantization, the difference $q^w - q^{\rm Wick}$ is L^2 bounded. We need then to estimate from below the selfadjoint part of $q(t,x,\xi)^w s(t,x,\xi)^{\rm Wick}$ and to check what remains of the simple equality $q(t,x,\xi)s(t,x,\xi) = |q(t,x,\xi)|$.

We develop two different methods for this purpose. The first one was given recently in [L5], and amounts to investigate closely the composition formula $q^{\rm Wick}s^{\rm Wick}$ and to extract the principal symbol in the Wick quantization of this product of operators. The second one is more elaborate and uses various tools of microlocal analysis to study the same product of operators : we construct a metric linked to the symbol $q(t,\cdot,\cdot)$ under scope and we get as close as we can of the non singular set of q, namely $\{(x,\xi), q(t,x,\xi) = 0, \text{ and } d_{x,\xi}q(t,x,\xi) \neq 0\}$. All the difficulties are somehow concentrated near this set, and we use then the Fefferman-Phong inequality ([FP], [H1]) for general second order pseudo-differential operators to get semi-boundedeness for $\operatorname{Re} q^{\rm Wick}s^{\rm Wick}$. Anyhow, both methods are useful for us and we are able to prove the following

Theorem 0.1. *Let n be an integer and $q(t,x,\xi) \in C^1\Big([-1,1], C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)\Big)$ satisfying (0.2) for $s,t \in [-1,1], (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and such that for all multi-indices α,β,*

$$\sup_{|t|\le 1,\ (x,\xi)\in\mathbb{R}^{2n}} |(\partial_x^\alpha \partial_\xi^\beta q)(t,x,\xi)|(1+|\xi|)^{-1+|\beta|} = \gamma_{\alpha\beta}(q) < +\infty. \tag{0.4}$$

We assume also that there exists a constant D_0 such that, for $|\xi| \ge 1$,

$$|\xi|^{-1}\Big|\frac{\partial q}{\partial x}(t,x,\xi)\Big|^2 + |\xi|\Big|\frac{\partial q}{\partial \xi}(t,x,\xi)\Big|^2 \le D_0 \frac{\partial q}{\partial t}(t,x,\xi), \qquad \text{when} \quad q(t,x,\xi) = 0. \tag{0.5}$$

Then, there exist positive constants ρ, C depending only on n and on a finite number of $\gamma_{\alpha\beta}(q)$ in (0.4) such that the estimate (0.1) is satisfied for $u(t,x) \in C_0^\infty\big((-\rho,\rho), \mathcal{S}(\mathbb{R}^n_x)\big)$ and $\mathcal{H} = L^2(\mathbb{R}^n)$.

In our remark 1.1 of [L2], we stated that the existence of a Lipschitz continuous function, homogeneous of degree 0, $\theta(x,\xi)$, so that

$$(t - \theta(x,\xi))\ q(t,x,\xi) \ge 0 \tag{0.6}$$

would imply solvability of the operator $D_t - iq(t,x,D_x)$, for q satisfying (0.4), homogeneous of degree 1 with respect to ξ. It is proven in section 4 of [L5] that (0.6) implies (0.5).

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1. Gaussian mollifiers for characteristic functions

We set for $\xi \in \mathbb{R}$,

$$\sigma_0(\xi) = \int_{\mathbb{R}} \operatorname{sign}(\eta) 2^{1/2} e^{-2\pi|\xi-\eta|^2} d\eta = \int_0^\xi 2^{3/2} e^{-2\pi t^2} dt. \tag{1.1}$$

Note that σ_0 is odd, $\sigma_0(+\infty) = 1$ and its derivative σ_0' is in $\mathcal{S}(\mathbb{R})$ and positive. We consider now a smooth real-valued function $b(\mathbf{x}, \lambda)$ defined on $\mathbb{R}^d \times [1, \infty)$, in the symbol class $S(\lambda^{1/2}, |d\mathbf{x}|^2\lambda^{-1})$. It means that b satisfies the estimates

$$\sup_{\mathbf{x}\in\mathbb{R}^d, \lambda\geq 1} |\partial_{\mathbf{x}}^k b(\mathbf{x}, \lambda)| \lambda^{-\frac{1}{2}+\frac{k}{2}} = \gamma_k(b) < \infty, \tag{1.2}$$

for any integer k. We omit below the dependence of b on the parameter λ and refer to

$$\gamma_k(b) \quad \text{as the semi} - \text{norms of } b. \tag{1.3}$$

We set-up then, for $(\mathbf{x}, \xi) \in \mathbb{R}^d \times \mathbb{R}$, $\beta \in \mathbb{R}$,

$$j(\mathbf{x}, \xi) = \iint_{\mathbb{R}^d \times \mathbb{R}} \operatorname{sign}\big(\eta + b(\mathbf{y})\big)\ 2^{\frac{d+1}{2}} e^{-2\pi(|\mathbf{x}-\mathbf{y}|^2+|\xi-\eta|^2)} d\mathbf{y} d\eta, \tag{1.4}$$

$$\sigma(\beta, \mathbf{x}) = \int \sigma_0\big(\beta + b(\mathbf{x}+\mathbf{y}) - b(\mathbf{x})\big)\Gamma(\mathbf{y})d\mathbf{y}, \qquad \Gamma(\mathbf{y}) = 2^{d/2}e^{-2\pi|\mathbf{y}|^2}. \tag{1.5}$$

We have thus

$$j(\mathbf{x}, \xi) = \sigma\big(\xi + b(\mathbf{x}), \mathbf{x}\big). \tag{1.6}$$

Moreover,

$$b(\mathbf{x}+\mathbf{y}) - b(\mathbf{x}) = b'(\mathbf{x}) \cdot \mathbf{y} + \omega_0(\mathbf{x}, \mathbf{y})\mathbf{y}^2\lambda^{-1/2}, \tag{1.7}$$

where the bilinear form $\omega_0(\mathbf{x}, \mathbf{y}) = \int_0^1 (1-\theta) b''(\mathbf{x}+\theta\mathbf{y})\lambda^{1/2} d\theta$ satisfies the estimates

$$|\partial_{\mathbf{x}}^k \partial_{\mathbf{y}}^l \omega_0(\mathbf{x}, \mathbf{y})| \leq \lambda^{-\frac{k+l}{2}} \gamma_{k+l+2}(b), \tag{1.8}$$

following from (1.2). We have from Taylor's formula and (1.5), (1.7),

$$\sigma(\beta, \mathbf{x}) = \int \sigma_0\big(b'(\mathbf{x}) \cdot \mathbf{y} + \omega_0(\mathbf{x}, \mathbf{y})\mathbf{y}^2\lambda^{-1/2}\big)\Gamma(\mathbf{y})d\mathbf{y} + \beta \iint_0^1 \sigma_0'\big(\theta\beta + b(\mathbf{x}+\mathbf{y}) - b(\mathbf{x})\big)\Gamma(\mathbf{y})d\mathbf{y}d\theta,$$

which implies, since σ_0 is odd,

$$\begin{aligned} \sigma(\beta, \mathbf{x}) = \quad & \lambda^{-1/2} \iint_0^1 \sigma_0'\big(b'(\mathbf{x}) \cdot \mathbf{y} + \theta\omega_0(\mathbf{x}, \mathbf{y})\mathbf{y}^2\lambda^{-1/2}\big)\omega_0(\mathbf{x}, \mathbf{y}) y^2 \Gamma(\mathbf{y}) d\mathbf{y} d\theta \\ & + \beta \iint_0^1 \sigma_0'\big(\theta\beta + b(\mathbf{x}+\mathbf{y}) - b(\mathbf{x})\big)\Gamma(\mathbf{y}) d\mathbf{y} d\theta. \end{aligned} \tag{1.9}$$

On the other hand, from (1.5), (1.7), we get

$$\sigma(\beta, \mathbf{x}) = \sigma_0(\beta) + \iint_0^1 \sigma_0'\Big(\beta + \theta\big[b'(\mathbf{x}) \cdot \mathbf{y} + \omega_0(\mathbf{x}, \mathbf{y})\mathbf{y}^2\lambda^{-1/2}\big]\Big)\big[b'(\mathbf{x}) \cdot \mathbf{y} + \omega_0(\mathbf{x}, \mathbf{y})\mathbf{y}^2\lambda^{-1/2}\big]\Gamma(\mathbf{y})d\mathbf{y}d\theta. \tag{1.10}$$

We state the following lemmas and refer the reader to [L6] for the proofs.

Lemma 1.1. *Let b be a symbol satisfying (1.2). Then, if σ is defined by (1.5), we have*

$$\sigma(\beta, \mathbf{x}) = \beta\sigma_1(\beta, \mathbf{x}) + \lambda^{-1/2} r_0(\mathbf{x}) = \sigma_0(\beta) + \sigma_2(\beta, \mathbf{x}), \tag{1.11}$$

where $r_0 \in S(1, |d\mathbf{x}|^2\lambda^{-1})$ *with semi-norms depending only on the* γ_k *in* (1.2). *Moreover* $\sigma_1(\beta, \mathbf{x}) \geq 0$ *and for all* k,

$$\sup_{\beta\in\mathbb{R},\mathbf{x}\in\mathbb{R}^d,\lambda\geq 1} |(\partial_{\mathbf{x}}^k \sigma_1)(\beta,\mathbf{x})|\lambda^{k/2} < \infty, \qquad \sup_{\beta\in\mathbb{R},\mathbf{x}\in\mathbb{R}^d,\lambda\geq 1} |\beta|\ |(\partial_{\mathbf{x}}^k \sigma_2)(\beta,\mathbf{x})|\lambda^{k/2} < \infty, \tag{1.12}$$

depending only on the γ_k *in* (1.2). *Moreover, there exists a positive constant* c_0 *depending only on* $d, \gamma_1(b), \gamma_2(b)$, *such that, for all positive* C,

$$\inf_{|\beta|\leq C,\mathbf{x}\in\mathbb{R}^d} \sigma_1(\beta,\mathbf{x}) \geq c_0 e^{-4\pi C^2}. \tag{1.13}$$

Lemma 1.2. *Let* b *be a symbol satisfying (1.2) and* j *be defined by (1.4). Then, there exist positive constants* c_1, c_2, c_3, *depending only on* $d, \gamma_1(b), \gamma_2(b)$, *such that for all* $(\xi, \mathbf{x}, \lambda) \in \mathbb{R} \times \mathbb{R}^d \times [1, +\infty)$,

$$\lambda^{1/2}\big(\xi + b(\mathbf{x})\big)j(\mathbf{x},\xi) + c_3 \geq 0. \tag{1.14}$$

Moreover, if $|\xi + b(\mathbf{x})| \geq c_1$, *we have*

$$c_2^{-1}\lambda^{1/2}|\xi + b(\mathbf{x})| \leq \lambda^{1/2}\big(\xi + b(\mathbf{x})\big)j(\mathbf{x},\xi) \leq \lambda^{1/2}|\xi + b(\mathbf{x})|. \tag{1.15}$$

If $|\xi + b(\mathbf{x})| \leq c_1$

$$\lambda^{1/2}\big(\xi + b(\mathbf{x})\big)j(\mathbf{x},\xi) + c_3 \geq \lambda^{1/2}\big(\xi + b(\mathbf{x})\big)^2 c_0 e^{-4\pi {c_1}^2}, \tag{1.16}$$

where c_0 *is defined in lemma* 1.1.

2. An admissible non-conformal metric

Let n be an integer and $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ be the standard phase space with its symplectic form

$$\varsigma = \sum_{1\leq j\leq n} d\xi_j \wedge dx_j.$$

We equip the phase space with a positive definite quadratic form Γ_0 such that $\Gamma_0^\varsigma = \Gamma_0$: it means that there is a symplectic basis of $\mathbb{R}^{2n}$ in which the matrix of Γ_0 is the identity (see (18.5.7) in [H1] for a general definition of Γ_0^ς). We consider now a smooth real-valued function $q(X, \Lambda)$ defined on $\mathbb{R}^{2n} \times [1, \infty)$, in the symbol class $S(\Lambda, \Lambda^{-1}\Gamma_0)$. It means that q satisfies the estimates

$$\sup_{X\in\mathbb{R}^{2n},\Lambda\geq 1} |\partial_X^k q(X,\Lambda)|_{\Gamma_0}\Lambda^{-1+\frac{k}{2}} = \gamma_k(q) < \infty, \tag{2.1}$$

for any integer k (the norm of the multi-linear form $\partial_X^k q$ is evaluated with respect to Γ_0). As in the previous section, we omit below the dependence of q upon Λ as well as the index Γ_0 for the norms of multi-linear forms. We define an admissible metric g on $\mathbb{R}^{2n}$ as a mapping $X \mapsto g_X$ from $\mathbb{R}^{2n}$ to the set of positive definite quadratic forms such that g is slowly varying, temperate and such that, for each $X \in \mathbb{R}^{2n}$, $g_X \leq g_X^\varsigma$. The proper class of the symbol q is defined by the following metric, conformal to Γ_0,

$$G_X = \lambda(X)^{-1}\Gamma_0, \qquad \lambda(X) = 1 + |q'(X)|^2_{\Gamma_0} + |q(X)|. \tag{2.2}$$

It is known that G is admissible with constants depending only on $\gamma_k,\ k = 0, 1, 2$ in (2.1), ([H1], section 26.10) and that $q \in S(\lambda, G)$ with the same semi-norms as q in $S(\Lambda, \Lambda^{-1}\Gamma_0)$. We define a new metric by

$$g_X(T) = \frac{|dq(X)\cdot T|^2}{\lambda(X) + |q(X)|^2} + \frac{\Gamma_0(T)}{\lambda(X)^{1/2} + |q(X)|} \quad , \qquad T \in \mathbb{R}^{2n}. \tag{2.3}$$

The following four lemmas are proved in [L6].

Lemma 2.1. *Let q be a symbol satisfying* (2.1). *If G is defined by* (2.2) *and g by* (2.3), *we have for* $\Lambda \geq 1$,

$$\gamma_{01}(q)^{-1}\Lambda^{-1}\Gamma_0 \leq G_X \leq 2g_X \leq 4\Gamma_0 = 4\Gamma_0^\varsigma \leq 8g_X^\varsigma \leq 16G_X^\varsigma \leq 16\gamma_{01}(q)\Lambda\Gamma_0, \tag{2.4}$$

with $\gamma_{01}(q) = 1 + \gamma_1(q)^2 + \gamma_0(q)$. *Moreover, g is slowly varying and temperate.*

Remark 2.2. The metric G separates the phase space into specific regions, depending on the fact that the dominant term in the expression (2.2) of $\lambda(X)$ is $|q(X)|$, $|q'(X)|^2$ or 1. In fact, following lemma 26.10.2 in [H1], one gets G-elliptic regions in which $C|q(X)| \geq \lambda(X)$. In such places, the metric g is equivalent to G, i.e. the ratios $g_X(T)/G_X(T)$ are bounded above and below by fixed constants. This is also the case for the G-negligible regions, in which $\lambda(X)$ is bounded above. In fact, in both cases $\lambda(X)^{1/2} + |q(X)|$ is equivalent to $\lambda(X)$ and since $|q'(X)\cdot T|^2 \leq \gamma_1^2\lambda(X)\Gamma_0(T)$, we get the equivalence of g and G there. The metric g is not equivalent to G on G-non-degenerate regions, that is on places where $|q'(X)|^2$ is the dominant term in (2.2). For instance, if q were the linear form $\lambda^{1/2}\xi_1$, the metric g would be, with symplectic coordinates $(x_1, \xi_1, X') \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{2n-2}$,

$$g = \frac{|d\xi_1|^2}{1+\xi_1^2} + \frac{|dx_1|^2 + |dX'|^2}{\lambda^{1/2}(1+|\xi_1|)} \gg \frac{|dX|^2}{\lambda} = G,$$

when $|\xi_1| \ll \lambda^{1/2}$.

Lemma 2.3. *Let $g/2$ be the admissible metric defined in* (2.3). *We define the positive numbers μ by* $\mu(X)^2 = 4\ \inf\big[g_X^\varsigma(T)/g_X(T)\big]$. *We have, with a constant C depending only on the γ_k in* (2.1),

$$1 \leq \mu(X) \leq 4\lambda(X), \tag{2.5}$$

$$|q(X)| + 1 \geq \lambda(X)/2 \quad \Longrightarrow \quad G_X \leq 2g_X \leq 4(1+4\gamma_1^2)G_X, \tag{2.6}$$

$$|q'(X)|^2 \geq \lambda(X)/2 \ \text{ and } \ |q(X)| \leq \lambda(X)^{1/2} \quad \Longrightarrow \quad C^{-1} \leq \frac{\mu(X)^2}{\lambda(X)^{1/2}} \leq C, \tag{2.7}$$

$$|q'(X)|^2 \geq \lambda(X)/2 \ \text{ and } \ |q(X)| \geq \lambda(X)^{1/2} \quad \Longrightarrow \quad C^{-1} \leq \frac{\mu(X)^2}{|q(X)|^3\lambda(X)^{-1}} \leq C, \tag{2.8}$$

$$|q(X)| \leq C\mu(X)^2, \qquad |q'(X)\cdot T| \leq C\mu(X)^2 g_X(T)^{1/2}. \tag{2.9}$$

3. Symbol classes

Lemma 3.1. *Let q be a symbol satisfying* (2.1). *If g is defined by* (2.3), *μ in lemma 2.3, then $q \in S(\mu^2, g)$ with the same semi-norms as q in* (2.1).

Lemma 3.2. *Let f be a bounded smooth function of one real variable so that f' belongs to the Schwartz space $\mathcal{S}(\mathbf{R})$. Let q be a symbol satisfying* (2.1) *and g defined in* (2.3). *Take $\tilde\lambda(X) \in S(\lambda(X), G_X)$ so that $\tilde\lambda(X) \geq d_0\lambda(X)$ for some positive constant d_0 (e.g $\tilde\lambda(X) = \sqrt{1+|q'(X)|_{\Gamma_0}^4 + |q(X)|^2}$). We have*

$$a(X) = f\Big(\tilde\lambda(X)^{-1/2}\, q(X)\Big) \in S(1, g), \tag{3.1}$$

with semi-norms depending only on those of q in (2.1), *on the L^∞ norm of f, on semi-norms of f' in $\mathcal{S}(\mathbf{R})$ and on d_0.*

4. Wick quantization

Before defining the Wick quantization, we recall the usual quantization formula,

$$a(x, D_x)u(x) = \iint e^{2i\pi x\xi} a(x,\xi)\hat{u}(\xi)d\xi, \qquad \hat{u}(\xi) = \int e^{-2i\pi x\xi} u(y)dy,$$

and the Weyl formula

$$a^w u(x) = \iint e^{2i\pi(x-y)\xi} a(\frac{x+y}{2}, \xi)u(y)dyd\xi.$$

As in section 2 and 3, we assume that the phase space $\mathbb{R}^{2n}$ is equipped with a symplectic norm Γ_0. For simplicity of notations, we shall often write $|T|^2$ instead of $\Gamma_0(T)$. The following definition contains also some classical properties.

Definition 4.1. *Let $Y = (y, \eta)$ be a point in $\mathbb{R}^{2n}$. The operator Σ_Y is defined as $\left[2^n e^{-2\pi|\cdot - Y|^2}\right]^w$. This is a rank-one orthogonal projection: $\Sigma_Y u = (Wu)(Y)\tau_Y\varphi$ with $(Wu)(Y) = \langle u, \tau_Y\varphi\rangle_{L^2(\mathbb{R}^n)}$, where $\varphi(x) = 2^{n/4}e^{-\pi|x|^2}$ and $(\tau_{y,\eta}\varphi)(x) = \varphi(x-y)e^{2i\pi\langle x-\frac{y}{2}, \eta\rangle}$. Let a be in $L^\infty(\mathbb{R}^{2n})$. The Wick quantization of a is defined as*

$$a^{\mathrm{Wick}} = \int_{\mathbb{R}^{2n}} a(Y)\Sigma_Y dY. \tag{4.1}$$

The following two propositions are classical and proved in [L6].

Proposition 4.2. *Let a be in $L^\infty(\mathbb{R}^{2n})$. Then $a^{\mathrm{Wick}} = W^* a^\mu W$ and $1^{\mathrm{Wick}} = \mathrm{Id}_{L^2(\mathbb{R}^n)}$ where W is the isometric mapping from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$ given above, and a^μ the operator of multiplication by a in $L^2(\mathbb{R}^{2n})$. The operator $\pi_H = WW^*$ is the orthogonal projection on a closed proper subspace H of $L^2(\mathbb{R}^{2n})$. Moreover, we have*

$$\|a^{\mathrm{Wick}}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le \|a\|_{L^\infty(\mathbb{R}^{2n})}, \qquad a(X) \ge 0 \Longrightarrow a^{\mathrm{Wick}} \ge 0, \tag{4.2}$$

$$\|\Sigma_Y \Sigma_Z\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le 2^n e^{-\frac{\pi}{2}|Y-Z|^2}. \tag{4.3}$$

Proposition 4.3. *Let p be a symbol in $S(\Lambda, \Lambda^{-1}\Gamma_0)$ (see (2.1) for the definition of a class of symbols with a large parameter Λ). Then $p^{\mathrm{Wick}} = p^w + r(p)^w$, with $r(p) \in S(1, \Lambda^{-1}\Gamma_0)$ so that the mapping $p \mapsto r(p)$ is continuous. Moreover, $r(p) = 0$ if p is a linear form or a constant.*

Proposition 4.4. *Let $a \in L^\infty(\mathbb{R}^{2n}), b \in S(\Lambda, \Lambda^{-1}\Gamma_0)$, be real-valued functions. Then*

$$\mathrm{Re}\left(a^{\mathrm{Wick}} b^{\mathrm{Wick}}\right) = \left[ab - \frac{1}{4\pi}\, a'(Y)\cdot b'(Y)\right]^{\mathrm{Wick}} + S, \tag{4.4}$$

where $\|S\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le d_n \|a\|_{L^\infty}\gamma_2(b)$. Here $\gamma_2(b)$ is a semi-norm of b in $S(\Lambda, \Lambda^{-1}\Gamma_0)$, and d_n depends only on the dimension.

Proof. We have

$$\begin{aligned} a^{\mathrm{Wick}}\, b^{\mathrm{Wick}} &= \iint a(Y)\ b(Z)\ \Sigma_Y\Sigma_Z\ dYdZ \\ &= \iint a(Y)\ \Big[b(Y) + b'(Y)\cdot(Z-Y) + \int_0^1 (1-\theta)\ b''\big(Y+\theta(Z-Y)\big)d\theta(Z-Y)^2\Big]\ \Sigma_Y\Sigma_Z\ dYdZ \\ &= \int a(Y)b(Y)\Sigma_Y dY + \iint a(Y)\ b'(Y)\cdot(Z-Y)\ \Sigma_Y\Sigma_Z\ dYdZ + R, \end{aligned} \tag{4.5}$$

with

$$R = \iint \alpha(Y,Z)(Z-Y)^2\ \Sigma_Y\Sigma_Z\ dYdZ,$$

where the norm of the quadratic form $\alpha(Y,Z)$ is less than $\|a\|_{L^\infty}\gamma_2(b)$; here $\gamma_2(b)$ is a semi-norm of the symbol b. From (4.3) and Cotlar's lemma, using

$$\Sigma_Y\Sigma_Z\Sigma_{Y'}\Sigma_{Z'} = (\Sigma_Y\Sigma_Z)(\Sigma_Z\Sigma_{Y'})(\Sigma_{Y'}\Sigma_{Z'}),$$

one gets that

$$\|R\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C(n)\|a\|_{L^\infty}\gamma_2(b), \tag{4.6}$$

where $C(n)$ depends only on the dimension. We check now the second term in (4.5), using definition 4.1,

$$\int b'(Y)\cdot(Z-Y)\ \Sigma_Z\ dZ = b'(Y)\cdot\Big[\int(\overbrace{Z-X}^{\text{will give }0}+X-Y)2^n e^{-2\pi|X-Z|^2}\ dZ\Big]^w = b'(Y)\cdot L_Y^w, \tag{4.7}$$

where L_Y is the (vector-valued) linear form $X-Y$. Note that, from proposition 4.3, $L^w = L^{\text{Wick}}$. From (4.5), (4.7), we get

$$\operatorname{Re}\big(a^{\text{Wick}}b^{\text{Wick}}\big) = (ab)^{\text{Wick}} + \int a(Y)b'(Y)\cdot\ \operatorname{Re}(L_Y^w\Sigma_Y)dY + \operatorname{Re}R. \tag{4.8}$$

Now, since L_Y is a real linear form, we have

$$\operatorname{Re}(L_Y^w\Sigma_Y) = \Big[(X-Y)2^n e^{-2\pi|X-Y|^2}\Big]^w = \frac{1}{4\pi}\frac{\partial}{\partial Y}(\Sigma_Y). \tag{4.9}$$

An integration by parts, in the distribution sense, gives what we expect in proposition 4.4, except possibly for

$$-\frac{1}{4\pi}\int a(Y)\ \text{Trace}\ b''(Y)\Sigma_Y dY + \operatorname{Re}R. \tag{4.10}$$

The estimate of R in (4.6), $a\in L^\infty$, $b\in S(\Lambda,\Lambda^{-1}\Gamma_0)$ and the estimate of $\|a^{\text{Wick}}\|$ in (4.2) applied to the integral in (4.10) prove the statement on S in proposition 4.4, whose proof is now complete.□

5. A non negativity result

We consider in this section a smooth real-valued function $q(t,X,\Lambda)$ defined on $\mathbb{R}_t\times\mathbb{R}_X^{2n}\times[1,\infty)$ which satisfies (2.1) uniformly in t, i.e.

$$\sup_{t\in\mathbb{R},X\in\mathbb{R}^{2n},\Lambda\ge1} |\partial_X^k q(t,X,\Lambda)|_{\Gamma_0}\Lambda^{-1+\frac{k}{2}} = \gamma_k(q) < \infty, \tag{5.1}$$

where Γ_0 is a symplectic norm (see §2). Moreover, we assume that $\tau - iq$ satisfies condition (ψ) (from now on, we omit the dependence of q on Λ),

$$q(t,X)>0 \quad\text{and}\quad s>t \quad\Longrightarrow\quad q(s,X)\ge 0. \tag{5.2}$$

Let's consider, *for t fixed*, the function

$$\lambda(t,X) = 1+|q(t,X)|+|q'_X(t,X)|^2_{\Gamma_0}. \tag{5.3}$$

We have, according to (2.2),

$$q(t,X)\in S(\lambda(t,X),\frac{\Gamma_0}{\lambda(t,X)} = G_X^{(t)}). \tag{5.4}$$

According to in section 2, the metric $G^{(t)}$ is slowly varying on $\mathbb{R}^{2n}_X$, satisfies the uncertainty principle ($G \le G^{\sigma}$), and is temperate. All the metrics $G^{(t)}$ are conformal and have the same "median symplectic" norm Γ_0, according to lemma 2.1. The metric $G^{(t)}$ defines the proper class of the symbol $q(t,\cdot)$: this is a metric on the phase space $\mathbb{R}^{2n}$, depending on $t \in \mathbb{R}$. We shall refer below to $G^{(t)}$ as the proper metric of the symbol q at the level t. We define now the bounded measurable functions

$$\begin{aligned}&\theta(X) = \inf\{t \in (-1,+1),\ q(t,X) > 0\} \quad \text{with} \quad \theta(X) = 1 \quad \text{if this set is empty,}\\ &s(t,X) = 1,\ \text{if } t > \theta(X), \qquad s(t,X) = 0,\ \text{if } t = \theta(X), \qquad s(t,X) = -1,\ \text{if } t < \theta(X).\end{aligned} \tag{5.5}$$

We get from (5.2) and (5.5) that, for $t \in (-1,1)$,

$$q(t,X)s(t,X) = |q(t,X)|. \tag{5.6}$$

We consider $J(t)$ the following increasing (with t) bounded selfadjoint operator on $L^2(\mathbb{R}^n)$:

$$J(t) = \big(s(t,X)\big)^{\mathrm{Wick}}. \tag{5.7}$$

We can now state the main result of this section,

Theorem 5.1. *Let q be a function satisfying (5.1-2), $Q(t) = q(t,X)^w$, $Q_0(t) = q(t,X)^{\mathrm{Wick}}$ and $J(t)$ be the operator given in (5.7). Then there exists $\tilde\gamma, \tilde\gamma_0$ depending only on a finite number of γ_k in (5.1) such that*

$$\operatorname{Re} Q(t)J(t) + \tilde\gamma \ge 0, \qquad \operatorname{Re} Q_0(t)J(t) + \tilde\gamma_0 \ge 0, \tag{5.8}$$

where $2\operatorname{Re} A = A + A^*$ for a bounded operator A on $L^2(\mathbb{R}^n)$. Moreover, the mapping $t \mapsto J(t)$ from $(-1,1)$ to $\mathcal{L}(L^2(\mathbb{R}^n))$ is non-decreasing.

Proof. The first inequality in (5.8) implies the second one since J is L^2 bounded as well as $Q - Q_0$ from proposition 4.3. We prove now the first inequality. The operator $J(t)$ is non-decreasing with t since the function $s(t,X)$ is nondecreasing of t and the Wick quantization is non-negative (second property in (4.2)). We set

$$J(t,X) = \int_{\mathbb{R}^{2n}} s(t,Y)\ 2^n\ e^{-2\pi|X-Y|^2} dY, \tag{5.9}$$

so that $J(t,X)$ is the Weyl symbol of $J(t)$. We obtain that $J(t,X) \in S(1,\Gamma_0)$ with semi-norms bounded by constants depending only on the dimension n. Since $q(t,X) \in S(\Lambda, \Lambda^{-1}\Gamma_0)$, the real part of the operator $Q(t)J(t)$ is given, up to L^2 bounded terms, by $\big(q(t,X)J(t,X)\big)^w$. From now on , we suppose that the variable t is fixed. We consider a partition of unity subordinated to the metric $G_X^{(t)}$ defined in (5.4). The following lemma is classical for an admissible metric (see section 18.4 in [H1]).

Lemma 5.2. *Let t be a number in $(-1,1)$. There exists a sequence $(X_\nu)_{\nu\in\mathbb{N}}$ of points in the phase space $\mathbb{R}^{2n}$ and positive numbers ρ_0, N_0, such that the following properties are satisfied ($G_\nu = \lambda_\nu^{-1}\Gamma_0$, $\lambda_\nu = \lambda(X_\nu)$, will stand for $G^{(t)}_{X_\nu}$ defined in (5.4)). We define $U_\nu, U_\nu^*, U_\nu^{**}$ as the G_ν balls with center X_ν and radius $\rho_0, 2\rho_0, 4\rho_0$. There exist two families of non-negative smooth functions on $\mathbb{R}^{2n}$, $(\chi_\nu)_{\nu\in\mathbb{N}}$, $(\psi_\nu)_{\nu\in\mathbb{N}}$ such that*

$$\sum_\nu \chi_\nu(X) = 1,\ \operatorname{supp} \chi_\nu \subset U_\nu, \qquad \psi_\nu \equiv 1 \text{ on } U_\nu^*,\ \operatorname{supp} \psi_\nu \subset U_\nu^{**}. \tag{5.10}$$

Moreover, $\chi_\nu, \psi_\nu \in S(1, G_\nu)$ with semi-norms bounded independently of ν (in fact depending only on the γ_k in (5.1)). The overlap of the balls U_ν^{**} is bounded, i.e.

$$\bigcap_{\nu\in\mathcal{N}} U_\nu \ne \emptyset \quad \Longrightarrow \quad \#\mathcal{N} \le N_0.$$

Moreover, $G_X \sim G_\nu$ all over U_ν^{**} (i.e. the ratios $G_X(T)/G_\nu(T)$ are bounded above and below by a fixed constant, provided that $X \in U_\nu^{**}$), so that $\psi_\nu q \in S(\lambda_\nu, G_\nu)$ uniformly (in fact with semi-norms depending only on the γ_k in (5.1)*).*

We have, using the above notations,

$$
\begin{aligned}
q(t,X)J(t,X) = &\sum_\nu \chi_\nu(X)q(t,X)\int_{\mathbb{R}^{2n}} s(t,Y)\ \psi_\nu(Y)2^n\ e^{-2\pi|X-Y|^2}dY \\
&+\sum_\nu \chi_\nu(X)q(t,X)\int_{\mathbb{R}^{2n}} s(t,Y)\ \big(1-\psi_\nu(Y)\big)2^n\ e^{-2\pi|X-Y|^2}dY.
\end{aligned}
\tag{5.11}
$$

We examine first the second term in (5.11)

$$
r_\nu(X) = \chi_\nu(X)q(t,X)\int_{\mathbb{R}^{2n}} s(t,Y)\ \big(1-\psi_\nu(Y)\big)2^n\ e^{-2\pi|X-Y|^2}dY. \tag{5.12}
$$

We obtain immediately from lemma 5.2 and (5.4)

$$
|r_\nu^{(k)}(X)T^k| \le \lambda_\nu |T|^k e^{-\pi\rho_0^2\lambda_\nu}\ C(k,n), \tag{5.13}
$$

where $C(k,n)$ depends on the γ_k in (5.1), on the dimension n, on k, but is *uniform with respect to* ν. Since the support of $r_\nu \subset U_\nu$, and these sets have a bounded overlap , (5.13) implies that

$$
\sum_\nu \chi_\nu(X)q(t,X)\int_{\mathbb{R}^{2n}} s(t,Y)\ (1-\psi_\nu(Y))2^n\ e^{-2\pi|X-Y|^2}dY = \sum_\nu r_\nu(X) \in S(1, G_X^{(t)}), \tag{5.14}
$$

and thus gives rise to a $L^2(\mathbb{R}^n)$ bounded operator. We are left with the first terms in the right-hand side of (5.11). We focus our attention on the non-degenerate indices: for these indices ν, with a constant C_3 independent of ν,

$$
C_3|q'_X| \ge \lambda_\nu^{1/2} \text{ for any } X \in U_\nu^{**} \qquad \text{and} \qquad \inf_{X\in U_\nu^{**}} |q(X)| \le C_1^{-1}\lambda_\nu.
$$

Then, for $X \in U_\nu^*$, the symbol q can be written as

$$
q = \lambda_\nu^{1/2}\big(\xi_1 + b_0(x_1, x', \xi')\big)\ e_0(x,\xi), \tag{5.15}
$$

for a suitable choice (depending on ν) of linear symplectic coordinates ($\xi_1 \in \mathbb{R}, x_1 \in \mathbb{R}$ are dual variables, $\xi' \in \mathbb{R}^{n-1}, x' \in \mathbb{R}^{n-1}$ are dual variables ; we note below $X' = (x', \xi') \in \mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$ and $Y' = (y', \eta') \in \mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$, and $Y = (y_1, \eta_1, Y') \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{2n-2}$). Here, we know that $b_0(x_1, x', \xi')$ satisfies the estimates of $S(\lambda_\nu^{1/2}, G_\nu)$ on U_ν^{**}, the symbol e_0 satisfies the estimates of $S(1, G_\nu)$ on U_ν^{**} and is elliptic i.e. $e_0(x,\xi) \ge m_0 > 0$ on U_ν^{**}. Then, there is no difficulty extending the symbols b_0 and e_0 to $\mathbb{R}^{2n}$: we set $b = \psi_\nu b_0$ and $e = e_0\psi_\nu$ in such a way that

$$
\chi_\nu q = \chi_\nu \lambda_\nu^{1/2}\big(\xi_1 + b(x_1, x', \xi')\big)\ e(x,\xi), \tag{5.16}
$$

with $b(x_1, x', \xi') \in S(\lambda_\nu^{1/2}, G_\nu)$, the symbol $e \in S(1, G_\nu)$ and is elliptic on U_ν^*, i.e. $e(x,\xi) \ge m_0 > 0$ there and $e \ge 0$ everywhere. Going back to the first term in the right-hand-side of (5.11) for non degenerate indices, and noticing that $s(t,Y)\psi_\nu(Y) = \psi_\nu(Y)\ \operatorname{sign}\big(\eta_1 + b(y_1, Y')\big)$ we check

$$
\begin{aligned}
&\chi_\nu(X)q(t,X)\int_{\mathbb{R}^{2n}} s(t,Y)\ \psi_\nu(Y)2^n\ e^{-2\pi|X-Y|^2}dY = \\
&e(X)\chi_\nu(X)\lambda_\nu^{1/2}\big(\xi_1 + b(x_1, X')\big)\int_{\mathbb{R}^{2n}} \operatorname{sign}\big(\eta_1 + b(y_1, Y')\big)2^n\ e^{-2\pi|X-Y|^2}dY \\
&\quad -\chi_\nu(X)q(t,X)\int_{\mathbb{R}^{2n}} \operatorname{sign}\big(\eta_1 + b(y_1, Y')\ (1-\psi_\nu(Y))2^n\ e^{-2\pi|X-Y|^2}dY.
\end{aligned}
\tag{5.17}
$$

We get rid of the last term, which is similar to (5.12), by using the same type of estimates as in (5.13), (5.14). It turns out eventually that the remaining terms in (5.11) are, $\mathbf{x}$ and $\mathbf{y}$ standing for (x_1, X') and (y_1, Y'),

$$e(X)\chi_\nu(X)\lambda_\nu^{1/2}\big(\xi_1+b(\mathbf{x})\big)\int_{\mathbb{R}^d}\int_{\mathbb{R}}\operatorname{sign}\big(\eta_1+b(\mathbf{y})\big)2^{d/2}\ e^{-2\pi|\mathbf{x}-\mathbf{y}|^2}2^{1/2}\ e^{-2\pi|\xi_1-\eta_1|^2}d\mathbf{y}d\eta_1. \tag{5.18}$$

We note first that the function b is defined on $\mathbb{R}^d$, $d=2n-1$, with the norm induced by Γ_0 and satisfies the estimates

$$|b^{(k)}(\mathbf{x})|\le\tilde\gamma_k\ \lambda_\nu^{\frac12-\frac k2}, \tag{5.19}$$

where the $\tilde\gamma_k$ are uniform in ν and depend only on the γ_k in (5.1). Our first important point is that from (1.4-5), we get

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}}\operatorname{sign}\big(\eta_1+b(\mathbf{y})\big)2^{d/2}\ e^{-2\pi|\mathbf{x}-\mathbf{y}|^2}2^{1/2}\ e^{-2\pi|\xi_1-\eta_1|^2}d\mathbf{y}d\eta_1=\sigma(\xi_1+b(\mathbf{x}),\mathbf{x})=j(\mathbf{x},\xi_1).$$

This implies, using (1.14) in lemma 1.2,

$$\lambda_\nu^{1/2}\big(\xi_1+b(\mathbf{x})\big)\int_{\mathbb{R}^d}\int_{\mathbb{R}}\operatorname{sign}\big(\eta_1+b(\mathbf{y})\big)2^{d/2}\ e^{-2\pi|\mathbf{x}-\mathbf{y}|^2}2^{1/2}\ e^{-2\pi|\xi_1-\eta_1|^2}d\mathbf{y}d\eta_1\ge-c_3, \tag{5.20}$$

where c_3 is the constant of lemma 1.2 (and thus is uniform in ν and depend only on the γ_k in (5.1)). Eventually, we are left with the function

$$A_\nu(X)=\chi_\nu(X)q(t,X)\ \sigma\big(\xi_1+b(\mathbf{x}),\mathbf{x}\big), \tag{5.21}$$

which is bounded from below (see (5.20)). We can prove that $A_\nu\in S(\mu^2,g)$ (see [L6]). Since g is an admissible metric, Hörmander's generalization of the Fefferman-Phong inequality (theorem 18.6.8 in [H1]) proves that A_ν^w semi-bounded from below. The proof of theorem 5.1 is complete. □

6. Energy estimates

Let $q(t,X,\Lambda)$ be a smooth function on $\mathbb{R}_t\times\mathbb{R}_X^{2n}\times[1,\infty)$, supported in $\mathbf{B}=\{|t|\le1\}\times\{|X|\le\Lambda^{1/2}\}$, satisfying (5.1). We assume that $\tau-iq$ satisfies Nirenberg-Treves' condition (ψ) i.e. that (5.2) is satisfied. Let $\chi_0:\mathbb{R}\to[0,1]$ be a smooth function, equal to 1 on $[-1,1]$, vanishing outside $(-2,2)$ and $\omega=1-\chi_0$. We set, with $s(t,X)$ defined in (5.5),

$$\mathcal{T}=\sum_{j=1}^{2n}\frac{\partial q}{\partial X_j}\frac{\partial s}{\partial X_j}=\frac{\partial q}{\partial X}\cdot\frac{\partial s}{\partial X}=\overbrace{\chi_0(|q'_X|^2)\mathcal{T}}^{\mathcal{T}_0}+\overbrace{\omega(|q'_X|^2)\mathcal{T}}^{\mathcal{T}_1}. \tag{6.1}$$

Lemma 6.1. *Let q, s and $\mathcal{T}$ be as above. The distribution derivative $\partial s/\partial t$ is a positive measure satisfying*

$$\langle\frac{\partial s}{\partial t},\Psi(t,X)\rangle_{\mathcal{S}'(\mathbb{R}^{2n+1}),\mathcal{S}(\mathbb{R}^{2n+1})}=2\int_{\mathbb{R}^{2n}}\Psi(\theta(X),X)dX. \tag{6.2}$$

Moreover, we have the following inclusions,

$$\operatorname{supp}\mathcal{T}\subset\{(t,X)\in\mathbf{B},q(t,X)=0\},\qquad\operatorname{supp}\mathcal{T}_1\subset\{(t,X)\in\mathbf{B},q(t,X)=0\text{ and }|q'_X(t,X)|\ge1\}=\mathbf{K}. \tag{6.3}$$

The open set $\Omega=\{q'_X(t,X)\ne0\}\cap\{|q(t,X)|<1\}$ is a neighborhood of the compact $\mathbf{K}$ and the Lebesgue measure of $\Omega\cap\{q(t,X)=0\}$ is zero. The restriction $s_{|\Omega}$ of s to Ω is the L^∞ function $q/|q|$. We have

$$\mathcal{T}_{1|\Omega}=\omega(|q'_X|^2)q'_X\cdot\frac{\partial}{\partial X}\left[\frac{q}{|q|}\right]=2\delta(q)|q'_X|^2\omega(|q'_X|^2). \tag{6.4}$$

Proof. The expression (6.2) is a consequence of (5.5). Moreover, from (5.5) and (5.6), the restriction of s to the open set $\{q(t,X)>0\}$ (resp.$\{q(t,X)<0\}$) is 1 (resp.-1). Thus the support of $\partial s/\partial X_j$ is included in $\{q(t,X)=0\}$. Since the restriction of q to the open set $\mathbf{B}^c$ is zero, (6.3) is proved. If (t,X) is a point of Ω such that $q(t,X)=0$, since $q'_X(t,X)\neq 0$ there is a neighborhood V of this point such that $\mathcal{L}(V\cap\{q=0\})=0$ ($\mathcal{L}$ stands for the Lebesgue measure). This proves that the compact sets

$$\{(t,X)\in\mathbf{B},\ 2^{j-1}\leq|q'_X(t,X)|\leq 2^j\}\cap\{(t,X),q(t,X)=0\}$$

are of Lebesgue measure 0 for all $j\in\mathbb{Z}$, and so is their denumerable union $\Omega\cap\{q=0\}$. From (5.6), we get that the restriction $s_{|\Omega}$ of s to Ω is the L^∞ function $q/|q|$. This gives (6.4). Note that since Ω is a neighborhood of the support of T_1, (6.4) determines completely T_1. The proof of lemma 6.1 is complete. □

Lemma 6.2. *Let q and s be as above. We define, using* (4.1),

$$Q_0(t)=\int_{\mathbb{R}^{2n}}q(t,X)\Sigma_X dX=q(t,\cdot)^{\text{Wick}}.$$

Let $u(t,x)$ be a function in $C_0^\infty(\mathbb{R}_t,\mathcal{S}(\mathbb{R}^n_x))$, and set $u(t)(x)=u(t,x)$, and for $(t,X)\in\mathbb{R}\times\mathbb{R}^{2n}$

$$\Phi(t,X)=[Wu(t)](X)=\langle u(t),\tau_X\varphi\rangle_{L^2(\mathbb{R}^n)}. \tag{6.5}$$

The function Φ belongs to $C_0^\infty(\mathbb{R},\mathcal{S}(\mathbb{R}^{2n}))$ and, with $D_t=(2i\pi)^{-1}\partial/\partial t$, ω defined above, Ω in lemma 6.1, $\Psi\in C_0^\infty(\Omega,[0,1])$, $\Psi\equiv 1$ on a neighborhood of $\mathbf{K}$ (see (6.3)*), we have*

$$\operatorname{Re}\langle D_t u, iJ(t)u(t)\rangle_{L^2(\mathbb{R}^{n+1})}=\frac{1}{2\pi}\int_{\mathbb{R}^{2n}}|\Phi(\theta(X),X)|^2dX, \tag{6.6}$$

$$\begin{aligned}\operatorname{Re}\langle Q_0(t)u(t),J(t)u(t)\rangle_{L^2(\mathbb{R}^{n+1})}\geq&\iint_{\mathbb{R}_t\times\mathbb{R}^{2n}_X}|q(t,X)||\Phi(t,X)|^2dtdX\\&-\frac{1}{2\pi}\langle\delta(q)|q'_X|^2\omega(|q'_X|^2),\Psi(t,X)|\Phi(t,X)|^2\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}-\tilde{\gamma}_1\|u\|^2_{L^2(\mathbb{R}^{n+1})},\end{aligned} \tag{6.7}$$

where $\tilde{\gamma}_1$ is a constant depending only on the dimension and the semi-norms of q.

Proof. Let us first notice that from (4.1) and (6.2) the left-hand-side of (6.6) is

$$-\frac{1}{4\pi}\iint\frac{\partial}{\partial t}\Big[\langle\Sigma_X u(t),u(t)\rangle_{L^2(\mathbb{R}^n)}\Big]\ s(t,X)\ dtdX=\frac{1}{2\pi}\int_{\mathbb{R}^{2n}}|\Phi(\theta(X),X)|^2dX.$$

We use the expression of Q_0 and proposition 4.4 to write, with $L^2(\mathbb{R}^{n+1})=L^2(\mathbb{R}_t,L^2(\mathbb{R}^n))$ dot products,

$$\begin{aligned}\operatorname{Re}\langle Q_0(t)u(t),J(t)u(t)\rangle&=\langle\operatorname{Re}[J(t)Q_0(t)]\ u(t),u(t)\rangle\\&=\langle\Big[|q(t,\cdot)|-\frac{1}{4\pi}\frac{\partial q}{\partial X}(t,\cdot)\cdot\frac{\partial s}{\partial X}(t,\cdot)\Big]^{\text{Wick}}u(t),u(t)\rangle+\langle S(t)\ u(t),u(t)\rangle,\end{aligned}$$

where $\|S(t)\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\leq d_n\gamma_2(q)$. We get then the following inequality, using (6.1), (6.4) and (6.5), with Ψ as in lemma 6.1,

$$\begin{aligned}\operatorname{Re}\langle Q_0(t)u(t),J(t)u(t)\rangle\geq&\iint_{\mathbb{R}_t\times\mathbb{R}^{2n}_X}|q(t,X)||\Phi(t,X)|^2dtdX\\&-\frac{1}{2\pi}\langle\delta(q)|q'_X|^2\omega(|q'_X|^2),\Psi(t,X)|\Phi(t,X)|^2\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}\\&-\frac{1}{4\pi}\ \langle\chi_0(|q'_X|^2)\frac{\partial q}{\partial X}(t,X)\cdot\frac{\partial s}{\partial X}(t,X),|\Phi(t,X)|^2\rangle_{\mathcal{S}'(\mathbb{R}^{2n+1}),\mathcal{S}(\mathbb{R}^{2n+1})}\\&-d_n\gamma_2(q)\ \|u\|^2_{L^2(\mathbb{R}^{n+1})}.\end{aligned}$$

To obtain (6.7), we need only to check the duality bracket with χ_0. This term is

$$
\begin{aligned}
&\frac{1}{4\pi}\iint s(t,X)\ \frac{\partial}{\partial X}\cdot\Big[\chi_0(|q'_X|^2)\frac{\partial q}{\partial X}(t,X)|\Phi(t,X)|^2\Big]dtdX\\
(6.8)\qquad &=\frac{1}{4\pi}\iint s(t,X)\ \frac{\partial}{\partial X}\cdot\Big[\chi_0(|q'_X|^2)\frac{\partial q}{\partial X}(t,X)\Big]|\Phi(t,X)|^2dtdX\\
&\quad+\frac{1}{4\pi}\iint s(t,X)\ \ \chi_0(|q'_X|^2)\frac{\partial q}{\partial X}(t,X)\cdot\frac{\partial}{\partial X}\big[\langle\Sigma_X u(t),u(t)\rangle_{L^2(\mathbb{R}^n)}\big]\ dtdX.
\end{aligned}
$$

We calculate

$$
(6.9)\qquad \frac{\partial}{\partial X}\cdot\Big[\chi_0(|q'_X|^2)\frac{\partial q}{\partial X}(t,X)\Big]=\chi_0'(|q'_X|^2)2q''_{XX}(q'_X,q'_X)+\chi_0(|q'_X|^2)\ \mathrm{Tr}q''_{XX}.
$$

From (6.1) and the fact that the support of χ_0 is bounded by 2, we get that (6.9) is bounded by a semi-norm of q. This proves that the absolute value of the first term in the right-hand-side of (6.8) is bounded above by the product of a semi-norm of q with $\|u\|^2_{L^2(\mathbb{R}^{n+1})}$. We claim that, from Cotlar's lemma and (4.3),

$$
(6.10)\qquad \Big\|\int_{\mathbb{R}^{2n}}\alpha(Y)\frac{\partial}{\partial Y_j}(\Sigma_Y)\ dY\Big\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\le\|\alpha\|_{L^\infty(\mathbb{R}^{2n})}d_n,
$$

where d_n depends only on the dimension : in fact, from (4.4), the Weyl symbol of $\Sigma_Y\Sigma_Z$ is

$$
p_{YZ}(X)=e^{-\pi|X-Y|^2}e^{-\pi|X-Z|^2}e^{-2i\pi[X-Y,X-Z]}2^n.
$$

This implies that the Weyl symbol of $\frac{\partial}{\partial Y_j}(\Sigma_Y)\frac{\partial}{\partial Z_j}(\Sigma_Z)=\frac{\partial^2}{\partial Y_j\partial Z_j}\Sigma_Y\Sigma_Z$ is

$$
q_{YZ}(X)=p_{YZ}(X)L_j(Y-X,Z-X),
$$

where L_j is a polynomial of degree 2. Now, we have

$$
(6.11)\qquad |q_{YZ}(X)|\le 16\pi 2^{n/2}\sqrt{|p_{YZ}(X)|}\le 16\pi 2^n e^{-\frac{\pi}{4}|Y-Z|^2}e^{-\pi|X-\frac{Y+Z}{2}|^2},
$$

so that the $\mathcal{L}(L^2(\mathbb{R}^n))$ norm of $\frac{\partial}{\partial Y_j}(\Sigma_Y)\frac{\partial}{\partial Z_j}(\Sigma_Z)$ is bounded above by the $L^1(\mathbb{R}^{2n})$ norm of its symbol q_{YZ}, which is estimated by $16\pi 2^n\ e^{-\frac{\pi}{4}|Y-Z|^2}$ from (6.11). Cotlar's lemma implies then (6.10). We note that

$$
s(t,X)\chi_0(|q'_X|^2)\frac{\partial q}{\partial X}(t,X)
$$

is bounded by 2, so that (6.11) implies that the absolute value of the second term in the right-hand-side of (6.8) is bounded above by $\pi^{-1}nd_n\|u\|^2_{L^2(\mathbb{R}^{n+1})}$. This concludes the proof of lemma 6.2.□

Theorem 6.3. *Let q,Q_0,J,u be as in lemma 6.2. We assume that there exists a constant D_0, such that*

$$
(6.12)\qquad q(t,X)=0\quad\text{ and }\quad |q'_X(t,X)|^2\ge 1\quad\Longrightarrow\quad |q'_X(t,X)|^2\le D_0q'_t(t,X).
$$

Then, there exist ε_0,T_0 positive constants depending only on the semi-norms of q and on D_0 such that, assuming supp $u\subset\{|t|\le T_0\}$, the following estimate holds (with $L^2(\mathbb{R}^{n+1})$ dot products and norms)

$$
(6.13)\qquad \mathrm{Re}\langle D_tu+iQ_0(t)u,iJ(t)u+i\frac{\varepsilon_0 t}{T_0}u\rangle\ge\frac{\varepsilon_0}{8\pi T_0}\|u\|^2.
$$

Thus, there exists a positive constant γ, depending only on the semi-norms of q and on D_0, such that, for $u \in C_0^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^n))$ with supp $u \subset \{|t| \le T_0\}$,

$$\gamma \|D_t u + iQ(t)u\| \ge \|u\|. \tag{6.14}$$

Proof. Let (t_0, X_0) be a point in $\mathbf{K}$ (see (6.3)). From (6.12), $q_t'(t_0, X_0) > 0$, so that the implicit function theorem give that, in an open neighborhood of (t_0, X_0),

$$q(t,X) = e(t,X)(t - \theta(X)) \quad \text{with} \quad e > 0 \quad \text{and} \quad e, \theta \in C^\infty.$$

This implies that, on this neighborhood,

$$\delta(t - \theta(X)) = \delta(q) q_t'(t,X). \tag{6.15}$$

Eventually, (6.15) makes sense and is satisfied in an open neighborhood $\tilde{\Omega}$ of $\mathbf{K}$. Thus, setting $\Omega_0 = \Omega \cap \tilde{\Omega}$, where Ω is defined in lemma 6.1, we obtain, with ω defined before (6.1) and $\Psi \in C_0^\infty(\Omega_0, [0,1])$, $\Psi \equiv 1$ in a neighborhood of $\mathbf{K}$, Φ given by (6.5),

$$\langle \delta(q) q_t'(t,X), \Psi(t,X) |\Phi(t,X)|^2 \rangle_{\mathcal{D}'(\Omega_0), \mathcal{D}(\Omega_0)} \le \int_{\mathbb{R}^{2n}} |\Phi(\theta(X), X)|^2 dX. \tag{6.16}$$

Moreover, from the assumption (6.12), we have

$$\begin{aligned} &\frac{1}{2\pi} \langle \delta(q) |q_X'|^2 \omega(|q_X'|^2), \Psi(t,X) |\Phi(t,X)|^2 \rangle_{\mathcal{D}'(\Omega_0), \mathcal{D}(\Omega_0)} \\ &\le \frac{D_0}{2\pi} \langle \delta(q) q_t', \omega(|q_X'|^2) \Psi(t,X) |\Phi(t,X)|^2 \rangle_{\mathcal{D}'(\Omega_0), \mathcal{D}(\Omega_0)} \le \frac{D_0}{2\pi} \int_{\mathbb{R}^{2n}} |\Phi(\theta(X), X)|^2 dX. \end{aligned} \tag{6.17}$$

We have the identity, for positive constants $\varepsilon_0, \varepsilon_1$ smaller than 1 to be precised later,

$$\begin{aligned} &\mathrm{Re}\langle D_t u + iQ_0(t)u, iJ(t)u + i\frac{\varepsilon_0 t}{T_0} u\rangle = \\ &\frac{1}{2\pi} \int_{\mathbb{R}^{2n}} |\Phi(\theta(X), X)|^2 dX + \frac{\varepsilon_0}{4\pi T_0} \|u\|^2 + (1-\varepsilon_1) \mathrm{Re}\langle Q_0(t)u, J(t)u\rangle + \varepsilon_1 \mathrm{Re}\langle Q_0(t)u, J(t)u\rangle + \mathrm{Re}\langle Q_0(t)u, \frac{\varepsilon_0 t}{T_0} u\rangle. \end{aligned}$$

Using theorem 5.1 to estimate from below the third term in the right-hand-side above (with factor $(1-\varepsilon_1)$), we get

$$\begin{aligned} &\mathrm{Re}\langle D_t u + iQ_0(t)u, iJ(t)u + i\frac{\varepsilon_0 t}{T_0}\ u\rangle \ge \\ &\quad \frac{1}{2\pi} \int_{\mathbb{R}^{2n}} |\Phi(\theta(X), X)|^2 dX + \Big[\frac{\varepsilon_0}{4\pi T_0} - \tilde{\gamma}_0(1-\varepsilon_1)\Big] \|u\|^2 + \varepsilon_1 \mathrm{Re}\langle Q_0(t)u, J(t)u\rangle + \mathrm{Re}\langle Q_0(t)u, \frac{\varepsilon_0 t}{T_0} u\rangle. \end{aligned}$$

We use now (6.7) to estimate from below the third term in the right-hand-side of the inequality above and we obtain,

$$\begin{aligned} &\mathrm{Re}\langle D_t u + iQ_0(t)u, iJ(t)u + i\frac{\varepsilon_0 t}{T_0} u\rangle \ge \\ &\frac{1}{2\pi} \int_{\mathbb{R}^{2n}} |\Phi(\theta(X), X)|^2 dX + \Big[\frac{\varepsilon_0}{4\pi T_0} - \tilde{\gamma}_0(1-\varepsilon_1) - \tilde{\gamma}_1 \varepsilon_1\Big] \|u\|^2 \\ &+ \iint_{\mathbb{R}_t \times \mathbb{R}^{2n}_X} \Big(\varepsilon_1 |q(t,X)| + \frac{\varepsilon_0 t}{T_0} q(t,X)\Big) |\Phi(t,X)|^2 dt dX - \varepsilon_1 \frac{1}{2\pi} \langle \delta(q) |q_X'|^2 \omega(|q_X'|^2), \Psi(t,X) |\Phi(t,X)|^2 \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}. \end{aligned}$$

We use (6.17) to estimate from below the last term above to get

$$\mathrm{Re}\langle D_t u + iQ_0(t)u, iJ(t)u + i\frac{\varepsilon_0 t}{T_0}u\rangle \geq \frac{1}{2\pi}(1-\varepsilon_1 D_0)\int_{\mathbb{R}^{2n}} |\Phi(\theta(X),X)|^2 dX + \Big[\frac{\varepsilon_0}{4\pi T_0} - \tilde{\gamma}_0(1-\varepsilon_1) - \tilde{\gamma}_1\varepsilon_1\Big]\|u\|^2 + \iint_{\mathbb{R}_t\times\mathbb{R}^{2n}_X}(\varepsilon_1 - \frac{\varepsilon_0|t|}{T_0})|q(t,X)||\Phi(t,X)|^2 dtdX.$$

We choose $\varepsilon_1 \leq \min(1, 1/D_0)$ and we obtain, using that u vanishes on $|t| \geq T_0$ and so does Φ (see (6.5)),

$$\mathrm{Re}\langle D_t u + iQ_0(t)u, iJ(t)u + i\frac{\varepsilon_0 t}{T_0}u\rangle \geq \Big[\frac{\varepsilon_0}{4\pi T_0} - \tilde{\gamma}_0 - \tilde{\gamma}_1\Big]\|u\|^2 + \iint_{\mathbb{R}_t\times\mathbb{R}^{2n}_X}(\varepsilon_1 - \varepsilon_0)|q(t,X)||\Phi(t,X)|^2 dtdX.$$

Eventually, one can take

$$\varepsilon_0 = \varepsilon_1 = \frac{1}{2}\min(1, 1/D_0), \qquad T_0 = \min(1, \frac{\varepsilon_0}{\tilde{\gamma}_0 + \tilde{\gamma}_1}\frac{1}{8\pi})$$

to obtain

$$\mathrm{Re}\langle D_t u + iQ_0(t)u, iJ(t)u + i\frac{\varepsilon_0 t}{T_0}u\rangle \geq \frac{\varepsilon_0}{8\pi T_0}\|u\|^2,$$

which implies (6.13). Since J is bounded with norm less than 1, we get (6.14) with $\gamma = (1+\varepsilon_0)8\pi T_0/\varepsilon_0$. This completes the proof of Theorem 6.3 and thus of Theorem 0.1.

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