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A sharp result on the poles localization for a Gevrey convex body.

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Abstract

In this talk we extend to Gevrey- s obstacles with $1 \leq s < 3$ a result on the poles free zone due to J. Sjöstrand [8] for the analytic case.

1 Introduction and statements.

Let \mathcal{O} a strictly convex body in \mathbf{R}^n . Let $P = -\Delta$ in $\mathbf{R}^n \setminus \mathcal{O}$, the resolvent

$$(P - \lambda^2)^{-1} : L^2(\mathbf{R}^n \setminus \mathcal{O}) \rightarrow H^2(\mathbf{R}^n \setminus \mathcal{O}) \cap H_0^1(\mathbf{R}^n \setminus \mathcal{O}), \Im \lambda > 0, \quad (1)$$

extends to a meromorphic operator :

$$(P - \lambda^2)^{-1} : L_{\text{compact}}^2(\mathbf{R}^n \setminus \mathcal{O}) \rightarrow H_{\text{loc}}^2(\mathbf{R}^n \setminus \mathcal{O}) \cap H_{0,\text{loc}}^1(\mathbf{R}^n \setminus \mathcal{O}) \quad (2)$$

for $\lambda \in \mathbf{C}$ when n is odd or $\lambda \in \Lambda$, the logarithmic plane when n is even. The scattering poles are defined by the poles of this meromorphic extension.

Many works have been devoted to study the distribution of poles and it is still a subject in fast development. We shall concentrate only on the strictly convex case and in its recent progresses to situate our contribution.

The first (rigorous) results on this subject have started with the microlocal analysis of the mixed hyperbolic problem. In the C^∞ case, as a consequence of Melrose-Sjöstrand works we were able to say that under a logarithmic curve there were only finitely many poles. With the G^3 diffractive propagation of singularities due to Lebeau, it results that when \mathcal{O} is analytic this curve has the form $y = x^{1/3}$, see [1]. There were a lot of difficult microlocal analysis and a simple functional analysis.

Then Melrose and Zworsky proved with a completely different approach that the number of poles in a ball of radius r is an $\mathcal{O}(r^n)$ for large r .

But the behavior of the distributions of poles near the real line is still related to the microlocal analysis than can be developed. The recent works of T. Hargé-G. Lebeau [2] and J. Sjöstrand-M. Zworsky [10] involved much more precise

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estimates. These works featured a specific to strictly convex obstacles, complex scaling method and a lot of refined estimates.

We investigate here the problem : what is the best constant C for which there are finitely many scattering poles in :

$$\{k \in \mathbf{C}; \Re k \geq 1, \Im k \geq -(C - \varepsilon)(\Re k)^{1/3}\} \quad (3)$$

for any $\varepsilon > 0$.

When the body is C^∞ , Hargé-Lebeau [2] gives a constant $C_{0,\infty}$ which controls the poles free region :

$$C_{0,\infty} = 2^{-1/3} \cos(\pi/6) \zeta_1 \inf_{\nu \in S\partial\mathcal{O}} Q(\nu)^{2/3}. \quad (4)$$

Where ζ_1 is the first zero of the Airy function, $Q(\nu)$ is the second fundamental form; which will be non degenerated since the obstacle is strictly convex.

The result was also recovered by [10] and new counting of poles results were obtained.

If the body is analytic J. Sjöstrand [8] refined this result by replacing $C_{0,\infty}$ by $C_{0,a}$, $C_{0,a} \geq C_{0,\infty}$ given by :

$$C_{0,a} = 2^{-1/3} \cos(\pi/6) \zeta_1 \lim_{T \rightarrow \infty} \frac{1}{T} \inf_{\gamma} \int_0^T Q(\dot{\gamma}(t))^{2/3} dt \quad (5)$$

where γ varies in the set of geodesics with speed 1.

On the other hand in [1], the authors proved in the analytic case that a non degenerate, isolated, simple closed geodesic γ generates infinitely many resonances in the set :

$$\{\zeta; -\Im \zeta < B(\Re \zeta)^{1/3}\} \quad B > B_\gamma = 2^{-1/3} \zeta_1 \cos(\pi/6) \frac{1}{d_\gamma} \int_0^{d_\gamma} \rho_\gamma(s)^{2/3} ds \quad (6)$$

where d_γ is the length of γ and ρ_γ is the curvature of γ in \mathbf{R}^n .

The starting point of our work is the remark that the improvement obtained in [8] was due to the addition of microlocal G^3 tangential weight, after the complex scaling.

This was for us a surprise since using only tangential weights we can't obtain the G^3 diffraction.

It is not known if the G^3 diffraction holds for G^3 boundaries. In our approach [6] we got G^s diffraction for an obstacle $G^{s'}$ with $s \geq 2s' + 1$. Moreover the method (inspired by the proofs of V. Ivrii and L. Hörmander in the C^∞ case) was based on L^2 estimates with very irregular p.d.o. resulting by the division by the equation of Gevrey- s weighted Fourier Integral Operators; such operators were still of infinite orders but due to the disappearance of the distinction phase-symbol they could be analyzed only with the Weyl calculus of L. Hörmander [3], the index $s = 3$ was the critical value for which the principle of uncertainty was valid.

Tangential G^s weights cause no problems with a G^s operator calculus. For micro-hyperbolic problems it is possible to work with G^s singularities with G^s equations (see [4] and [5]).

Sjöstrand [8] had also to push forward the TFBI calculus in both the C^∞ and analytic contexts.

We believed then that the result of [8] could be true for some Gevrey obstacles if it were possible to make a Gevrey TFBI calculus with the properties required by [8]. The Gevrey methods of [4] were too rough and insufficient here.

The work [8] uses analytic TFBI, an appropriate complex scaling and L^2 estimates. If we push forward our Gevrey TFBI methods allowing in particular to make Gevrey complex canonical transformations and Töeplitz operators associated with I-Lagrangian submanifolds, we can obtain the result of [8] for Gevrey obstacles.

The main difference of the Gevrey TFBI analysis with respect to the analytic situation is that we must stay much closer to the real and weaken accordingly the imaginary parts of the phase functions.

We can now state our result :

Theorem 1 *Let \mathcal{O} a G^s strictly convex obstacle with and $1 \leq s < 3$. Then for every $\varepsilon > 0$ there are only finitely many resonances in in the region*

$$\{k \in \mathbf{C}; \Re k \geq 1, \Im k \geq -(C_{0,a} - \varepsilon)(\Re k)^{1/3}\}. \quad (7)$$

Remark 1 *We note the following remarks :*

- (i) *Here s can be as close as we wish to 3 at the contrary to the propagation of singularities theorems.*
- (ii) *The limiting case $s = 3$ is still open, for may be technical reasons but difficult to overcome unless we change the G^3 definition in keeping the same $n!^3$ but assuming that the A in A^n can be chosen as small as different technical steps require it. We believe that this is cheating.*
- (iii) *If the methods of [9] are really microlocal enough to give the propagation of singularities that will force us to change our philosophy, but this is far to be clear for us.*
- (iv) *The results of [8] give much more estimates in the counting of poles, they are true here but we don't recall here these results since they really belong to [8].*
- (v) *We don't know if a Gevrey 3 singularity of the boundary is really able to produce infinitely many poles, which could let us guess that the constant $C_{0,a}$ is valid only for G^3 boundaries. But with the microlocal analysis used here, G^3 is the natural limit.*

2 The Gevrey TFBI calculus.

Let us start with some basic definitions. The realizations of the identity were introduced by J. Sjöstrand in various forms and contexts. A Gevrey- s realization of the identity is an operator like :

$$Tu(\alpha) = \int a(\alpha, y, \lambda) e^{i\lambda\Phi(y, \alpha, \lambda)} u(y) dy \quad (8)$$

with $\alpha = (\alpha_x, \alpha_\xi) \in T^*\mathbf{R}^n$, $y \in \mathbf{R}^n$ and

$$\Phi(y, \alpha, \lambda) = \varphi(y, \alpha, \lambda) + i\omega\psi(y, \alpha, \lambda). \quad (9)$$

φ and ψ are two real suitable G^s functions, $\psi \geq 0$, $\omega = \omega_0 \lambda^{-1+1/s}$ with a large parameter $\lambda \geq 1$ and a small parameter ω_0 . In the following we shall admit that $\alpha \in \Lambda_t$ where Λ_t is a smooth family of I-Lagrangian submanifolds of \mathbf{C}^{2n} still remaining close to the real.

P is a Gevrey-s pseudo-differential operator of order (m, k) with symbol p in the λ -Weyl quantization if it is given by :

$$Pu(x, \lambda) = \left(\frac{\lambda}{2\pi}\right)^n \int p\left(\frac{x+y}{2}, \xi, \lambda\right) e^{i\lambda(x-y)\xi} u(y, \lambda) dy. \quad (10)$$

where p is a Boutet de Monvel-Kr ee symbol namely we have the estimates :

$$|D_x^\beta D_\xi^\alpha p(x, \xi, \lambda)| \leq C \lambda^k A^{|\alpha|+|\beta|} < \xi >^{m-|\alpha|} |\alpha|!^s |\beta|!^s. \quad (11)$$

Such symbols have almost analytic extensions in a neighborhood of the reals satisfying :

$$|D_x^\beta D_\xi^\alpha \bar{\partial}_{x,\xi} p(x, \xi, \lambda)| \leq C \lambda^k A^{|\alpha|+|\beta|} < \xi >^{m-|\alpha|} |\alpha|!^s |\beta|!^s \exp(-A_1(|\Im x| + |\Im \xi| / < \Re \xi >)^{-(1/(s-1))}). \quad (12)$$

As a consequence of the moments problem of L. Carleson.

The first step (the easy one) is to conjugate a realization of the identity with a pseudo-differential acting on the right to do this we compose on the left with ${}^t P$ which is analogous to P .

We have the result :

Proposition 2 *For given functions $\Phi(y, \alpha, \lambda)$, $p(x, \xi, \lambda)$, and $\psi \geq 0$ homogeneous of degree $1/s$ in α_ξ . If $d_{T^*\mathbf{R}^n}(\alpha) = |\Im \alpha_x| + |\Im \alpha_\xi| / < \Re \alpha_\xi > \leq c_1(\lambda |\alpha_\xi|)^{-1+1/s}$ with c_1 and ω_0 small enough. Then*

$$P(e^{i\lambda\Phi(\cdot, \alpha, \lambda)} a(\alpha, \cdot, \lambda))(x, \alpha) = e^{i\lambda\Phi(x, \alpha, \lambda)} c(x, \alpha, \lambda) + \mathcal{O}(e^{-c(\lambda |\alpha_\xi|)^{1/s}}). \quad (13)$$

In (13) the \mathcal{O} is taken in $\mathcal{S}(\mathbf{R}_x^n)$, $c(x, \alpha, \lambda)$ is Gevrey-s symbol of order 0.

The proof uses the Kuranishi trick and appropriate deformations of the integral paths. The main point when working in Gevrey to have remainders like in (13), is to stay at a fractional distance to the reals.

Let Λ an I-Lagrangian submanifolds of \mathbf{C}^{2n} , we introduce a new phase function

$$\Phi^*(y, \alpha, \lambda) = \varphi^*(y, \alpha, \lambda) + i\omega\psi^*(y, \alpha, \lambda) \quad (14)$$

which will be adjusted later.

We set $T_\Lambda u = Tu|_\Lambda$ and we introduce :

$$S_\Lambda v(x, \lambda) = \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_\Lambda b(x, \alpha, \lambda) e^{i\lambda\Phi^*(x, \alpha, \lambda)} v(\alpha, \lambda) d\alpha. \quad (15)$$

We need to compute :

$$S_\Lambda T_\Lambda = \sigma(x, D_x/\lambda, \lambda) = \left(\frac{\lambda}{2\pi}\right)^{3n/2} \int_\Lambda \int_\Lambda b(x+u/2, \alpha, \lambda) a(x-u/2, \alpha, \lambda) e^{i\lambda(\Phi^*(x+u/2, \alpha, \lambda) - \Phi^*(x-u/2, \alpha, \lambda) - u\xi)} d\alpha du. \quad (16)$$

The integral (??) will be carried out by the stationary phase method. First we parameterize Λ by $\alpha \in \mathbf{R}^{2n} \rightarrow \alpha(t, x_n, \alpha)$ at this time t and x_n , will just be small parameters and we assume that $\alpha(t, x_n, \alpha) = \alpha + \lambda^{-1+1/s} \mathcal{O}(t)$.

This time contrary to the C^∞ and analytic context the computations are not carried out by a direct application of the stationary phase method, but we can prove :

Proposition 3 *Let M a G^s compact riemaniann manifold. Let $\Phi(x, \alpha, \lambda) = \varphi(x, \alpha, \lambda) + i\omega\psi(x, \alpha, \lambda)$ and $\Phi^*(x, \alpha, \lambda) = \varphi^*(x, \alpha, \lambda) + i\omega\psi^*(x, \alpha, \lambda)$, with :*

$$\begin{aligned}\varphi(x, \alpha, \lambda) &= \exp_{\alpha_x}^{-1} x \cdot \alpha_\xi + (a/2) |\alpha_\xi|_{\alpha_x} d(x, \alpha_x)^2 \\ \varphi^*(x, \alpha, \lambda) &= -\exp_{\alpha_x}^{-1} x \cdot \alpha_\xi + (a'/2) |\alpha_\xi|_{\alpha_x} d(x, \alpha_x)^2 \\ \psi(x, \alpha, \lambda) &= \psi^*(x, \alpha, \lambda) = |\alpha_\xi|_{\alpha_x}^{1/s} d(x, \alpha_x)^2\end{aligned}\quad (17)$$

where a and a' are two real numbers such that $a + a' \neq 0$.

Let T_Λ and S_Λ given (??). If Λ is a G^s I-Lagrangian submanifold of the complexified $\widetilde{T^*M}$ of T^*M such that for each point $\alpha \in \Lambda$, $d_{T^*M}(\alpha) = \mathcal{O}(t(|\alpha_\xi|\lambda)^{-1+1/s})$ for a small parameter t . If t and ω_0 are small enough and λ is large, we may choose the symbols a and b such that

$$S_\Lambda T_\Lambda = I + R_\Lambda. \quad (18)$$

In (18) $R_\Lambda = \mathcal{O}(e^{-c\lambda^{1/s}})$ in $C^\infty(M \times M)$.

Remark 2 *By a continuous deformation of contours if Λ_t is a smooth family of I-Lagrangian submanifolds only $R_{\Lambda,t}$ will change.*

It is important to remark here that $\psi(x, \alpha, \lambda)$ and $\psi^*(x, \alpha, \lambda)$ are too weak to insure a non degenerate critical point at $u_c = 0$, $(x, \xi) = \alpha(t, x_n, \alpha_c)$ in the integral (??), this why the trick to add the parameters a and a' is necessary.

As Λ is I-Lagrangian there is function $H(\alpha)$, unique up to a constant such that

$$dH = -\Im(\alpha_\xi d\alpha_x). \quad (19)$$

The computation of

$$(T_\Lambda S_\Lambda)v(\alpha, \lambda) = \int_\Lambda k(\alpha, \beta, \lambda)v(\beta) d\beta \quad (20)$$

is much more difficult since while $S_\Lambda T_\Lambda$ is a pseudo-differential operator, $T_\Lambda S_\Lambda$ is some sort of twisted Töeplitz operator.

Anyway we can prove :

Proposition 4 *If Λ_t is a a smooth family of I-Lagrangian manifolds of $\widetilde{T^*M}$ such that for each point $\alpha \in \Lambda_t$, $d_{T^*M}(\alpha) = \mathcal{O}(t(|\alpha_\xi|\lambda)^{-1+1/s})$ then*

$$k(\alpha, \beta, \lambda) = e^{i\lambda\Psi(\alpha, \beta, \lambda)} c(\alpha, \beta, \lambda) + \mathcal{O}(e^{-c(\lambda \min(|\alpha_\xi|, |\beta_\xi|)^{1/s})}) \quad (21)$$

where Ψ is a phase function such that $d_\alpha(-\Im\Psi)|_\Lambda = d_\alpha H$, and $d_\beta(-\Im\Psi)|_\Lambda = d_\beta H$ when $\alpha = \beta$.

Proposition 5

$$-H(\alpha) + \Im\Psi(\alpha, \beta) + H(\beta) \sim \lambda^{-1+1/s} |\alpha_\xi|^{1/s} (|\alpha_\xi - \beta_\xi|^2 |\alpha_\xi|^{-2} + |\alpha_x - \beta_x|^2). \quad (22)$$

The result (22) is necessary but at the contrary of the analytic case is non sufficient to prove that $k(\alpha, \beta, \lambda)$ is the kernel of a bounded operator in $L_m^2(\Lambda) = L^2(\Lambda, e^{-2\lambda H(\alpha)} d\alpha)$ uniformly with respect to the parameters λ and $|\alpha_\xi|$. This is true as a consequence of the oscillation of the real part of $\Psi(\alpha, \beta, \lambda)$.

Proposition 6 *If $L_m^2(\Lambda) = L^2(\Lambda, e^{-2\lambda H(\alpha)} |\alpha_\xi|^m d\alpha)$, then $k(\alpha, \beta, \lambda)$ is the kernel of a bounded operator in $L_m^2(\Lambda)$.*

This result in fact follows from perturbations arguments and by an application of the theory of Fourier Integral operators of Melin-Sjöstrand.

We have now completed all the needed modifications of the TFBI calculus to adapt it to a Gevrey situation.

We need to verify the two next results :

Proposition 7 (Proposition 1.1 of [8]) *Let B_Λ be a suitable approximate version of the orthogonal projector from $L_m^2(\Lambda)$ to $L_m^2(\Lambda) \cap \text{Image of } T_\Lambda$. Let $\mathcal{H}_m = \{u, T_\Lambda u \in L_m^2\}$ we have for all $m \in \mathbf{R}$:*

$$(i) \ B_\Lambda^2 = B_\Lambda + R_\Lambda \text{ with an } R_\Lambda = \mathcal{O}(\lambda^{-\infty}) \text{ in } \mathcal{L}(L_{-m}^2, L_m^2)$$

$$(ii) \ \|T_\Lambda u - B_\Lambda T_\Lambda u\|_{L_m^2} \leq \mathcal{O}(\lambda^{-\infty}) \|u\|_{\mathcal{H}_{-m}}$$

We don't comment here this result since the construction of B_Λ and the proof of this proposition involve non trivial but asymptotic in the $\mathcal{O}((\lambda|\alpha_\xi|)^{-\infty})$ category and no analyticity is involved. We have also :

Proposition 8 (Proposition 1.3 of [8]) *Let P_1 and P_2 two Gevrey-s pseudo-differential operators of order m_1 and m_2 then :*

$$(P_1 u, P_2 u)_{\mathcal{H}_0} = (P_1|_\Lambda T_\Lambda u, P_2|_\Lambda T_\Lambda u)_{L_0^2} + \mathcal{O}(\lambda^{-1}) \|u\|_{\mathcal{H}_{\tilde{m}_1}} \|u\|_{\mathcal{H}_{\tilde{m}_2}} \quad (23)$$

if $m_1 + m_2 = \tilde{m}_1 + \tilde{m}_2$.

So, up to an error term as in (23), the computation of $\|Pu\|_{\mathcal{H}_0}$ involves only the multiplication of $T_\Lambda u$ by the restriction to Λ of p , which is like applying a complex canonical transformation to P and we have here a classical property of Töeplitz operators.

3 The link with the localization of poles.

First of all $h = \lambda^{-1}$ and the asymptotic are semi-classical with respect to this small parameter.

It is impossible in the framework of this talk to repeat the quite difficult estimates used in [8] to obtain his count of resonances near the real line. Let us just outline the proof to see where the preceding results are used.

The first step is to use a set geodesic coordinates. Let $z_0 \in \partial\mathcal{O}$ and $x' = (x_1, \dots, x_{n-1})$ so that we have a diffeomorphism :

$$s : \text{neigh.}_{\mathbf{R}^{n-1}}(0) \rightarrow \text{neigh.}_{\partial\mathcal{O}}(z_0) \quad (24)$$

the normal geodesic coordinates will be given by

$$x = s(x') + x_n n(s(x')) = s(x') + x_n \nabla d(s(x')) \quad (25)$$

where d is the distance to $\partial\mathcal{O}$.

Then in these coordinates :

$$\begin{aligned} -h^2\Delta &= (hD_{x_n})^2 + R(x', hD_{x'}) + 2x_n Q(x', hD_{x'}) \\ &+ \mathcal{O}(x_n^2 (hD_{x'})^2) + \mathcal{O}(h)hD_x + \mathcal{O}(h^2) \end{aligned} \quad (26)$$

where $R(x', \xi')$ is the dual of the first fundamental form of $\partial\mathcal{O}$; $Q(x', \xi')$ is the dual of the second fundamental form .

Then the authors of [9] perform a complex scaling by $\gamma(x) = x' + e^{i\pi/3}x_n n(x')$ in the neighborhood of $\partial\mathcal{O}$. If $\Gamma = \gamma(\mathbf{R}^n \setminus \mathcal{O})$. Let P be given by the formula :

$$\begin{aligned} P \stackrel{\text{def}}{=} -h^2\Delta|_{\Gamma} &= e^{-2i\pi/3}((hD_{x_n})^2 + 2x_n Q(x', hD_{x'})) \\ &+ R(x', hD_{x'}) + \mathcal{O}(x_n^2 (hD_{x'})^2) + \mathcal{O}(h)hD_x + \mathcal{O}(h^2), \end{aligned} \quad (27)$$

it was proved in [9] that the poles of the meromorphic extension of $(-\Delta - \lambda^2)^{-1}$ in $0 < -\arg \lambda < \pi/(3C)$ are the square roots of the eigenvalues of $-\Delta|_{\Gamma}$ in $0 < -\arg \lambda < 2\pi/(3C)$ if C is large enough. (It is a remarkable reduction of the problem of counting poles to the problem of counting the eigenvalues of a non self-adjoint operator). They use then the Weyl formula to count the scattering poles. The other remarkable fact is that using a properties of Töeplitz operators the estimates follows from estimates for ordinary differential equations (so ζ_1 appears naturally); a fact which was before a consequence of the constructive method for the G^3 propagation of singularities of G. Lebeau.

Then we perform a TFBI with the family of I-Lagrangian submanifolds $\Lambda_{t(x_n)} = \exp(t(x_n)H_G^{\mathfrak{S}\sigma})(T^*\partial\mathcal{O})$ where G is used to make averaging along long trajectories on the boundary. The effect of this complex stretching is :

$$p(\exp(tH_G^{\mathfrak{S}\sigma}(\rho))) = p(\rho) + t(dp(\rho), H_G^{\mathfrak{S}\sigma}(\rho)) + \mathcal{O}(t^2). \quad (28)$$

If p is real valued we have :

$$p(\exp(tH_G^{\mathfrak{S}\sigma}(\rho))) = p(\rho) - it(H_p^\sigma, dG)(\rho) + \mathcal{O}(t^2). \quad (29)$$

$t = t(x_n)$ will be $h^{2/3}$ for small x_n (e.g. when we are close to the boundary). This is indeed a G^3 complex stretching.

So in [8] the author has to consider the new symbol :

$$\begin{aligned} P(x, \xi', hD_{x_n}; h) &= e^{-2i\pi/3}((hD_{x_n})^2 + 2x_n Q(y', \eta')) + R(y', \eta') \\ &- ih^{2/3}H_R G(y', \eta') + \mathcal{O}(x_n^2 \langle \eta' \rangle^2 + h \langle \eta' \rangle) + \mathcal{O}(h)hD_{x_n} \\ &+ \mathcal{O}(x_n h^{2/3}) + \mathcal{O}(h^{4/3}) \end{aligned} \quad (30)$$

if $(x', \xi') = \exp(t(x_n)H_G^{\mathfrak{S}\sigma})(y', \eta')$. This will lead the author of [8] to make sharp estimates on the lower bounds of :

$$\int_0^\infty \|T_{\Lambda_{t(x_n)}}(P - \omega_0)u\|_{L^2(\Lambda_{t(x_n)}, e^{-2H_{t(x_n)}/h} L(dx' d\xi'))}^2 dx_n \quad (31)$$

with a point $\omega_0 \in \mathbf{C}$.

Roughly speaking the Töeplitz calculus (23) allows in some estimates to replace P in (32) by the $P(x, \xi', hD_{x_n}; h)$ and make estimates for a second order differential equation depending on parameters.

Let :

$$\zeta_j(y', \eta') \stackrel{\text{def}}{=} (2Q(y', \eta'))^{2/3} \zeta_j \quad (32)$$

the scaled zeros of the Airy function, they are modified by the weight G into

$$\tilde{\zeta}_j(y', \eta') = \zeta_j(y', \eta') + \frac{1}{\cos(\pi/6)} H_R G(y', \eta'). \quad (33)$$

In [8], the author used the set Σ of points at the energy level $\Re\omega_0$ defined by :

$$\Sigma = \{(y', \eta'); R(y', \eta') = \Re\omega_0\} \quad (34)$$

He proves then that G can be adjusted such that :

$$\tilde{\zeta}_1 = \frac{1}{T} \int_0^T \zeta_1 \circ \exp(tH_R) dt \quad (35)$$

on Σ , this defines the G^3 weight used in the estimates.

The reason for which we are restricted to $s < 3$ is due to the fact that in some regions $t(x_n)$ is indeed $h^{2/3}$, we had in the previous section to make $t(x_n)h^{-1+1/s}$ small if $\partial\mathcal{O}$ is G^s .

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