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[1] **Review on formal and moderate cohomology.** Let M be a real manifold, and let $\mathbb{R}$-Cons(M) denote the category of $\mathbb{R}$-constructible sheaves on M, $D^b_{\mathbb{R}-c}(\mathbb{C}_M)$ its derived category. Recall first the functors $Thom(.,\mathcal{D}b_M)$ of [4] and the dual functor $.\overset{w}{\otimes} C^\infty_M$ of [6], defined on the category $\mathbb{R}-Cons(M)$, with values in the category $\mathrm{Mod}(\mathcal{D}_M)$ of $\mathcal{D}_M$-modules on M. (The first functor is contravariant).

They are characterized as follows. Denote by $\mathcal{D}b_M$ the sheaf of Schwartz's distributions on M and by $\mathcal{C}^\infty_M$ the sheaf of $\mathcal{C}^\infty$ functions on M. Let Z (resp. U) be a closed (resp. open) subanalytic subset of M. Then these two functors are exact and moreover :

$$\begin{aligned} \mathcal{T}\mathrm{hom}(\mathbb{C}_Z, \mathcal{D}b_M) &= \Gamma_Z(\mathcal{D}b_M)\ , \\ \mathbb{C}_U \overset{w}{\otimes} \mathcal{C}^\infty_M &= \mathcal{I}^\infty_{M\setminus U}\ , \end{aligned}$$

where $\Gamma_Z(\mathcal{D}b_M)$ denotes as usual the subsheaf of $\mathcal{D}b_M$ of sections supported by Z and $\mathcal{I}^\infty_{M\setminus U}$ denotes the ideal of $\mathcal{C}^\infty_M$ of sections vanishing up to order infinity on $M \setminus U$.

These functors being exact, they extend naturally to the derived category $D^b_{\mathbb{R}-c}(\mathbb{C}_X)$. We keep the same notations to denote the derived functors.

Now let X be a complex manifold and denote by $\overline{X}$ the complex conjugate manifold and by $X_\mathbb{R}$ the real underying manifold. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X, let $\mathcal{D}_X$ be the sheaf of finite order holomorphic differential operators on X. The functors of moderate and formal cohomology (see [4], [6]) are defined for $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_{X_\mathbb{R}})$ by :

$$\begin{aligned} \mathcal{T}\mathrm{hom}(F, \mathcal{O}_X) &= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{T}\mathrm{hom}(F, \mathcal{D}b_{X_\mathbb{R}})), \\ F \overset{w}{\otimes} \mathcal{O}_X &= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{w}{\otimes} \mathcal{C}^\infty_{X_\mathbb{R}})\ . \end{aligned}$$

Laplace transform. Consider a complex vector space $\mathbb{E}$ of complex dimension n, and denote by $j : \mathbb{E} \hookrightarrow \mathbb{P}$ its projective compactification. Let $D^b_{\mathbb{R}-c,\mathbb{R}^+}(\mathbb{C}_\mathbb{E})$ denote the full triangulated subcategory of $D^b_{\mathbb{R}-c}(\mathbb{C}_\mathbb{E})$ consisting of $\mathbb{R}^+$-conic objects (i.e. objects whose cohomology is $\mathbb{R}$-constructible and locally constant on the orbits of the action of $\mathbb{R}^+$ on $\mathbb{E}$).

Let $F \in D^b_{\mathbb{R}-c,\mathbb{R}^+}(\mathbb{C}_\mathbb{E})$ and set for short

$$\begin{aligned} \mathcal{T}\mathrm{Hom}(F, \mathcal{O}_\mathbb{E}) &= R\Gamma(\mathbb{P}; \mathcal{T}\mathrm{hom}(j_! F, \mathcal{O}_\mathbb{P})), \\ W\mathrm{Tens}(F, \mathcal{O}_\mathbb{E}) &= R\Gamma(\mathbb{P}; j_! F \overset{w}{\otimes} (\mathcal{O}_\mathbb{P}). \end{aligned}$$

These are objects of the bounded derived category $D^b(W(\mathbb{E}))$ of the category of modules over the Weyl algebra $W(\mathbb{E})$. Let $\mathbb{E}^*$ denote the dual vector space

[1] This paper is summary of [7]

to E. One denotes by $F^\wedge$ the Fourier-Sato transform of the sheaf F, an object of $D^b_{\mathbb{R}-c,\mathbb{R}^+}(\mathbb{C}_{\mathbb{E}^*})$. (See [5] for an exposition.) One identifies $D^b(W(\mathbb{E}^*))$ to $D^b(W(\mathbb{E}))$ by the usual Fourier transform.

Theorem 0.1 *The classical Laplace transform extends naturally as isomorphisms in $D^b(W(\mathbb{E}))$:*

$$L_{\mathbb{E}} : T\mathrm{Hom}(F, \mathcal{O}_{\mathbb{E}}) \simeq T\mathrm{Hom}(F^\wedge[n], \mathcal{O}_{\mathbb{E}^*}), \tag{0.1}$$

$$L_{\mathbb{E}} : W\mathrm{Tens}(F, \mathcal{O}_{\mathbb{E}}) \simeq W\mathrm{Tens}(F^\wedge[n], \mathcal{O}_{\mathbb{E}^*}) \ . \tag{0.2}$$

Applications 1. Let M be a real vector space of dimension n such that $\mathbb{E}$ is a complexification of M. As a particular case of the theorem, we obtain a characterization of the Laplace transform of the space of distributions on M supported by (not necessarily convex) cones. Let γ denote a closed subanalytic cone in M and let $\Gamma_\gamma \mathcal{S}'_M$ denote the space of tempered distributions supported by γ. One has $\Gamma_\gamma \mathcal{S}'_M \simeq T\mathrm{Hom}(\mathbb{C}_\gamma[-\mathrm{n}], \mathcal{O}_{\mathbb{E}})$. Hence, we get that the Laplace transform of distributions induces an isomorphism :

$$L_{\mathbb{E}} : \Gamma_\gamma \mathcal{S}'_M \simeq T\mathrm{Hom}((\mathbb{C}_\gamma)^\wedge, \mathcal{O}_{\mathbb{E}^*}) \ .$$

When γ is proper and convex, this result is well known, since $(\mathbb{C}_\gamma)^\wedge \simeq \mathbb{C}_U$ where U is the open convex tube $\mathrm{int}\gamma^0 \times \sqrt{-1}\mathrm{M}^*$, the interior of the polar cone to γ, and the right hand side denotes the space of holomorphic functions in this tube, tempered up to infinity. When $\gamma = M$, one recovers the isomorphism between $\mathcal{S}'_M$ and $\mathcal{S}'_{\sqrt{-1}M^*}$.

Let us consider now the case where γ is a non degenerate quadratic cone. Let (x', x'') denote the coordinates on $M = \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ with $p, q \geq 1$, and let $\gamma = \{x; x'^2 - x''^2 \leq 0\}$. Let (u', u'') denote the dual coordinates on M^*, and let $\lambda = \{(u', u''); u'^2 - u''^2 \geq 0\}$. One checks easily that $(\mathbb{C}_\gamma)^\wedge \simeq \mathbb{C}_\lambda[-q]$. We get the isomorphism :

$$L_{\mathbb{E}} : \Gamma_\gamma \mathcal{S}'_M \simeq H^q T\mathrm{Hom}(\mathbb{C}_{\lambda \times \sqrt{-1}M^*}, \mathcal{O}_{\mathbb{E}^*}) \ .$$

This last result is essentially due to Faraut-Gindikin [2] (in a different langage and with a different proof).

Applications 2. Denote by $\mathcal{O}^t_{\mathbb{E}}$ the conic sheaf on $\mathbb{E}$ associated to the presheaf $U \mapsto T\mathrm{Hom}(\mathbb{C}_U, \mathcal{O}_{\mathbb{E}})$. One easily deduces from the main theorem that the Laplace transform induces an isomorphism :

$$(\mathcal{O}^t_{\mathbb{E}})^\wedge[n] \simeq \mathcal{O}^t_{\mathbb{E}^*} \ .$$

This gives a new proof of a result of Hotta-Kashiwara [3] and Brylinski-Malgrange-Verdier [1] on the Fourier-Sato transform of the sheaf of holomorphic solutions of a monodromic $\mathcal{D}$-module.

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