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A trace formula for resonances and application to semi-classical Schrödinger operators

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A trace formula for resonances and application to semi-classical Schrödinger operators.

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Résumé. On décrit une formule de trace [S] pour les résonances, qui est valable en toute dimension et pour les perturbations à longue portée du Laplacien. On établit une nouvelle application à l'éxistence de nombreuses résonances pour des opérateurs de Schrödinger semi-classiques.

0. Introduction.

A very useful formula in the study of resonances is the so called Poisson formula in Lax-Phillips theory. Let $P_0 = -\Delta$, where Δ is the standard Laplace operator on $\mathbf{R}^n$, and let P_1 be some other operator which is equal to $-\Delta$ outside a compact set, and which satisfies suitable additional assumptions that will not be recalled here. Then assuming also that the dimension n is odd:

$$2\text{"}\mathrm{tr}\,(\cos t\sqrt{P} - \cos t\sqrt{P_0})\text{"} = \sum e^{it\lambda_j},\ t > 0. \tag{0.1}$$

Here λ_j are the resonances of P_1 in the upper half-plane, while below we shall rather take the opposite convention and define resonaces as the corresponding complex conjugates. The quotation marks indicate that the definition of the left hand side needs some work, since P_1 and P_0 do not act in the same Hilbert space in general. This Poisson type formula was proved first by Lax-Phillips [LP2]and then with successive extensions by Bardos-Guillot-Ralston [BGR], Melrose [M], Sjöstrand-Zworski [SZ3]. These results were obtained in the frame work of the Lax-Phillips scattering theory. An analogous reult was obtained for hyperbolic surfaces by Guillopé-Zworski [GZ], and the proof uses more general scattering theory and among other things the Birman-Krein formula for the scattering phase. This trace formula has lead to a number of results on the existence of infinitely many resonances, and also on results on lower bounds on the number of resonances in various zones. See [BLRa], [I], [M2], [SaZ], [SaZ], [SZ2,3], [P].

In this exposé, we explain a trace formula valid in all dimensions, and even for long-range perturbations of the Laplacian. A detailed proof can be found in [S], where it is also explained how the new trace-formula permits to recover most (and perhaps all) the lower bounds on the density of resonances near the real axis, for certain compactly supported perturbations of the Laplacian, and to extend them to the case of even dimensions. L.Nedelec [N] has recently studied certain Schrödinger operators with linear matrix valued potentials and used a trace formula directly based on Lidskii's theorem. Discussions around that work were useful for [S].

The new trace formula is stated and proved in the frame work of dilation analytic operators, but the main new elements could be easily adaptable to other frame works, such as the one developed with Helffer in [HS]. The trace formula is stated in section 1 and we refer to [S] for proofs and further details. In section 2, we establish a very general application to semi-classical Schrödinger operators.

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1. The trace formula.

In the following we shall consider two operators, P_0, P_1 acting on two different Hilbert spaces. It will be practical to write $P_\cdot$, where $\cdot = 0, 1$ and use the same system of notation for the various objects associated to $P_\cdot$. We will use essentially the same abstract framework as in [SZ], and we will use the same type of analytic distorsions as in that work. (See also [AC], [Hu] for analytic distorsions.) In the following h will be a small parameter with $0 < h \leq h_0$.

Consider two complex separable Hilbert spaces $\mathcal{H}_\cdot$ with the orthogonal decomposition

$$\mathcal{H}_\cdot = \mathcal{H}_{\cdot,R_0} \oplus L^2(\mathbf{R}^n) \setminus B(0,R_0), \tag{H1}$$

where $R_0 > 0$, $\mathcal{H}_{\cdot,R_0}$ is some Hilbert space, and $B(0,R_0)$ denotes the open ball in $\mathbf{R}^n$ of center 0 and radius R_0. The corresponding othogonal projections are denoted by $1_{B(0,R_0)}$ and $1_{\mathbf{R}^n\setminus B(0,R_0)}$. If $\chi \in C_b(\mathbf{R}^n)$ (the space of bounded continous functions) is constant on $B(0,R_0)$, then there is a natural way of defining the multiplication operator $\chi : \mathcal{H}_\cdot \to \mathcal{H}_\cdot$. (See [SZ].)

Let $P_\cdot = P_\cdot(h)$ be an unbounded self-adjoint operator: $\mathcal{H}_\cdot \to \mathcal{H}_\cdot$ with domain $\mathcal{D}_\cdot = \mathcal{D}(P_\cdot)$. Assume

$$1_{\mathbf{R}^n\setminus B(0,R_0)}\mathcal{D}_\cdot = H^2(\mathbf{R}^n \setminus B(0,R_0)) \text{ uniformly in } h. \tag{H2}$$

Here we equip $\mathcal{D}_\cdot$ with the norm $\|(P_\cdot + i)u\|_{\mathcal{H}_\cdot}$, and H^2 with the norm $\|\langle hD\rangle^2 u\|_{L^2}$, where $\langle \xi \rangle = \sqrt{1+\xi^2}$. The precise meaning of the uniformity requirement is then that the restriction operator in (H2) should be uniformly bounded $\mathcal{D}_\cdot \to H^2$ and have a uniformly bounded right inverse.

Further, assume:

$$1_{B(0,R_0)}(P_\cdot + i)^{-1} \text{ is compact,} \tag{H3}$$

$$\begin{gathered} 1_{\mathbf{R}^n\setminus B(0,R_0)}P_\cdot u = Q_\cdot u = \sum_{|\alpha|\leq 2} a_{\cdot,\alpha}(x;h)(hD)^\alpha u, \text{ where} \\ a_{\cdot,\alpha}(x;h) \text{ is independent of } h \text{ for } |\alpha| = 2,\ a_{\cdot,\alpha} \in C_b^\infty(\mathbf{R}^n) \text{ uniformly in } h. \end{gathered} \tag{H4}$$

Here we use standard multi-index notation and $C_b^\infty(\mathbf{R}^n)$ denotes the space of C^∞-functions which are bounded with all thier derivatives. Further assume,

$$\begin{gathered} \sum_{|\alpha|=2} a_{\cdot,\alpha}(x)\xi^\alpha \geq \frac{1}{C}|\xi|^2, \\ \sum a_{\cdot,\alpha}(x;h)\xi^\alpha \to \xi^2, \text{ uniformly in } h \text{ when } |x| \to \infty. \end{gathered} \tag{H5}$$

$$|[a_{\cdot,\alpha}(x;h)]_0^1| \leq C\langle x\rangle^{-\widetilde{n}}, \text{ where } \widetilde{n} > n. \tag{H6}$$

Here we use the notation $[a_\cdot]_0^1 = a_1 - a_0$, $\langle x \rangle = \sqrt{1+|x|^2}$.

Let T be a torus of the form $\mathbf{R}^n/(R\mathbf{Z}^n)$ with $R >> R_0$, and view $B(0,R_0)$ as a subset of T in the natural way. Define the Hilbert spaces $\mathcal{H}^\sharp_\cdot$ as the orthogonal sums $\mathcal{H}_{\cdot,R_0} \oplus L^2(T\setminus B(0,R_0))$. There is then a natural but non-unique way of defining self-adjoint reference operators $P^\sharp_\cdot : \mathcal{H}^\sharp_\cdot \to \mathcal{H}^\sharp_\cdot$ which satisfy the natural modifications of (H2-4) and the first part of (H5) and which "coincide with $P_\cdot$ near $\overline{B(0,R_0)}$" (in a precise sense as in [SZ], [S]). We assume

$$\#(\sigma(P^\sharp_\cdot)\cap[-\lambda^2,\lambda^2]) = \mathcal{O}((\lambda/h)^{n_\cdot}),\ \lambda\geq 1,\ \text{where } n_\cdot \geq n. \tag{H7}$$

Here the $n_\cdot$ do not depend on the choice of the $P^\sharp_\cdot$ above. Put $n_{\max} = \max(n_0,n_1)$. The following result is proved in [S]:

Proposition 1.1. *Let $\frac{2m}{n_{\max}} > 1$ and let $f \in C^\infty(\mathbf{R})$ be independent h with $\partial^k f(E) = \mathcal{O}(\langle E\rangle^{-m-k})$ for every $k\in\mathbf{N}$. Let $\chi \in C_0^\infty(\mathbf{R}^n)$ be equal to 1 near $\overline{B(0,R_0)}$ and view χ, $1-\chi$ as multiplication operators in $\mathcal{H}_\cdot$ or in L^2 depending on the context. Then $\chi f(P_\cdot)$, $f(P_\cdot)\chi$, $[(1-\chi)f(P_\cdot)(1-\chi)]_0^1$ are of trace class and*

$$\text{"tr}\,[f(P_\cdot)]_0^1\text{"} = [\text{tr}(\chi f(P_\cdot)\chi + (1-\chi)f(P_\cdot)\chi + \chi f(P_\cdot)(1-\chi))]_0^1 + \text{tr}([(1-\chi)f(P_\cdot)(1-\chi)]_0^1)$$

is independent of χ and $= \mathcal{O}(h^{-n_{\max}})$.

We now introduce dilation analyticity and make the assumption:

$$\begin{aligned}&\exists\theta_0\in]0,\pi[,\ \epsilon>0,\ R\geq R_0,\ \text{such that the coefficients } a_{\cdot,\alpha}(x;h) \text{ extend}\\ &\text{holomorphically in } x \text{ to } \{r\omega;\ \omega\in\mathbf{C}^n,\ \text{dist}(\omega,S^{n-1})<\epsilon,\ r\in\mathbf{C},\ |r|>R,\\ &\arg r\in[-\epsilon,\theta_0+\epsilon]\} \text{ and (H6), (H4) and the second half of (H5) extend to this set.}\end{aligned} \tag{H8}$$

Realizing $P_\cdot$ on a suitable contour Γ_{θ_0} as in [SZ], [S] which coincides with $e^{i\theta_0}\mathbf{R}^n$ near infinity in $\mathbf{C}^n$, we can define the resonances of $P_\cdot$ as the eigenvalues in $e^{i]-2\theta_0,0]}]0,+\infty[$. They form a discrete set $\text{Res}\,(P_\cdot)$, where the elements are counted with their natural multiplicity. Let $0<\epsilon_0<2\pi-2\theta_0$ and let $W \subset\subset \Omega \subset\subset e^{i]-2\theta_0,\epsilon_0]}]0,+\infty[$ be relatively open, independent of h and assume that Ω is simply connected. Let $I_\pm = W\cap\mathbf{R}_\pm$, $J_\pm = \Omega\cap\mathbf{R}_\pm$ and assume that I_+, J_+ are intervals. Let W_-,Ω_- be the intersections of W,Ω with $e^{i]-2\theta_0,0]}]0,+\infty[$. The main result of [S] is:

Theorem 1.1. *Let $f=f(z;h)$ be holomorphic in Ω with $|f(z;h)|\leq 1$, for $z\in\Omega\setminus W$. Let $\chi_+\in C_0^\infty(J_+)$ be equal to 1 near $\overline{I}_+$ and independent of h. Then*

$$\text{"tr}\,[(\chi_+ f)(P_\cdot;h)]_0^1\text{"} = [\sum_{\lambda\in\text{Res}\,(P_\cdot)\cap W_-} f(\lambda;h)]_0^1 - [\sum_{\mu\in\sigma(P_\cdot)\cap I_-} f(\mu;h)]_0^1 = \mathcal{O}(h^{-n_{\max}}).$$

Here $\sigma(P_\cdot)$ denotes the spectrum of the (undistorted) operator $P_\cdot$, known to be discrete on $\mathbf{R}_-$.

Remark 1.2. It is well-known that $\#\sigma(P_\cdot)\cap J_- = \mathcal{O}(h^{-n_\cdot})$ and the proof of the trace theorem above gives the essentially well-known fact that $\#\text{Res}\,(P_\cdot)\cap\Omega_- = \mathcal{O}(h^{-n_\cdot})$. This means that the remainder appears to be quite optimal in general.

Remark 1.3. If $P_\cdot$ are independent of h and satisfy all the assumptions with $h=1$, then $h^2 P_\cdot$ satisfy all the assumptions of the theorem, which then gives a trace formula with remainder for resonances which tend to infinity in a sector.

In [S] we used the trace theorem above to deduce a Poisson formula with remainder, valid in all dimensions. This Poisson formula was then used as in [SZ2], to extend the lower bounds there on the resonance density in logarithmic neighborhoods of the reals, to the case of even dimension. The passage over a Poisson formula was for convenience only, since we could then use the arguments of [SZ2], we suspect strongly that a direct use of the trace theorem could have given the same results quite as easily.

2. Application to semiclassical Schrödinger operators.

We shall establish the existence of $\sim h^{-n}$ resonances in specific bounded subsets of the complex plane for semiclassical Schrödinger operators $-h^2\Delta + V(x)$ under very weak assumptions. The method will consist in combining the traceformula with a semi-classical trace formula of D.Robert [R] and with the existence of analytic singularities in certain measures, related to phase space volumes. We start by discussing such measures:

Let V_0, V_1 be continuous real valued functions on $\mathbf{R}^n$ which tend to 0 at infinity and which satisfy:

$$\exists \widetilde{n} > n;\ |V_1(x) - V_0(x)| \leq C\langle x\rangle^{-\widetilde{n}}. \tag{2.1}$$

For $E>0$, let $\nu_{+,\cdot}(E) = \int_{V_\cdot(x)\geq E} dx$, $\mu_{+,\cdot}(E) = -\frac{d}{dE}\nu_{+,\cdot}(E)$ (in the sense of distributions). Notice that $\nu_{+,\cdot}(E)$ is a decreaing function and consequently $\mu_{+,\cdot}$ is a positive measure (of locally finite mass) on $\mathbf{R}_+ =]0,+\infty[$ with support in $]0, \sup V_\cdot]$. Similarly for $E<0$, we put $\nu_{-,\cdot}(E) = \int_{V_\cdot(x)\leq E} dx$, $\mu_{-,\cdot}(E) = \frac{d}{dE}\nu_{-,\cdot}(E)$ so that $\nu_{-,\cdot}$ is an increasing function and $\mu_{-,\cdot}$ a positive measure with support in $[\inf V_\cdot, 0[$.

For $\phi \in C_0^\infty(\mathbf{R})$, we put

$$\langle \mu, \phi\rangle = \int [\phi \circ V_\cdot(x)]_0^1 dx = \int (\phi(V_1(x)) - \phi(V_0(x)))dx.$$

We have

$$|\langle \mu, \phi\rangle| \leq (\sup|\phi'|) \int |V_1(x) - V_0(x)| dx,$$

so μ is a distribution on $\mathbf{R}$ of order ≤ 1, with

$$\operatorname{supp} \mu \subset [\min_{j=0,1} \inf V_j(x), \max_{j=0,1} \sup V_j(x)],$$

$$\int \mu(E)dE = 0.$$

(Here the integral is to be interpreted in the sense of distribution theory as $\langle \mu, 1\rangle$.) We observe that

$$\mu_{|\mathbf{R}_\pm} = [\mu_{\pm,\cdot}]_0^1.$$

For $f \in C_0^\infty(\mathbf{R})$, we put:

$$\langle \omega, f \rangle = \iint [f(\xi^2 + V_\cdot(x))]_0^1 dx d\xi.$$

The integrand has uniformly compact support in ξ: If $\operatorname{supp} f \subset [a,b]$, $V_\cdot(x) \geq -c$, the support of the integrand is contained in the set $|\xi| \leq \sqrt{b+c}$. It follows that

$$|\langle \omega, f \rangle| \leq \sup |f'| \operatorname{vol} B_{\mathbf{R}^n}(0, \sqrt{b+c}) \int |V_1(x) - V_0(x)| dx,$$

so ω is a distribution on $\mathbf{R}$ of order ≤ 1 with support in $[\min \inf V_\cdot(x), +\infty[$.

For $R \geq 1$, let $V_\cdot^R(x) = 1_{B(0,R)}(x) V_\cdot(x)$ and define $\nu_{\pm,\cdot}^R$, $\mu_{\pm,\cdot}^R$, μ^R, ω^R as above but with $V_\cdot$ replaced by $V_\cdot^R$. Clearly $\nu_{\pm,\cdot}^R \to \nu_{\pm,\cdot}$ locally uniformly, when $R \to +\infty$, and $\mu_{\pm,\cdot}^R \to \mu_{\pm,\cdot}$, $\mu^R \to \mu$, $\omega^R \to \omega$ in the sense of distributions. We have

$$\langle \omega^R, f \rangle = [\iint_{|x| \leq R} f(\xi^2 + V_\cdot(x)) dx d\xi]_0^1 = \int f(E) d\rho_R(E),$$

so that $\omega^R = \frac{d}{dE}\rho_R$, where

$$\rho_R(E) = [\iint_{|x| \leq R,\, \xi^2 + V_\cdot(x) \leq E} dx d\xi]_0^1 = \operatorname{vol} B_{\mathbf{R}^n}(0,1) \int (E-t)_+^{n/2} d\nu_R(t), \tag{2.2}$$

where

$$\nu_R(t) = -[\int_{V_\cdot(x) \geq t,\, |x| \leq R} dx]_0^1.$$

It is clear that

$$\int \phi(t) d\nu_R(t) = \int_{|x| \leq R} [\phi(V_\cdot(x))]_0^1 dx = \int \phi(E) \mu^R(E) dE,$$

so $d\nu_R = \mu^R dE$.

Rewrite (2.2) as a convolution

$$\rho_R = \operatorname{vol} B_{\mathbf{R}^n}(0,1) (\cdot)_+^{n/2} * \mu_R,$$

differentiate w.r.t. E:

$$\omega^R = C_n (\cdot)_+^{\frac{n}{2}-1} * \mu_R, \; C_n = \frac{n}{2} \operatorname{vol} B_{\mathbf{R}^n}(0,1),$$

and let $R \to +\infty$:

$$\omega = C_n (\cdot)_+^{\frac{n}{2}-1} * \mu. \tag{2.3}$$

In general, if $\alpha > -1$, and if $H_\alpha(x) = H(x) x^\alpha$, where $H(x) = 1_{\mathbf{R}_+}(x)$, then $\widehat{H}_\alpha(\xi) = C_\alpha (\xi - i0)^{-1-\alpha}$, where $C_\alpha \neq 0$. Let $E_\alpha \in \mathcal{S}'(\mathbf{R})$ be the inverse Fourier transform of

$\widehat{E}_\alpha = \frac{1}{C_\alpha}(\xi - i0)^{1+\alpha}$. Then we have $\operatorname{supp} E_\alpha \subset [0, +\infty[$, $H_\alpha * E_\alpha = E_\alpha * H_\alpha = \delta$. If $1 + \alpha \in \mathbf{N}$, then E_α is a constant times a derivative of δ, and in general, we know that $E_\alpha(x) = \widetilde{C}_\alpha x^{-2-\alpha}$, for $x > 0$. We can now invert (2.3) and get,

$$\mu = \frac{1}{C_n} E_{\frac{n}{2}-1} * \omega. \tag{2.4}$$

If near some point $t_0 \in \mathbf{R}$, we know that μ extends to a holomorphic function, then the same holds for ω and vice versa. Since μ, ω are real,

$$\mathrm{WF}_a(\mu) = \mathrm{WF}_a(\omega),$$

where WF_a denotes the analytic wavefront set.

Let us consider the extension of $\omega(E)$ from $]\max\sup V_\cdot(x), +\infty[$ to the complex domain $\mathbf{C}\backslash] - \infty, \max\sup V_\cdot(x)]$, given by

$$\omega_+(E) = C_n \int (E - t)^{\frac{n}{2}-1} \mu(t) dt,$$

where we take the branch of the square root which is positive on $]0, \infty[$. Consider two cases depending on the parity of n:

Case 1: n is even ≥ 2. Then $\frac{n}{2} - 1 \in \mathbf{N}$ and $\omega_+(E)$ is a polynomial of degree $\leq \frac{n}{2} - 1$.

Case 2: n is odd. If $E_0 \in \mathbf{R}$, then when $E \to E_0$, $\Im E < 0$, we see that the factor $(E-t)^{\frac{n}{2}-1}$ converges to $(E_0 - t)^{\frac{n}{2}-1}$ when $t < E_0$, and to $i^{2-n}|E_0 - t|^{\frac{n}{2}-1}$, when $t > E_0$. When E converges to the real axis through the lower halfplane, we therefore get the limit (in the sense of distributions):

$$\omega_+(E - i0) = C_n((\cdot - i0)^{\frac{n}{2}-1} * \mu)(E) = \omega(E) + i^{2-n}(1_{]-\infty,0]}|\cdot|^{\frac{n}{2}-1}) * \mu(E),\ E < 0.$$

In both cases, we see that,

$$\mathrm{WF}_a(\omega_+(E - i0) - \omega(E)) = \mathrm{WF}_a(\mu).$$

We now discuss resonances close to analytic singularities of μ on the open positive half-axis. Let $P_j = -h^2\Delta + V_j(x)$, $j = 0, 1$, $V_j \in C^\infty(\mathbf{R}^n; \mathbf{R})$ satisfy the general assumptions for our trace formula. Recall that the assumption (2.1) follows from those general assumptions, so we can define μ as above. Then we have:

Theorem 2.1. *Let $0 < E_0 \in \operatorname{sing\,supp}_a(\mu)$. Then for every complex neighborhood W of E_0, there exists $h_0 = h_0(W) > 0$, and $C = C(W)$, such that when $0 < h < h_0$: $\sum_0^1 \#(\operatorname{Res}(P_j) \cap V) \geq \frac{1}{C(W)} h^{-n}$.*

Here $\operatorname{sing\,supp}_a$ denotes the analytic singular support and $\operatorname{Res}(P_j)$ denotes the set of resonances of P_j in $e^{i]-2\theta_0,0]}]0, +\infty[$, where θ_0 is the dilation angle that appears in the general assumptions of the trace formula. The proof of the theorem will give a slightly more precise conclusion, which may be of independent interest.

Corollary 2.2 *Let* $P_1 = -h^2\Delta + V_1(x)$, $V_1 \in C^\infty(\mathbf{R}^n;\mathbf{R})$ *satisfy all the general assumptions that one of the operators should satisfy for the trace formula. Let* $0 < E_0 \in \operatorname{sing\,supp}_a(\nu_{+,1})$. *Then for every complex neighborhood* W *of* E_0, *there exist* $h_0 = h_0(W) > 0$, $C = C(W) > 0$, *such that for* $0 < h \le h_0(W)$: $\#(\mathrm{Res}\,(P_1)\cap W) \ge \frac{1}{C(W)}h^{-n}$.
Proof of the Corollary. It suffices to construct $P_0 = -h^2\Delta + V_0(x)$, $V_0 \in C^\infty(\mathbf{R}^n;\mathbf{R})$ so that (P_1, P_0) satisfies the general assumptions for the trace formula and such that in addition

$$\sup\operatorname{supp}\nu_{+,0} < E_0, \tag{2.5}$$

$$\begin{aligned}&\exists \text{ a complex neighborhood } W_0 \text{ of } E_0 \text{ such that}\\ &\mathrm{Res}\,(P_0)\cap W_0 = \emptyset \text{ when } h>0 \text{ is small enough.}\end{aligned} \tag{2.6}$$

We shall produce V_0 from V_1 by a cutoff and regularization. Put $K(x) = C_n e^{-x^2/2}$, with $C_n > 0$ chosen so that the integral of K over $\mathbf{R}^n$ is equal to 1. Put $K_\lambda(x) = \lambda^{-n}K(\lambda^{-1}x)$, for $\lambda > 0$. We make an x-dependent choice of λ; $\lambda(R,x) = R\langle R^{-1}x\rangle^{-N_0}$, where $R \ge 1$ is a large parameter and N_0 is fixed, but sufficiently large depending on the dimension n. Put $K_R(x,y) = K_{\lambda(R,x)}(x-y)$. Let $\chi \in C_0^\infty(B(0,2);[0,1])$ be equal to 1 for $|x| < 1$. For R large enough, put

$$V_0(x) = \int K_R(x,y)(1-\chi(R^{-1}x))V_1(y)dy.$$

If $\theta > 0$ is small enough independently of R, we check that in the domain $|\Im x| < \theta\langle\Re x\rangle$, $V_0(x)$ is holomorphic and satisfies

$$|V_0(x)| \le \epsilon(R),$$

$$|V_1(x) - V_0(x)| \le C(R)\langle x\rangle^{-n-1}$$

We clearly have (2.5) and using that the resonances near E_0 can be viewed as the eigenvalues of $P_{0|e^{i\theta/2}\mathbf{R}^n}$, we see that (2.6) holds. #

Notice that if $V_1(x) > 0$ for some real x, then the Corollary can be applied with $E_0 = \sup V_1(x)$.
Proof of Theorem 2.1. We have previously seen that $\mathrm{WF}_a(\mu) = \mathrm{WF}_a(\omega)$ and since μ and ω are real, it is clear that $(E_0,1)$, $(E_0,-1) \in \mathrm{WF}_a(\omega)$. Considering the definition of $\mathrm{WF}_a(\omega)$ by means of the FBI-transform, we see that there are sequences $(\alpha_j,\beta_j) \to (E_0,1)$ in $\mathbf{R}^2$, $\lambda_j \to +\infty$, $\epsilon_j \to 0$, such that

$$\left|\int e^{i\lambda_j(\beta_j(\alpha_j-E)+\frac{i}{2}(\alpha_j-E)^2)}\chi(E)\omega(E)dE\right| \ge e^{-\epsilon_j\lambda_j}. \tag{2.7}$$

Here $\chi \in C_0^\infty(\mathbf{R}_+)$ is equal to 1 near E_0 and has its support in a small neighborhood of E_0. Let $a, b, a/b$ be small and positive and let

$$\Omega =]E_0 - b, E_0 + b[+i] - a, a],\ \ W =]E_0 - b/2, E_0 + b/2[+i] - a/2, a].$$

Let I, J be the intersections of W, Ω respectively with the real axis, and choose $\chi \in C_0^\infty(J)$ equal to 1 on I. Let

$$f_j(E) = e^{i\lambda_j(\beta_j(\alpha_j - E) + \frac{i}{2}(\alpha_j - E)^2)}.$$

Then $|f_{j|\Omega\setminus W}| \leq e^{-\frac{1}{C_0}\lambda_j}$, and (2.7) reads:

$$|\int (f_j\chi)(E)\omega(E)dE| \geq e^{-\epsilon_j\lambda_j}.$$

The trace formula gives

$$\mathrm{tr}\,[(f_j\chi)(P_\cdot)]_0^1 = [\sum_{\mathrm{Res}(P_\cdot)\cap W_-} f(\lambda)]_0^1 + \mathcal{O}(1)h^{-n}e^{-\frac{1}{C_0}\lambda_j}. \tag{2.8}$$

On the other hand, by a traceformula of Robert [R], that we can apply under our slightly different hypothesies (and we sketch a proof of that at the end of this section), we have

$$\mathrm{tr}\,[(f_j\chi)(P_\cdot)]_0^1 = \frac{1}{(2\pi h)^n}\int (f_j\chi)(E)\omega(E)dE + \mathcal{O}_j(h^{1-n}). \tag{2.9}$$

Combining (2.8,9) we get

$$[\sum_{\lambda\in\mathrm{Res}(P_\cdot)\cap W_-} f_j(\lambda)]_0^1 = \frac{1}{(2\pi h)^n}\int (f_j\chi)(E)\omega(E)dE + \mathcal{O}(h^{-n})e^{-\frac{1}{C_0}\lambda_j} + \mathcal{O}_j(h^{1-n}),$$

so

$$|[\sum_{\lambda\in\mathrm{Res}(P_\cdot)\cap W_-} f_j(\lambda)]_0^1| \geq \frac{(e^{-\epsilon_j\lambda_j} - \mathcal{O}(1)e^{-\frac{1}{C_0}\lambda_j})}{(2\pi h)^n} + \mathcal{O}_j(h^{1-n}).$$

Here we first fix j large enough, and then let h be small enough, and conclude that,

$$|[\sum_{\lambda\in\mathrm{Res}(P_\cdot)\cap W_-} f(\lambda)]_0^1| \geq \frac{1}{Ch^n},$$

where $f = f_j$ (with j fixed) is independent of h. From this we get the required lower bound on the number of resonances in W_- and since we can choose W as small as we like, we get the theorem. #

We next consider resonances generated by analytic singularities on the negative axis.

Theorem 2.3. *We make the same assumptions as prior to Theorem 2.1, and assume that the angle of scaling θ_0 is $> \frac{\pi}{2}$. Let $0 > E_1 \in \mathrm{sing\,supp}_a(\mu)$. Let $\gamma : [0,1] \to \mathbf{C}$ be a C^1 curve with $E_0 =_{\mathrm{def}} \gamma(0) \in]\sup\mathrm{supp}\,\mu, +\infty[$, $\gamma(1) = E_1$, $\Im\gamma(t) < 0$ for $0 < t < 1$. Also assume that γ is injective and $\gamma'(t) \neq 0$, $\forall t$. Then for every neighborhood W of $\gamma([0,1])$, there exist $C(W)$, $h_0(W) > 0$, such that*

$$\sum_0^1 \#((\mathrm{Res}(P_\cdot) \cap W) \geq \frac{1}{C(W)}h^{-n},\ 0 < h \leq h_0(W).$$

We do not know if the corresponding analogue of Corollary 2.2 is valid with the same degree of generality. If V_1 is sufficiently short range, the assumptions of Theorem 2.3 will be satisfied with $V_0 = 0$, and we then have an analogue of Corollary 2.2, that we leave to the reader to formulate.

Proof of Theorem 2.3. We first need some preparations in order to solve a $\overline{\partial}$-problem, and which will affect the choices of some domains. Let $\Omega_0 \subset \arg^{-1}(]-\pi, \epsilon_0])$ be a small simply connected relatively open neighborhood of $\gamma([0,1[)$, which near E_1 coincides with $J_- + i]-\alpha, 0[$ for some small open interval J_- containing E_1. We also choose Ω_0 so that it satisfies the geometric assumptions for "Ω" in the trace formula. Let J_+ be the open interval of intersection with $\mathbf{R}_+$. Let $\Omega_\epsilon = \Omega_0 \cup (J_- + i[0, \epsilon[)$. We also arrange so that Ω_0 has smooth boundary except at the corners at the end points of J_-.

Let K be a compact subset of $\Omega_0 \cap \Im^{-1}(]-\infty, 0[)$. Let $f_\epsilon \geq 0$ be a continuous function on $\partial\Omega_\epsilon$ which is $= 0$ everywhere except on some subinterval of $\Omega_0 \cap \arg^{-1}(\epsilon_0)$ (assumed to be non-empty by construction), where f_ϵ is non-vanishing and independent of ϵ. Let $u_\epsilon \geq 0$ be the harmonic function on Ω_ϵ, with boundary value f_ϵ. Then $u_\epsilon > 0$ in the interior of Ω_ϵ (where we consider Ω_ϵ as a subset of $\mathbf{C}$). In particular $\inf_K u_\epsilon > 0$.

Let $R_\epsilon = J_- + i[0, \epsilon[$. If $\epsilon > 0$ is small enough, we have

$$\sup_{R_\epsilon} u_\epsilon < \inf_K u_\epsilon - \frac{1}{C_0},$$

for some $C_0 > 0$ which is independent of ϵ. Consider

$$v_\epsilon = u_\epsilon - \sup_{R_\epsilon} u_\epsilon - \frac{1}{2C_0},$$

so that v_ϵ is harmonic on Ω_ϵ and

$$\sup_{R_\epsilon} v_\epsilon \leq -\frac{1}{2C_0}, \quad \inf_K v_\epsilon \geq \frac{1}{2C_0}.$$

We take $W = W_\epsilon \subset\subset \Omega_\epsilon$ as in the trace formula, relatively open with the property that

$$v_\epsilon \leq -\frac{1}{2C_0} \text{ on } \Omega \setminus W, \ \gamma([0,1]) \subset W.$$

We may also arrange so that $W \cap]-\infty, 0[= I_-$, $W \cap]0, +\infty[= I_+$ are intervals. As we saw in the beginning of this section, we have $(E_1, 1)$, $(E_1, -1) \in \mathrm{WF}_a(\omega_-)$, where we put

$$\omega_-(E) = \omega(E) - \omega_+(E - i0).$$

Consequently there are sequences $(\alpha_j, \beta_j) \to (E_1, -1)$ in $\mathbf{R}^2$, $\epsilon_j \to 0$, $\lambda_j \to +\infty$, such that

$$|\int e^{i\lambda_j((\alpha_j - E)\beta_j + \frac{i}{2}(\alpha_j - E)^2)} \chi_-(E)\omega_-(E) dE| \geq e^{-\epsilon_j \lambda_j}. \tag{2.10}$$

Here we let $\chi_- \in C_0^\infty(J_-)$ be equal to 1 near $\overline{I}_-$.

Consider

$$\widetilde{f}_j(E)=e^{i\lambda_j((\alpha_j-E)\beta_j+\frac{i}{2}(\alpha_j-E)^2)}$$

as a function on Ω_0 and on Ω_ϵ. This function has to be modified before applying our trace formula. Let $\psi\in C^\infty(\mathbf{R};[0,1])$ have its support in $]-2,\infty[$ and be equal to 1 on $[-1,\infty[$. Consider the function $\widehat{f}_j(E)$ on Ω_0 with support in some fixed neighborhood of J_-, given there by

$$\widehat{f}_j(E)=\widetilde{f}_j(E)\psi(\frac{1}{\delta}\Im E).$$

Then $\frac{\partial}{\partial\overline{E}}\widehat{f}_j$ has its support in

$$\{E\in\Omega_0;\ -2\delta\leq\Im E\leq-\delta,\ E \text{ in the fixed neighborhood of } J_-\} \tag{2.11}$$

and we can split:

$$\frac{\partial\widehat{f}_j}{\partial\overline{E}}=r_{j,1}+r_{j,2},$$

where $r_{j,1}$, $r_{j,2}$ have their support in the same set, moreover $\operatorname{supp} r_{j,1}\subset K$, where $K\subset\subset\Omega_0$ is a subset of the set (2.11), and

$$|r_{j,1}|\leq e^{C_0\lambda_j},\ r_{j,2}\leq e^{-\frac{1}{C_0}\lambda_j}.$$

Having now fixed K, we choose $\Omega=\Omega_\epsilon$ as above, and solve

$$\frac{\partial}{\partial\overline{E}}h_{j,2}=r_{j,2}$$

with $h_{j,2}=\mathcal{O}(e^{-\frac{1}{C_0}\lambda_j})$ in Ω. Using the harmonic function $v=v_\epsilon$ above, we can solve

$$\frac{\partial}{\partial\overline{E}}h_{j,1}=r_{j,1},$$

with $h_{j,1}=\mathcal{O}(e^{C_1\lambda_j v})$.

Put $f_j=\widehat{f}_j-h_{j,1}-h_{j,2}$. Then f_j is holomorphic on Ω and $f_j=\mathcal{O}(e^{-\frac{1}{C_1}\lambda_j})$ in $\Omega\setminus W$ (with $W=W_\epsilon$ chosen as above). Moreover $f_j-\widetilde{f}_j=\mathcal{O}(e^{-\frac{1}{C_1}\lambda_j})$ in a fixed neighborhood of J_-. With a possibly new sequence ϵ_j, we get from (2.10) and the previous estimates:

$$|\int f_j(E)\chi_-(E)\omega_-(E)dE|\geq e^{-\epsilon_j\lambda_j}. \tag{2.12}$$

We shall now apply the trace formula. Let $\chi_+\in C_0^\infty(J_+)$ be equal to 1 near $\overline{I}_+$. Choose $\chi\in C_0^\infty(\Omega)$ with $\chi=1$ near W and with $\chi_{|J_\pm}=\chi_\pm$. Then we get

$$\operatorname{tr}[(\chi_+f_j)(P_\cdot)]_0^1=[\sum_{\lambda\in\operatorname{Res}P_\cdot\cap W_-}f_j(\lambda)]_0^1-[\sum_{\mu\in\sigma(P_\cdot)}(\chi_-f_j)(\mu)]_0^1+\mathcal{O}(h^{-n})\sup_{\Omega\setminus W_-}|f_j|,$$

which can be written

$$[\sum_{\lambda\in \mathrm{Res}\, P_.\cap W_-} f_j(\lambda)]_0^1 = \mathrm{tr}\,[(\chi_+ f_j)(P_.)]_0^1 + \mathrm{tr}\,[(\chi_- f_j)(P_.)]_0^1 + \mathcal{O}(h^{-n})e^{-\frac{1}{C_1}\lambda_j}. \qquad (2.13)$$

Now by the same semi-classical trace formula as before:

$$\mathrm{tr}\,[(\chi_\pm f_j)(P_.)]_0^1 = \frac{1}{(2\pi h)^n}\int (\chi_\pm f_j)(E)\omega(E)dE + \mathcal{O}_j(h^{1-n}).$$

We choose χ almost analytic at J_+ and J_-, use Stokes' formula and get

$$\int (\chi_+ f_j)(E)\omega(E)dE = -\int (\chi_- f_j)(E)\omega_+(E-i0)dE + \mathcal{O}(e^{-\frac{1}{C_1}\lambda_j}),$$

where the last term corresponds to an integral over $(\Omega\backslash W)\cap\Im^{-1}(]-\infty,0[)$ whose integrand contains $\frac{\partial}{\partial\overline{E}}\chi$. Recalling that $\omega_-(E)=\omega(E)-\omega_+(E-i0)$, we get:

$$[\sum_{\lambda\in \mathrm{Res}\, P_.\cap W_-} f_j(\lambda)]_0^1 = \frac{1}{(2\pi h)^n}\int (\chi_- f_j)(E)\omega_-(E)dE + \mathcal{O}(h^{-n})e^{-\frac{1}{C_1}\lambda_j} + \mathcal{O}_j(h^{1-n}).$$

Combining this with (2.12), we get

$$|[\sum_{\lambda\in \mathrm{Res}\, P_.\cap W_-} f_j(\lambda)]_0^1| \geq \frac{e^{-\epsilon_j\lambda_j}}{(2\pi h)^n} - \mathcal{O}(1)e^{-\frac{1}{C_1}\lambda_j}h^{-n} - \mathcal{O}_j(h^{1-n}),$$

and we can conclude as in the proof of Theorem 2.1. #

We end by outlining a proof of the semiclassical trace formula ([R]) under the slightly different assumptions that we need, following ideas of Dimassi [D] (see also [DS]). Let $P_. = -h^2\Delta + V_.$ with $V_. \in C_b^\infty(\mathbf{R}^n;\mathbf{R})$, $V_.(x)\to 0$ when $|x|\to\infty$, and assume (2.1). We shall then discuss a proof of the fact that for every $f\in C_0^\infty(\mathbf{R})$, independent of h:

$$\mathrm{tr}\,[f(P_.)]_0^1 = \frac{1}{(2\pi h)^n}\iint [f(p_.(x,\xi))]_0^1 dx d\xi + \mathcal{O}(h^{1-n}). \qquad (2.14)$$

Let $\chi\in C_0^\infty(\mathbf{R};[0,\infty[)$ be conveniently chosen, so that the infimum of the spectrum of

$$\widetilde{P}_. = -h^2\Delta + V_.(x) + \chi(hD)$$

is larger than $1+\sup\mathrm{supp}\, f$. Combining the resolvent identity,

$$(z-P_.)^{-1} = (z-\widetilde{P}_.)^{-1} + (z-\widetilde{P}_.)^{-1}\chi(hD)(z-P_.)^{-1}$$

with the operator Cauchy-Riemann-Green-Stokes formula [HS2], we get:

$$[f(P_.)]_0^1 = -\frac{1}{\pi}\int \frac{\partial\widetilde{f}(z)}{\partial\overline{z}}[(z-\widetilde{P}_.)^{-1}\chi(hD)(z-P_.)^{-1}]_0^1 L(dz), \qquad (2.15)$$

where $\widetilde{f}$ is an almost analytic extension of f with support close to that of f, and where $L(dz) = dxdy$ is the Lebesgue measure on $\mathbf{C}$. It follows from the estimates in the proof of Proposition 2.1 in [S], that the trace class norm of the bracket inside the integral is $\mathcal{O}(h^{-n}|\Im z|^{-N_0})$ for some fixed N_0 in trace class norm. The trace of the integral will therefore change only by a term $\mathcal{O}(h^\infty)$ if we restrict the integration to $|\Im z| \geq h^\delta$ for some fixed $\delta > 0$, that we can choose arbitrarily small. We are then in a region where we can apply symbolic calculus on the bracket in the integral, and if we write this bracket as the sum of two terms containing $[(z - \widetilde{P}.)^{-1}]_0^1$ and $[(z - P.)^{-1}]_0^1$ respectively, we see that the bracket in the integral is an h-pseudor with symbol in the class $S_{2\delta}(h^{-2\delta}\langle\xi\rangle^{-N_0}\langle x\rangle^{-\widetilde{n}})$, where we say that $a(x,\xi;h) \in S_{\widetilde{\delta}}(m(x,\xi;h))$ if $\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h) = \mathcal{O}(h^{-\widetilde{\delta}(|\alpha|+|\beta|)} m(x,\xi;h))$. Moreover we have a complete asymptotic expansion of the symbol of the bracket, and we can then get the leading terms in the asymptotics of the trace.

It is quite likely, that we can find large families of resonances also in the following situation: Let γ be a curve parametrized over $[0,1[$, with values in some Riemann surface, where ω is holomorphic. It is also assumed that $\gamma(0)$ lies on the positive real axis, and that $\gamma(t)$ approaches some limit point $\gamma(1)$ when $t \to 1$ and that ω is not holomorphic near $\gamma(1)$, but on one side of a C^2-curve which passes through $\gamma(1)$. The likely conclusion would then be that for every neighborhood of $\gamma([0,1])$, the corresponding projection will contain $\sim h^{-n}$ resonances when h is small enough.

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