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EQUATIONS AUX DERIVEES PARTIELLES

ON TRACE THEOREMS FOR PSEUDO-DIFFERENTIAL OPERATORS

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On trace theorems for pseudo-differential operators,

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1. Introduction

Our starting point will be the following diagram, where M is a C^∞ manifold, S a smooth hypersurface of M and A a pseudo-differential operator on M of order $m = -1 - 2\delta < -1$, Γ the trace operator on S.

$$\begin{array}{ccccc} & H_{comp}^{-\frac{1}{2}-\delta}(M) & \overset{A}{\longrightarrow} & H_{loc}^{\frac{1}{2}+\delta}(M) & \\ \Gamma^* & \uparrow & & \downarrow & \Gamma \\ & H_{comp}^{-\delta}(S) & \underset{A^\flat}{\longrightarrow} & H_{loc}^{\delta}(S) & \end{array} \tag{1.1}$$

Here the operator $A^\flat$, the restriction of the operator A on S, is well defined as

$$A^\flat = \Gamma A \Gamma^* \quad , \tag{1.2}$$

and is a pseudo-differential operator on S of order $m + 1 = -2\delta$. The following simple formula gives the expression of the principal symbol $a^\flat$ of $A^\flat$ in terms of the principal symbol a of A : when M is a Riemannian manifold, for $x \in S$ and $\xi \in T_x^*(S)$,

$$a^\flat(x,\xi) = \int_{\mathbb{R}} a(x, \xi + t\nu_x) dt \quad , \tag{1.3}$$

where ν_x is a unit conormal vector to S at x (we shall formulate below (1.3) using only the structure of differentiable manifold on M, assuming that A sends densities to functions). We have used here the Weyl quantization formula, such that the operator a^w, with Weyl symbol a is given by

$$(a^w u)(x) = \iint e^{2i\pi(x-y)\cdot\xi} a(\frac{x+y}{2}, \xi) u(y) dy d\xi \quad . \tag{1.4}$$

Formula (1.3) was given by Birman and Yafaev [BY1], in the case above $m < -1$, in connection with scattering theory for two-particle systems . As a matter of fact, one of the main objects of scattering theory, the scattering matrix , appears as the diagonal value of an integral operator [Y1]. Suppose that an unperturbed Hamiltonian H_0 is realized as multiplication by the variable λ in the space $\mathcal{H} = L^2(\Lambda; N)$, where Λ is an interval of $\mathbb{R}$ and N is an auxiliary Hilbert space. We consider an integral operator A given by

$$\langle \, AF \, , \, G \, \rangle_{\mathcal{H}} = \iint_{\Lambda\times\Lambda} \langle \, k(\mu,\nu) F(\nu) \, , \, G(\mu) \, \rangle_N \, d\mu d\nu \quad .$$

Since the kernel k is a distribution, its restriction to the diagonal $\mu = \nu$ requires some specific assumption. One has to justify the representation

$$\langle \, k(\lambda,\lambda) F(\lambda) \, , \, G(\lambda) \, \rangle_N = \lim_{\epsilon,\sigma \longrightarrow 0} \langle \, A\delta_\epsilon(H_o - \lambda) F \, , \, \delta_\sigma(H_o - \lambda) G \, \rangle_{\mathcal{H}} \quad , \tag{1.5}$$

where

$$\delta_\varepsilon(H_o-\lambda)=\frac{1}{2i\pi}[(H_o-\lambda-i\varepsilon)^{-1}-(H_o-\lambda+i\varepsilon)^{-1}] \quad . \tag{1.6}$$

One of our motivation for the study of the singular cases $m \geq -1$ is that, for multi-particle systems, the first Born approximation of the scattering matrix is the trace of a pseudo-differential operator of critical order -1. Naturally, diagram (1.1) does not make sense for $\delta = 0$ ($m = -1$). We should expect some restrictions on the symbol of the operator A for a diagram analogous to (1.1) to hold. Following (1.5), we shall define approximations Γ_ε (cf. [GK]) of the trace operator Γ and ask for the existence of the following limit

$$\lim_{\varepsilon,\sigma\longrightarrow 0}\Gamma_\sigma A\Gamma_\varepsilon^* \quad . \tag{1.7}$$

It turns out that an iff condition for the existence of (1.7) in the critical case $m = -1$ is the vanishing of the principal symbol on the conormal bundle $N^*(S)$ of the hypersurface S. In fact, we have prior knowledge of various examples [Y2] on the sphere S^{n-1} of $\mathbb{R}^n$. For instance, let $M = \mathbb{R}^n$, $S = S^{n-1}$ and

$$a(x,\xi)=\sum_{1\leq j,k\leq n}x_jx_k\frac{\partial^2F}{\partial\xi_j\partial\xi_k}(\xi) \quad , \tag{1.8}$$

where F is a homogeneous function of degree 1. It is possible to see directly that $A = a(x, D_x)$ has a restriction on S^{n-1} and that $A^\flat$ is given , up to compact operators, by the multiplication by $F(x) + F(-x)$. Moreover, when $m = 0$, the previous condition should be supplemented by the vanishing of the subprincipal symbol on $N^*(S)$. The study of the cases $m \geq 1$ unravels new invariants for pseudo-differential operators. These higher order invariants will materialize the obstructions to the restriction to a submanifold of some Lagrangian distribution. They do not coincide with the classical higher order invariants linked to the Weyl quantization (see the appendix of this paper and [He]), except for the first two, the principal and subprincipal symbols. Our problem has also close links with the transmission property introduced by Boutet de Monvel, and other studies of the Poisson operators (see e.g. [GH]). Last but not least, interesting links exist with the second microlocalization ([B], [BL],[DL]).

2. Definitions and preliminary results

If $M = \mathbb{R}^n$, S is the hyperplane $\{x_n = 0\}$ in diagram (1.1) , if the distribution kernel $T(x', x_n, y', y_n) \in \mathcal{D}'(\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R})$ of the operator A is a continuous function of (x_n, y_n) valued in $\mathcal{D}'(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ (as in the case $m < -1$), then the kernel of the restriction $A^\flat$ in (1.2) is $T(x', 0, y', 0)$. This motivates the following definition.

Definition 2.1. *Let Σ be an open set of $\mathbb{R}^{n-1}$, $\rho > 0$ and*

$$V=\Sigma\times\{x_n\in\mathbb{R},|x_n|<\rho\} \quad . \tag{2.1}$$

Let A be a continuous linear operator acting from the space of smooth compactly supported densities $C_o^\infty(V, |\Omega|)$ to $\mathcal{D}'(V)$. Its kernel is a distribution $T(x', x_n, y', y_n) \in \mathcal{D}'(V \times V)$. We say that A can be restricted to Σ if there exists $T^\flat \in \mathcal{D}'(\Sigma \times \Sigma)$ such that

$$\mathcal{D}'-\lim_{\varepsilon,\sigma\rightarrow 0}T(x',\varepsilon x_n,y',\sigma y_n)=T^\flat(x',y')\otimes 1 \quad . \tag{2.2}$$

When (2.2) is satisfied, we define $A^\flat$ as the operator with kernel $T^\flat$.

In order to give a definition of the restriction for a manifold, we need the following lemma

Lemma 2.2. *Let V be as in (2.1) and $T \in \mathcal{D}'(V)$ such that T can be restricted to Σ , with restriction $T^\flat$ in the following sense :*

$$\mathcal{D}'-\lim_{\varepsilon\rightarrow 0}T(x',\varepsilon x_n)=T^\flat(x')\otimes 1 \quad . \tag{2.3}$$

Let $\theta(x', x_n, t) \in C^\infty(V \times \{t \in \mathbb{R}, |t| < \rho\})$, such that $\theta(x', x_n, 0) \equiv 0$. Then

$$\mathcal{D}' - \lim_{\varepsilon \to 0} T(x', \varepsilon x_n)\theta(x', x_n, \varepsilon x_n) = 0 \quad . \tag{2.4}$$

Proof. Let K be a compact subset of Σ and $0 < \rho_o < \rho$. Let N be the order of the distribution T on $K \times \{|x_n| \le \rho_o\} = L$. We have, using Taylor's formula,

$$\theta(x', x_n, \varepsilon x_n) = \sum_{1 \le j \le N+1} \frac{1}{j!} \partial_t^j \theta(x', x_n, 0)(\varepsilon x_n)^j + \theta_N(x', x_n, \varepsilon x_n)(\varepsilon x_n)^{N+2} \quad .$$

Let $\Phi(x', x_n) \in C_o^\infty(V)$, supp $\Phi \subset L$. For $j \ge 1$, we examine the duality brackets when $\varepsilon \to 0$; we obtain first that, from (2.3),

$$\langle\ T(x', \varepsilon x_n)\partial_t^j \theta(x', x_n, 0)(\varepsilon x_n)^j \ ,\ \Phi(x', x_n)\ \rangle \to 0 \quad . \tag{2.5}$$

Moreover, setting $\psi_\varepsilon(x', x_n) = \theta_N(x', \frac{x_n}{\varepsilon}, x_n)(\frac{x_n}{\varepsilon})^{N+2}\Phi(x', \frac{x_n}{\varepsilon})\frac{1}{\varepsilon}$ (note that supp $\psi_\varepsilon \subset L$),

$$|\langle T(x', \varepsilon x_n)\theta_N(x', x_n, \varepsilon x_n)(\varepsilon x_n)^{N+2}, \Phi(x', x_n)\rangle| = \varepsilon^{N+2}|\langle T(x', x_n), \psi_\varepsilon(x', x_n)\rangle|$$
$$\le \varepsilon^{N+2} C \max_{|\alpha| \le N} \|\partial^\alpha \psi_\varepsilon\|_{L^\infty} \le \varepsilon^{N+2} C_1 \varepsilon^{-N-1} = C_1 \varepsilon \quad .$$

The proof of lemma 2.2 is complete.□

Lemma 2.3. *Let $T_1 \in \mathcal{D}'(\Sigma_1 \times (-\rho_1, \rho_1))$, where Σ_1 is an open set of $\mathbb{R}^{n-1}$, $\rho_1 > 0$ such that (2.3) is satisfied : $\mathcal{D}' - \lim_{\varepsilon \to 0} T_1(x_1', \varepsilon x_1'') = T_1^\flat(x_1') \otimes 1$. Let Σ_2 be an open set of $\mathbb{R}^{n-1}$ and $\rho_2 > 0$. Let κ_{12} be a diffeomorphism such that*

$$\kappa_{12} : \Sigma_2 \times (-\rho_2, \rho_2) \longrightarrow \Sigma_1 \times (-\rho_1, \rho_1) \qquad , \qquad \kappa_{12}(\Sigma_2 \times \{0\}) = \Sigma_1 \times \{0\} \quad . \tag{2.6}$$

Then, with κ^* standing for the pull-back,

$$\mathcal{D}' - \lim_{\varepsilon \to 0} (\kappa_{12}^* T_1)(x_2', \varepsilon x_2'') = \left[(\kappa_{12}^\flat)^* T_1^\flat\right](x_2') \otimes 1 \quad , \tag{2.7}$$

where $\kappa_{12}^\flat : \Sigma_2 \longrightarrow \Sigma_1$ is the diffeomorphism defined by the equality $(\kappa_{12}^\flat(x_2'), 0) = \kappa_{12}(x_2', 0)$.

Proof. We note first that (2.6) implies that, with $\kappa_{21} = \kappa_{12}^{-1}$, $\kappa_{21}(x_1', x_1'') = \left(x_2'(x_1', x_1''), x_1'' e(x_1', x_1'')\right)$, with a non-vanishing function e. Let Ψ be a C_o^∞ density. We have the following equalities for duality brackets:

$$\langle\ (\kappa_{12}^* T_1)(x_2', \varepsilon x_2'')\ ,\ \Psi(x_2', x_2'')\ \rangle = \langle\ (\kappa_{12}^* T_1)(x_2', x_2'')\ ,\ \Psi(x_2', \varepsilon^{-1} x_2'')\ \rangle\ \varepsilon^{-1}$$
$$= \langle\ T_1(x_1', x_1'')\ ,\ \Psi\left(x_2'(x_1', x_1''), \varepsilon^{-1} x_1'' e(x_1', x_1'')\right)|\kappa_{21}'(x_1', x_1'')|\ \rangle\ \varepsilon^{-1}$$
$$= \langle\ T_1(x_1', \varepsilon x_1'')\ ,\ \Psi\left(x_2'(x_1', \varepsilon x_1''), x_1'' e(x_1', \varepsilon x_1'')\right)|\kappa_{21}'(x_1', \varepsilon x_1'')|\ \rangle \quad .$$

Now, we have, using Taylor's formula,

$$\Psi\left(x_2'(x_1', \varepsilon x_1''), x_1'' e(x_1', \varepsilon x_1'')\right)|\kappa_{21}'(x_1', \varepsilon x_1'')| = \Psi\left(x_2'(x_1', 0), x_1'' e(x_1', 0)\right)|\kappa_{21}'(x_1', 0)| + \varepsilon x_1'' \theta_o(x_1', x_1'', \varepsilon x_1'') \quad .$$

Using lemma 2.2, one gets

$$\lim_{\varepsilon \to 0} \langle\ (\kappa_{12}^* T_1)(x_2', \varepsilon x_2'')\ ,\ \Psi(x_2', x_2'')\ \rangle = \langle\ T_1^\flat(x_1') \otimes 1\ ,\ \Psi\left(x_2'(x_1', 0), x_1'' e(x_1', 0)\right)|\kappa_{21}'(x_1', 0)|\ \rangle \quad ,$$

which implies (2.7), since, with obvious matrix notation, one has

$$\kappa'_{21}(x'_1,0) = \begin{pmatrix} \dfrac{\partial x'_2}{\partial x'_1}(x'_1,0) & \dfrac{\partial x'_2}{\partial x''_1}(x'_1,0) \\ 0 & \dfrac{\partial x''_2}{\partial x''_1}(x'_1,0) \end{pmatrix} = \begin{pmatrix} (\kappa^\flat_{21})'(x'_1) & \dfrac{\partial x'_2}{\partial x''_1}(x'_1,0) \\ 0 & e(x'_1,0) \end{pmatrix} \quad . \tag{2.8}$$

The proof of lemma 2.3 is complete. □

Lemma 2.3 ensures that the following definition is consistent.

Definition 2.4. *Let M be a C^∞ manifold and S a smooth hypersurface of M. Let $T \in \mathcal{D}'(M)$. We shall say that T can be restricted to S, with restriction $T^\flat \in \mathcal{D}'(S)$ if any point $m \in S$ has a neighborhood U in M, such that there exists a diffeomorphism*

$$\kappa : \Sigma \times (-\rho,\rho) \longrightarrow U \quad , \quad (\rho > 0) \quad , \kappa(\Sigma \times \{0\}) = S \cap U \quad , \tag{2.9}$$

so that the pullback $\kappa^*(T_{|U})$ can be restricted to Σ in the sense of lemma 2.2. The restriction $T^\flat$ is defined as the unique distribution on S such that

$$T^\flat_{|S\cap U} = (\kappa^\flat)_*\left\{\left[\kappa^*(T_{|U})\right]^\flat\right\} \quad , \tag{2.10}$$

where $\kappa^\flat$ is the diffeomorphism induced by κ between Σ and $(S \cap U)$, $(\kappa^\flat)_*$ its push-forward.

Definition 2.5. *Let M be a C^∞ manifold and S a smooth hypersurface of M. Let A be a pseudo-differential operator acting from the space of smooth compactly supported densities $C^\infty_o(M, |\Omega|)$ to $C^\infty(M)$. Its kernel is a distribution $T(x,y) \in \mathcal{D}'(M \times M)$. We say that A can be restricted to S when T can be restricted to $S \times S$ in the sense of definition 2.4 (extended to a product manifold). We define in that case $A^\flat$ as the operator with kernel $T^\flat$, the restriction of T to $S \times S$.*

When S is the hyperplane $\{x_n = 0\}$, and A is a pseudo-differential operator on $\mathbb{R}^n$ with symbol $a(x', x_n, \xi', \xi_n)$, the existence of a restriction requires to look at the properties of the conormal distribution (with respect to $x_n = y_n$)

$$a^\natural(x',\xi';x_n,x_n-y_n) = \int_{\mathbb{R}} a(x',x_n,\xi',\xi_n)\, e^{i(x_n - y_n)\xi_n}\, d\xi_n \quad . \tag{2.11}$$

It is clear that the singularities of $a^\natural$ are getting worse as the order m of a increases. In general, $a^\natural$ belongs to no better space than the Besov space $\mathcal{B}^{-m-1/2}_{2,\infty}$. Property (2.2) amounts to check the following limit, in the distribution sense ($a^\natural$ is a distribution valued in a symbol class)

$$\mathcal{D}' - \lim_{\varepsilon,\sigma \to 0} a^\natural(x',\xi';\varepsilon x_n, \varepsilon x_n - \sigma y_n) \quad . \tag{2.12}$$

It is important to notice that this averaging procedure is more restrictive than the examination of the operator $A\Gamma^*$. In fact, in this situation, $\Gamma^* u = u(x') \otimes \delta(x_n)$, so that when A is simply the multiplication by x_n^k one has $x_n^k \delta(x_n) = 0$ for any $k \geq 1$ but

$$\lim(\varepsilon x_n)^k \delta(\varepsilon x_n - \sigma y_n) = 0 \Longleftrightarrow k > 1 \quad ,$$

and $\varepsilon x_n \delta(\varepsilon x_n - \sigma y_n)$ has no limit in $\mathcal{D}'$ when $\varepsilon, \sigma \to 0$. This means that the two Poisson operators (restriction of $A\Gamma^*$ to $\pm x_n > 0$) could be zero without existence of the limit in (2.12). In this case, the iterated limits $\lim\limits_{\varepsilon \to 0}[\lim\limits_{\sigma \to 0} a^\natural]$ and $\lim\limits_{\sigma \to 0}[\lim\limits_{\varepsilon \to 0} a^\natural]$ are 0, but the double limit in (2.12) does not exist.

We should say for a start that we shall only consider classical polyhomogeneous pseudo-differential operators with symbols $a \in S^m = S^m_{1,0}$(see definition 18.1.1 in [H]) and

$$a \sim \sum_{0\le j} a_{m-j} \qquad \text{i.e.} \qquad a - \sum_{0\le j<N} a_{m-j} \in S^{m-N} \quad , \tag{2.13}$$
$$\text{with} \quad a_k \in S^k \quad \text{and} \quad a_k(x,\lambda\xi) = \lambda^k a_k(x,\xi) \quad \text{for} \quad \lambda \ge 1 \quad \text{and} \quad |\xi| \ge 1 \ .$$

We want also to describe geometrically the link between the symbol of an operator and of its restriction. If U is an open set in $\mathbb{R}^n$ and A is a pseudo-differential operator of order m on U, sending densities to functions (Au is a function whenever u is a density), the symbol a of the operator A appears as a density with respect to the fiber variable. We write it as $a(x,\xi)|d\xi|$. When S is the hyperplane $\{x_n = 0\}$, and $m < -1$, the expression of the symbol of $A^\flat$, say $a^\flat$, is

$$a^\flat(x',\xi') = \int_{\mathbb{R}} a(x',0;\xi',\xi_n)\, d\xi_n \quad . \tag{2.14}$$

In fact, this expression makes sense when a is a density and we shall write

$$a^\flat(x',\xi')|d\xi'| = \Big[\int_{\mathbb{R}} a(x',0;\xi',\xi_n)\, d\xi_n\Big]|d\xi'| \quad , \tag{2.15}$$

so that $a^\flat|d\xi'|$ is a density characterized by the identity

$$\int a^\flat(x',\xi')\Phi(\xi')d\xi' = \iint a(x',0;\xi',\xi_n)\Phi(\xi')d\xi' d\xi_n \quad . \tag{2.16}$$

But the points $(x',0;\xi',\xi_n)$ are precisely those in $T^*(U)$ such that $x = (x',0) \in S$, (ξ',ξ_n) is a cotangent vector at x, that is a linear form on $T_x(U)$ that can be restricted to $T_x(S)$, which is a subspace of $T_x(U)$. One sees thus that (2.14) is a geometrical expression which leads us to the following more abstract description.

Let M be a smooth manifold and Ω_M the density bundle on M . The cotangent bundle will be denoted by $T^*(M)$. The topological dual of $C^\infty_o(M)$ is the space of distribution densities $\mathcal{D}'(M,\Omega_M)$. Let S be a smooth hypersurface of M (smooth submanifold of codimension 1), and $j : S \to M$ the canonical injection. Let Γ be the restriction operator

$$\begin{array}{rccc} \Gamma : & C^\infty_o(M) & \longrightarrow & C^\infty_o(S) \\ & u & \mapsto & \Gamma(u) = u \circ j \end{array} \quad . \tag{2.17}$$

Let Γ^* be the adjoint operator

$$\Gamma^* : \mathcal{D}'(S,\Omega_S) \longrightarrow \mathcal{D}'(M,\Omega_M) \quad , \tag{2.18}$$

defined by duality

$$\langle\, \Gamma^*(v) \ , \ \phi \,\rangle_{\mathcal{D}'(M,\Omega_M)\ ,\ C^\infty_o(M)} = \langle\, v \ , \ \Gamma(\phi) \,\rangle_{\mathcal{D}'(S,\Omega_S)\ ,\ C^\infty_o(S)} \quad . \tag{2.19}$$

We introduce

$$T^*_S(M) = \{(x,\xi) \in T^*(M) \quad \text{with} \quad x \in S\} \quad , \tag{2.20}$$

The conormal bundle of S will be denoted by

$$N^*(S) = \{(x,\xi) \in T^*_S(M)\ ,\ \xi_{|T_x(S)} = 0\ \} \quad . \tag{2.21}$$

We consider the submersion Π

$$\begin{array}{ccc} & \Pi & \\ T^*_S(M) & \longrightarrow & T^*(S) \\ (x,\xi) & \mapsto & (x,\xi_{|T_x(S)}) \end{array} \quad , \tag{2.22}$$

whose fibers are one-dimensional affine spaces as lines in the vector space $T^*_x(M)$. A symbol a on $T^*(M)$ will be a smooth section of the density bundle with respect to the second variable over $T^*(M)$. This means that for each $x \in M$, $a(x)$ is a density on the vector space $T^*_x(M)$: a symbol a can then be represented as

$$a(x,\xi)|d\xi| \quad . \tag{2.23}$$

So if $x \in M$ and $\Phi \in C^\infty_o(T^*_x(M))$

$$\langle\, a \, , \, \Phi \,\rangle\,(x) = \int_{T^*_x(M)} a(x,\xi)\Phi(\xi)d\xi \quad . \tag{2.24}$$

Once we have a symbol a on $T^*(M)$, such that, for each $x \in S$ the density $a(x)$ is in L^1, we can associate to it the following symbol $a^\flat$ on $T^*(S)$. Let $x \in S$ and $\varphi \in C^\infty_o(T^*_x(S))$

$$\langle\, a^\flat \, , \, \varphi \,\rangle\,(x) = \int_{T^*_x(M)} a(x,\xi)(\varphi \circ \pi_x)(\xi)d\xi \quad , \tag{2.25}$$

where π_x is the restriction of Π to $T^*_x(M)$. We shall write in this situation,

$$a^\flat(x,\xi) = \int_{\pi_x^{-1}\{\xi\}} a \quad . \tag{2.26}$$

In the sequel of this paper, we shall denote by

$$\Psi^m(M) = \Psi^m_{phg}(M,\Omega_M,\Omega^0_M) \tag{2.27}$$

the set of polyhomogeneous pseudo-differential operators of order m on the manifold M, acting on distribution-densities and transforming them into distributions. An operator $A \in \Psi^m$, with $m < -1$, always admits a restriction on a hypersurface and $A^\flat = \Gamma A \Gamma^*$. It is not difficult to formulate our results when the order of our operator is below 1, since only classical invariants enter the game.

Theorem 2.6.
Let M be a C^∞ manifold and S a smooth hypersurface. Let A be a pseudo-differential operator in $\Psi^m(M)$. If $-1 \leq m < 0$, a necessary and sufficient condition for A to admit a restriction on S is

$$a_m = 0 \quad \text{on } N^*(S) \text{ , the conormal bundle of } S \quad . \tag{2.28}$$

If $0 \leq m < 1$, a necessary and sufficient condition for A to admit a restriction on S is

$$\left.\begin{array}{l} \partial^\alpha a_m = 0 \quad \text{for } |\alpha| \leq 1 \\ a^s_{m-1} = 0 \end{array}\right\} \text{on } N^*(S) \quad , \tag{2.29}$$

where a^s_{m-1} is the subprincipal symbol. Were these conditions to be satisfied, the operator $A^\flat$ belongs to $\Psi^{m+1}(S)$ with a principal symbol $a^\flat$ given by the absolutely converging integral (2.26)(or (1.3)) in which $a = a_m$.

Moreover, if $-1 \leq m < 0$ and (2.28) is satisfied, $A^\flat = \lim_\nu \Gamma A_\nu \Gamma^*$, where A_ν is an operator whose symbol $a_\nu \in S^{-\infty}$ converges to the symbol a of A in S^m. This means that a_ν converges in C^∞ and is bounded in S^m. If $0 \leq m < 1$ and (2.29) is fulfilled, the same result holds.

One could derive from the appendix a stronger version of this theorem, using the refined principal symbol, availaible for operators in $\Psi^m_{phg}(M,\Omega^\rho,\Omega^\sigma)$ when $\rho+\sigma=1$. The proof of theorem 2.6 is simple and relies only on classical invariance properties for pseudo-differential operators : the principal symbol is invariant and the subprincipal symbol is invariant on the double characteristic set of the principal symbol. One can then straighten the hypersurface S into the hyperplane $\{x_n=0\}$ in a chart coordinate, and study (2.11) near the diagonal $x_n=y_n$. It is then a matter of routine to write the Taylor expansion of $a(x',x_n,\xi',\xi_n)$ at points of $N^*(S)=\{(x',0,0,\xi_n)\}$. When $m<0$, a zero-order expansion is enough to isolate the singularity, whereas for $0\le m<1$, a first order expansion is necessary, leading to condition (2.29). Although it is not difficult to go on with this method for studying (2.11) when $m\ge 1$, the invariance of the conditions obtained is not immediate so we postpone it to the next section.

3. Operators of order larger than 1 and higher order invariants

The following elementary lemma is useful for the understanding of our problem.

Lemma 3.1 *Let m be a real number. Let $a(x,\xi)$ be a polyhomogeneous symbol on $(\mathbb{R}^{n-1}_{x'}\times\mathbb{R}^1_{x_n})\times(\mathbb{R}^{n-1}_{\xi'}\times\mathbb{R}^1_{\xi_n})$ (cf. (2.13)). We note $\equiv$ the equality modulo continuous functions of (x_n,y_n) valued in a symbol class S^{m+1} in the variables (x',ξ'), Γ the gamma function. The distribution $a^\natural$ is defined in (2.11).*
If $m>-1$ is not an integer,

$$(3.1)\qquad a^\natural(x',\xi';x_n,x_n-y_n)\equiv\sum_{|\alpha|+j\le m+1}\Gamma(m+1-j-|\alpha|)\frac{\xi'^\alpha}{\alpha!}$$
$$\Big\{(x_n-y_n)_+^{-m-1+j+|\alpha|}e^{-\frac{i\pi(m+1-j-|\alpha|)}{2}}\left[\partial^\alpha_{\xi'}a_{m-j}(x',x_n,0,-1)-e^{i\pi(m-j-|\alpha|)}\partial^\alpha_{\xi'}a_{m-j}(x',x_n,0,1)\right]$$
$$+(x_n-y_n)_-^{-m-1+j+|\alpha|}e^{\frac{i\pi(m+1-j-|\alpha|)}{2}}\left[\partial^\alpha_{\xi'}a_{m-j}(x',x_n,0,-1)-e^{-i\pi(m-j-|\alpha|)}\partial^\alpha_{\xi'}a_{m-j}(x',x_n,0,1)\right]\Big\}\ .$$

If $m\ge -1$ is an integer ,

$$(3.2)\qquad a^\natural(x',\xi';x_n,x_n-y_n)\equiv\sum_{|\alpha|+j\le m+1}i^{(m-j-|\alpha|-1)}\frac{\xi'^\alpha}{\alpha!}$$
$$\Big\{\ \left(\partial_z^{m+1-j-|\alpha|}\{\ln|z|\}\right)\Big|_{z=x_n-y_n}\left[\partial^\alpha_{\xi'}a_{m-j}(x',x_n,0,-1)-e^{i\pi(m-j-|\alpha|)}\partial^\alpha_{\xi'}a_{m-j}(x',x_n,0,1)\right]$$
$$+\tfrac{i\pi}{2}\ \left(\partial_z^{m+1-j-|\alpha|}\{\mathrm{sign} z\}\right)\Big|_{z=x_n-y_n}\left[\partial^\alpha_{\xi'}a_{m-j}(x',x_n,0,-1)+e^{i\pi(m-j-|\alpha|)}\partial^\alpha_{\xi'}a_{m-j}(x',x_n,0,1)\right]\ \Big\}.$$

Moreover, the existence of $\lim_{\varepsilon,\sigma\to 0}a^\natural(x',\xi';\varepsilon x_n,\varepsilon x_n-\sigma y_n)$ *in* $\mathcal{D}'\Big(\mathbb{R}_{x_n}\times\mathbb{R}_{y_n},S^{m+1}(\mathbb{R}^{n-1}_{x'}\times\mathbb{R}^{n-1}_{\xi'})\Big)$ *is equivalent to*

$$(3.3)\qquad \partial^k_{x_n}\partial^\alpha_{\xi'}a_{m-j}(x',0,0,\xi_n)=0\quad\texttt{for}\ \ |\alpha|+k\le[m]-j+1\ .$$

Since the vector fields $\partial_{x'}$ and ∂_{ξ_n} are tangent to the conormal of the hyperplane $\Sigma\equiv\{x_n=0\}$, condition (3.3) amounts to require that

$$(3.4)\qquad N^*(\Sigma)\subset\Xi_{[m]-j+2}(a_{m-j})=\{(x,\xi)\in\mathbb{R}^{2n},\partial^\alpha_x\partial^\beta_\xi a_{m-j}=0\quad\texttt{for}\ |\alpha|+|\beta|\le[m]-j+1\}\ .$$

When $m<-1$, this condition is empty. When $-1\le m<0$, (3.4) is (2.28) for the hyperplane, whereas for $0\le m<1$, it is (2.29) : note that, in these coordinates , if $N^*(\Sigma)\subset\Xi_2(a_m)$, the subprincipal symbol on $N^*(\Sigma)$ is a_{m-1}, even when one uses the ordinary quantization. In fact,

$$\sum_{1\le l\le n}\frac{\partial^2a_m}{\partial x_l\partial\xi_l}(x',0,0,\xi_n)=\frac{\partial^2a_m}{\partial x'\partial\xi'}(x',0,0,\xi_n)+\frac{\partial^2a_m}{\partial x_n\partial\xi_n}(x',0,0,\xi_n)=0\ \ ,$$

since $\partial_{\xi'} a_m(x',0,0,\xi_n)$ and $\partial_{x_n} a_m(x',0,0,\xi_n)$ are identically zero. One has to clarify the invariance of property (3.4) whenever $m \geq 1$.

The reader will find in the appendix (§5) a complete discussion of the pointwise higher order invariants. We rely in this section on some of the calculations detailed in this appendix. We shall use the Weyl quantization formula (1.4) and recall that one of the nice feature of this quantization is that the subprincipal symbol appears simply as a_{m-1} in the polyhomogeneous expansion of the Weyl symbol. Analogously, the sub^{2}principal symbol is a_{m-2} and is invariantly defined on

$$\Xi_4(a_m) \cap \Xi_2(a_{m-1}) = \{\partial^\alpha a_m = 0 \ , \ |\alpha| \leq 3 \ , \ \partial^\beta a_{m-1} = 0 \ , \ |\beta| \leq 1\} \quad . \tag{3.5}$$

This does not correspond to the next invariant we expect, since condition (3.4) suggests that an invariant $a_{m-2,S}$ linked to the operator A and to the Lagrangian $N^*(S)$ should exist on $N^*(S)$, provided that

$$N^*(S) \subset \Xi_3(a_m) \cap \Xi_2(a_{m-1}) \quad . \tag{3.6}$$

More generally, we are going to construct inductively higher order invariants $a_{m-k,S}$ defined on $N^*(S)$ provided that

$$N^*(S) \subset \Xi_{k+1}(a_m) \cap \Xi_k(a_{m-1}) \cap \ldots \cap \Xi_{k+1-l}(a_{m-l,S}) \cap \ldots \cap \Xi_2(a_{m-k+1,S}) \quad . \tag{3.7}$$

A key observation for this purpose is an invariance property (proposition 3.2 below) of symbols of pseudo-differential operators with respect to diffeomorphisms leaving invariant the hyperplane $\{x_n = 0\}$.

Let M be a C^∞ manifold and S a smooth hypersurface of M. Let $m_o \in S$, U a chart-neighborhood of m_o in M. Let $\kappa_1 : V_1 \longrightarrow U$, $\kappa_2 : V_2 \longrightarrow U$ be two chart coordinates (V_j open sets in $\mathbb{R}^n$) such that for some hyperplanes Σ_1, Σ_2,

$$\kappa_1(\Sigma_1 \cap V_1) = S \cap U = \kappa_2(\Sigma_2 \cap V_2). \tag{3.8}$$

Then, up to an affine transformation, $\kappa_2^{-1} \circ \kappa_1$ leaves the hyperplane $\{y_n = 0\}$ invariant. Let $A \in \Psi^m(M)$ (see (2.27)) so that

$$U^*_{\kappa_1} A U_{\kappa_1} = A_1 \qquad , \qquad U^*_{\kappa_2} A U_{\kappa_2} = A_2 \ , \tag{3.9}$$

where U^*_κ stands for the pullback and U_κ for the push-forward related to the diffeomorphism κ. The symbols $a^{(1)}$ and $a^{(2)}$ of A_1 and A_2 in each of these charts have an expansion

$$a^{(1)} \sim \sum_{j \geq 0} a^{(1)}_{m-j} \qquad , \qquad a^{(2)} \sim \sum_{j \geq 0} a^{(2)}_{m-j} \tag{3.10}$$

with the following invariance property . Assume that

$$\partial^\alpha a^{(1)}_{m-l} = 0 \quad \text{on} \quad N^*(\Sigma_1) \quad \text{for} \quad |\alpha| \leq (k-l) \quad \text{and} \quad 0 \leq l < k \ . \tag{3.11}$$

Then the same holds for the polyhomogeneous expansion of $a^{(2)}$ on $N^*(\Sigma_2)$ and we have

$$(\kappa_2^{-1} \circ \kappa_1)^* a^{(2)}_{m-k} = a^{(1)}_{m-k} \quad \text{on} \quad N^*(\Sigma_1) \ , \tag{3.12}$$

that is

$$(\kappa_2)_* a^{(2)}_{m-k} = (\kappa_1)_* a^{(1)}_{m-k} \quad \text{on} \quad N^*(S) \ . \tag{3.13}$$

We identify a diffeomorphism $\kappa : Y \rightarrow X$ with the canonical homogeneous mapping from $T^*(Y)$ to $T^*(X)$ sending (y,η) to $(\kappa(y), {}^t\kappa'(y)^{-1}\eta)$. Pullback and push-forward of symbols are defined accordingly. From

(3.11), we can *define* for any diffeomorphism $\kappa : V \longrightarrow U$ such that $\kappa(\Sigma \cap V) = S \cap U$, where Σ is an hyperplane

$$a_{m-k,S} = (\kappa)_* b_{m-k} \quad . \tag{3.14}$$

The operator B is a pseudo-differential operator whose symbol $b \sim \sum_{j\geq 0} b_{m-j}$

$$U_\kappa^* A U_\kappa = B \ . \tag{3.15}$$

We precise the previous claims in the following

Proposition 3.2 *Let V_1, V_2 be open sets in $\mathbb{R}^n$, Σ_1, Σ_2 hyperplanes in $\mathbb{R}^n$. Let $\kappa_{21} : V_1 \to V_2$ a C^∞ diffeomorphism such that $\kappa_{21}(\Sigma_1 \cap V_1) = \Sigma_2 \cap V_2$. Let $A_1 \in \Psi^m(V_1)$ (see (2.27)) with a polyhomogeneous Weyl symbol $\sum_{j\geq 0} a^{(1)}_{m-j}$. We set $A_2 = U_{\kappa_{21}} A_1 U^*_{\kappa_{21}}$. The operator A_2 belongs to $\Psi^m(V_2)$ with a polyhomogeneous Weyl symbol $\sum_{j\geq 0} a^{(2)}_{m-j}$. Let $0 \leq j_o \leq k_o$ be non-negative integers such that*

$$\partial^\alpha a^{(2)}_{m-l} = 0 \qquad \text{on} \qquad N^*(\Sigma_2) \qquad \texttt{if} \qquad |\alpha| \leq k_o - l \qquad 0 \leq l < j_o \quad . \tag{3.16}$$

Then property (3.16) holds for $a^{(1)}$ on $N^*(\Sigma_1)$. Moreover, on $N^*(\Sigma_1)$,

$$\partial^\gamma(\kappa^*_{21}(a^{(2)}_{m-j_o}) - a^{(1)}_{m-j_o}) = 0 \ \texttt{ for } \ |\gamma| \leq k_o - j_o. \tag{3.17}$$

Proof. Note that property (3.11) is (3.16) with $j_o = k_o = k$. Using an affine change of coordinates, we can assume $\Sigma_1 = \Sigma_2 \equiv x_n = 0$ and that V_1, V_2 are neighborhoods of 0. Moreover, from the assumption $\kappa_{21}(\Sigma_1 \cap V_1) = \Sigma_2 \cap V_2$, we obtain, omitting the subscript on κ_{21}, that for

$$y \in V_1 \quad , y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \quad , \qquad \kappa(y) = \kappa(y', y_n) = (x'(y), y_n e(y)) \in \mathbb{R}^{n-1} \times \mathbb{R} \quad . \tag{3.18}$$

Since $\kappa'(0)$ leaves Σ invariant, a linear change of coordinates allows us to assume that $\kappa'(0) = \mathtt{Id}$. Then, we define

$$t(s) = 2\text{odd } \kappa(\frac{s}{2}) = (t_{(n-1)}(s), s_n e_o(s)) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \begin{pmatrix} s' \\ s_n \end{pmatrix} = \Lambda(s) \begin{pmatrix} s' \\ s_n \end{pmatrix} , \tag{3.19}$$

so that

$${}^t\Lambda(s)^{-1} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \qquad , \qquad {}^t\Lambda(s)^{-1} \begin{pmatrix} 0 \\ \eta_n \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \begin{pmatrix} 0 \\ \eta_n \end{pmatrix} = \eta_n \begin{pmatrix} 0 \\ * \end{pmatrix}. \tag{3.20}$$

Moreover, we set

$$\begin{gathered} \Omega(s) = \texttt{even } \kappa(\tfrac{s}{2}) = (*, t_n e_1(t)) \qquad (e_1 \ \texttt{odd}), \\ \chi(t) = (*, t_n e_1(t); 0, *) \qquad , \qquad J(t) = |\texttt{even } \kappa'(\tfrac{s}{2})|^{-1}. \end{gathered} \tag{3.21}$$

The conormal bundle $N^*(\Sigma)$ is the set of points $(x', 0; 0, \xi_n)$ and if one sets $a = a^{(2)}$ such that (3.16) holds, we see , following the tranformation law of the appendix ((5.6-11)), that the corresponding symbol $b = a^{(1)}$ on $T^*(V_1)$ at $N^*(\Sigma)$ (say at the point $(0, 0; 0, \eta_n)$) is such that, for a given j , $0 \leq j \leq j_o$,

$$b_{m-j} = a_{m-j} + \sum_{\substack{|\alpha| = j - l \\ |\alpha| \geq 1}} \frac{1}{\alpha!} \partial_t^\alpha \Big[D_\xi^\alpha a_{m-l}(*, t_n e_1(t); 0, *\eta_n) \ J(t) \Big]_{|\ t=0} \tag{3.22}$$

We note that in the sum of (3.22) we have $1 \leq |\alpha| = j - l$, so that $l \leq j - 1 < j_o$, which implies that (3.16) is satisfied for these l. Writing the Taylor expansion of $D_\xi^\alpha a_{m-l}$ at $(*, 0; 0, *\eta_n)$, we obtain

$$(3.23)\quad \partial_t^\alpha \left[D_\xi^\alpha a_{m-l}(*, t_n e_1(t); 0, *\eta_n) \right]_{| \, t=0} = \left\{ \partial_{t'}^{\alpha'} \partial_{t_n}^{\alpha_n} \left[\sum_{0 \leq p \leq j-l} \partial_{x_n}^p D_{\xi'}^{\alpha'} D_{\xi_n}^{\alpha_n} a_{m-l}(*, 0; 0, *\eta_n) t_n^p \omega_p(t) \right] \right\}_{| \, t=0}$$

so that , for a given p, α_n must be larger than p to get a non-zero term, and since $\partial_{x'}$ and ∂_{ξ_n} are tangent to $N^*(\Sigma)$, we must have also from (3.16)

$$(3.24)\qquad |\alpha'| + p > k_o - l \quad . \quad \texttt{But} \;\; j - l = |\alpha'| + \alpha_n \geq |\alpha'| + p > k_o - l \geq j - l \;\; ,$$

which is a contradiction. Eventually , one gets $b_{m-j} = a_{m-j}$ for all $0 \leq j \leq j_o$. Moreover, when (3.16) is satisfied for a, it is also satisfied for b : the Poisson brackets satisfy

$$(3.25)\qquad \sum_j \{b_{m-j}, \eta_n\} \sim \texttt{symbol of } i[B, D_{y_n}] \;\; \texttt{and} \;\; \sum_j \{b_{m-j}, y'\} \sim \texttt{symbol of } i[B, y']$$

so that , we get that for $|\beta| \leq k_o - j$, $\partial^\beta b_{m-j}$ appears as the $(j+1)^{th}$ term in the polyhomogeneous expansion of the symbol of the bracket of B with $|\beta|$ linear forms so that, applying the transformation law to this bracket, we get an expression of $\partial^\beta b_{m-j}$ like the one in (3.22), but involving at most $k_o - j$ more derivatives acting on each a_{m-j}:

$$(3.26)\quad \partial^\beta b_{m-j} = \partial^\beta a_{m-j} + \sum_{\substack{1 \leq |\alpha|, l, l', \gamma \\ |\alpha| + l = j, 0 \leq l' \leq l, \\ |\gamma| \leq |\beta| \, (\leq k_o - j)}} \frac{1}{\alpha!} \partial_t^\alpha \left[D_\xi^\alpha (c_{\gamma, r'} \partial^\gamma a_{m-l'})(*, t_n e_1(t); 0, *\eta_n) \; J(t) \right]_{| \, t=0} \; .$$

In this sum $l' \leq l = j - |\alpha| < j \leq j_o$, so that (3.16) applies to these l'. Writing Taylor expansion as in (3.23), and using the same notations as in (3.24) we get

$$(3.27)\qquad |\alpha'| + p > k_o - l' - |\gamma| \;\; . \;\; \texttt{But} \;\; j - l = |\alpha'| + \alpha_n \geq |\alpha'| + p > k_o - l' - |\gamma| \geq j - l,$$

since

$$k_o - j + l - l' \geq k_o - j \geq |\beta| \geq |\gamma|.$$

The proof of proposition 3.2 is complete.□

This proposition shows that the following definition does not depend on the choice of the diffeomorphism κ.

Definition 3.3. *Let M be a C^∞ manifold and S a smooth hypersurface. Let A be a pseudo-differential operator in $\Psi^m(M)$, with Weyl symbol $a \sim \sum_{0 \leq j} a_{m-j}$. Let U be a chart coordinate such that $\kappa : V \longrightarrow U$ is a diffeomorphism (V open set of $\mathbb{R}^n$). Assume that there exists a hyperplane Σ of $\mathbb{R}^n$ such that $\kappa(\Sigma \cap V) = S \cap U$. Assume that conditions (3.11) are fulfilled for the polyhomogeneous symbol $\kappa^*(a) \sim \sum_{j \geq 0} b_{m-j}$ with some integer k on $N^*(\Sigma)$. We define*

$$(3.28)\qquad a_{m-k,S} = \kappa_* b_{m-k} \;\; .$$

We shall say that the polyhomogeneous symbol a vanishes at the order $k+1$ on $N^*(S)$ if, for $0 \leq l \leq k$,

$$(3.29)\qquad b_{m-l} \;\; \texttt{vanishes at the order} \;\; k + 1 - l \;\; \texttt{on} \;\; N^*(\Sigma) \; : \; \partial^\alpha b_{m-l} = 0 \;\; \texttt{if} \;\; |\alpha| \leq k - l \; .$$

When (3.29) is satisfied, we will write symbolically

$$(3.30)\quad a_m^{\{k\}} = a_{m-1}^{\{k-1\}} = \ldots = a_{m-k+1}^{\{1\}} = a_{m-k}^{\{0\}} = 0 \quad \texttt{on} \;\; N^*(S) \;\; , \quad \texttt{or} \quad a^{\{\{k\}\}} = 0 \quad \texttt{on} \;\; N^*(S) \;\; .$$

Theorem 3.4. *Let M be a C^∞ manifold and S a smooth hypersurface. Let A be a pseudo-differential operator in $\Psi^m(M)$, with Weyl symbol $a \sim \sum_{0\leq j} a_{m-j}$. A necessary and sufficient condition for A to admit a restriction on S is*

$$a^{\{\{[m]+1\}\}} = 0 \quad on \quad N^*(S) \ . \tag{3.31}$$

Were these conditions to be satisfied, the operator $A^\flat$ belongs to $\Psi^{m+1}(S)$ with a principal symbol $a^\flat$ given by the absolutely converging integral (2.26)(or (1.3)) in which $a = a_m$.

Moreover, if (3.31) is satisfied, $A^\flat = \lim_\nu \Gamma A_\nu \Gamma^*$, where A_ν is an operator whose symbol $a_\nu \in S^{-\infty}$ converges to the symbol a of A in S^m. This means that a_ν converges in C^∞ and is bounded in S^m.

This theorem is now a direct consequence of definition 2.5 and statement (3.3) in lemma 3.1, after straightening the hypersurface S into a hyperplane. □

We want to go on and compute these new invariants $a_{m-k,S}$ in any coordinate system : if, in a chart coordinate U the hypersurface is described by $\Phi(x) = 0$, $d\Phi \neq 0$, we can find a diffeomorphism $\kappa : V \longrightarrow U$ so that $\Phi \circ \kappa$ appears as the n-th coordinate : $(\Phi\circ\kappa)(y) = y_n$. To carry out the computation, we can assume that the equation of S is given by

$$x_n = \varphi(x') \qquad , x' \in \mathbb{R}^{n-1}, \quad \varphi(0) = 0 \ .$$

so that we can consider the diffeomorphism

$$(x', x_n) = \kappa(y', y_n) = (y', y_n + \varphi(y')). \tag{3.32}$$

If we use the formulas and notations of the appendix, we have

$$t(s) = 2\text{odd}\ \kappa(\frac{s}{2}) = (s', s_n + \int_{-1/2}^{1/2} \varphi'(y' + \theta s')\ d\theta s'), \tag{3.33}$$

so that

$$\begin{aligned} t(s) = \Lambda(s)s &= \begin{pmatrix} I_{n-1} & 0 \\ \int_{-1/2}^{1/2} \varphi'(y'+\theta s')\ d\theta & 1 \end{pmatrix} \begin{pmatrix} s' \\ s_n \end{pmatrix}, \\ {}^t\Lambda(s)^{-1} &= \begin{pmatrix} I_{n-1} & -\int_{-1/2}^{1/2} \varphi'(y'+\theta s')\ d\theta \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{3.34}$$

and

$$ {}^t\Lambda(s)^{-1} \begin{pmatrix} 0 \\ \eta_n \end{pmatrix} = \eta_n \begin{pmatrix} -\int_{-1/2}^{1/2} \varphi'(y'+\theta s')\ d\theta \\ 1 \end{pmatrix} \quad , \quad \Omega(s) = \textbf{even}\ \kappa(\frac{s}{2}) = \ (y', y_n + \textbf{even}\varphi(y' + \frac{s'}{2})). \tag{3.35}$$

Since

$$t' = s' \quad \text{and} \quad J(t) \equiv 1 \ , \tag{3.36}$$

$$a_{m-k,S} = \kappa_* b_{m-k} = a_{m-k} +$$

$$\sum_{\substack{|\alpha|+l=k \\ |\alpha|\geq 1}} \frac{1}{\alpha!}\partial_{t'}^\alpha \Big[D_{\xi'}^\alpha a_{m-l}(y', \textbf{even}\ \varphi(y'+\frac{t'}{2}); -\int_{-1/2}^{1/2} \varphi'(y'+\theta t')\ d\theta\eta_n, \eta_n) \Big]_{|\ t'=0} \quad , \tag{3.37}$$

that is

$$a_{m-k,S} = \kappa_* b_{m-k} = a_{m-k}+$$

(3.38) $$\sum_{\substack{|\alpha|+l=k\\ |\alpha|\geq 1}} \frac{1}{\alpha!}\partial_{t'}^{\alpha}\left[D_{\xi'}^{\alpha} a_{m-l}\left(y',\varphi(y')+\sum_{j\geq 1}\frac{\varphi^{(2j)}(y')}{(2j)!}\frac{t'^{2j}}{2^{2j}}; -\sum_{r\geq 1}\frac{\varphi^{(2r+1)}(y')}{(2r+1)!}\frac{t'^{2r}}{2^{2r}}\eta_n,\eta_n\right)\right]_{|\,t'=0} .$$

Using a tensor notation for derivatives with respect to ξ', x', we get
(3.39)

$$a_{m-k,S}(y',\varphi(y'),-\eta_n\varphi'(y'),\eta_n) = \sum_{\substack{l\,,\ 0\leq l\leq k\\ p,q,\ p+q\leq \frac{k-l}{2}\\ \underbrace{|\gamma_1|+\ldots+|\gamma_p|}_{all\ |\gamma|\ even\geq 2}+\underbrace{|\gamma_{p+1}|+\ldots+|\gamma_{p+q}|}_{all\ |\gamma|\ odd\geq 3}=k-l+q}}$$

$$(4i\pi)^{-(k-l)}(-\eta_n)^q\frac{1}{p!q!}\partial_{x_n}^p\partial_{\xi'}^{k-l+q}a_{m-l}(y',\varphi(y'),-\eta_n\varphi'(y'),\eta_n)\prod_{1\leq j\leq p+q}\frac{\varphi^{(\gamma_j)}(y')}{\gamma_j!} .$$

4. Comments

Let u be a distribution on Ω, open set of $\mathbb{R}^n$ and S a smooth submanifold of codimension d of Ω . If there exists $s > d/2$ such that

(4.1) $$\mathrm{WF}_s u \cap N^*(S) = \emptyset \ ,$$

the microlocal version of the Sobolev theorem says that the restriction of u to S makes sense and that

(4.2) $$\mathrm{WF}_{s-\frac{d}{2}} j^*(u) \subset j^*(WF_s u) \ ,$$

where j is the embedding of S into Ω. It means that the pullback j^*, defined for smooth functions, has a unique extension to distributions satisfying (4.1). When we consider Lagrangian distributions

(4.3) $$u(x) = \int e^{i\Phi(x,\theta)}a(x,\theta)d\theta \ ,$$

where Φ is a non-degenerate phase function and a is a symbol of order m, so that

(4.4) $$\mathrm{WF}u \subset L = \{(x,\Phi'_x), \texttt{such that } \Phi'_\theta = 0\} \ ,$$

we could ask the same question of restriction, expecting a more detailed analysis from the particular structure of u. The regular cases above correspond to the microlocal Sobolev theorem, whereas the singular cases seem to be related to the second wave-front set of the distribution with respect to the Lagrangian manifold L, as remarked by Bony in this seminar.

Let Σ be a linear submanifold of $\mathbb{R}^n = \mathbb{R}^d_{x'} \times \mathbb{R}^{n-d}_{x''}$ given by the equations $\{x''=0\}$, so that its conormal set is the Lagrangian

(4.5) $$L = \{(x',x'',\xi',\xi'') \in \mathbb{R}^d\times\mathbb{R}^{n-d}\times\mathbb{R}^d\times\mathbb{R}^{n-d}\ ,\ x''=0\ ,\ \xi'=0\}.$$

The standard class of symbols S^M is, using Hörmander 's notations

(4.6) $$S(\Lambda^M, G = |dx|^2 + \frac{|d\xi|^2}{\Lambda^2}) \quad \texttt{with} \quad \Lambda = 1+|\xi|.$$

In a paper by Bony [B] (see also [BL]) the second microlocalization with respect to L is described as follows. First of all, one introduces a metric g larger than the classical G on the open set $|x''| < 1$, $|\xi''| > 1 + |\xi'|$:

$$g = |dx'|^2 + \frac{\Lambda^2 dx''^2}{\lambda^2} + \frac{|d\xi'|^2}{\lambda^2} + \frac{|d\xi''|^2}{\Lambda^2}, \quad \text{with } \lambda = 1 + |\xi'| + |x''|\Lambda \leq 2\Lambda. \tag{4.7}$$

The symbol class $S^{M,m}$ is defined as $S(\Lambda^M \lambda^m, g)$, i.e. $a(x,\xi) \in S^{M,m}$ means

$$|D_{x'}^{\alpha'} D_{x''}^{\alpha''} D_{\xi'}^{\beta'} D_{\xi''}^{\beta''} a| \leq C\Lambda^M \lambda^m \left(\frac{\Lambda}{\lambda}\right)^{|\alpha''|} \left(\frac{1}{\lambda}\right)^{|\beta'|} \left(\frac{1}{\Lambda}\right)^{|\beta''|}. \tag{4.8}$$

The metric g is slowly varying and satisfies $g \leq g^\sigma$ since $\lambda \geq 1$, but fails to be temperate globally. However, g is temperate uniformly on the unit balls of G, so that a modified quantization formula can be used to associate operators to these symbols ([BL]). Let a be a symbol in $S^M \subset S^{M,0}$. Writing the Taylor expansion of a at L, one gets

$$a(x', x'', \xi', \xi'') = \sum_{|\alpha|+|\beta| \leq [M]+d} \frac{1}{\alpha!\beta!} \partial_{x''}^{\alpha} \partial_{\xi'}^{\beta} a(x', 0, 0, \xi'') x''^{\alpha} \xi'^{\beta}$$
$$+ \int_0^1 \sum_{|\alpha|+|\beta|=[M]+d+1} ([M]+d+1)(1-\theta)^{[M]+d} \frac{1}{\alpha!\beta!} \partial_{x''}^{\alpha} \partial_{\xi'}^{\beta} a(x', \theta x', \theta\xi', \xi'') x''^{\alpha} \xi'^{\beta} d\theta$$

so that $\partial_{x''}^{\alpha} \partial_{\xi'}^{\beta} a(x', 0, 0, \xi'') \in S^{M-|\beta|,0}$ and $x''^{\alpha} \xi'^{\beta} = \Lambda^{-|\alpha|} (x''\Lambda)^{\alpha} \xi'^{\beta} \in S^{-|\alpha|,|\alpha|+|\beta|}$. Eventually, we obtain

$$S^M \subset S^{M,0} \subset S^{M,0} + S^{M-1,1} + S^{M-2,2} + \ldots + S^{M-[M]-d,[M]+d} + S^{M-[M]-d-1,[M]+d+1}. \tag{4.9}$$

The obstructions to the restriction of an operator with symbol a to the linear subspace $\{x'' = 0\}$ will come from the terms in $S^{M,0}, S^{M-1,1}, S^{M-2,2}, \ldots, S^{M-[M]-d,[M]+d}$. It seems plausible that a second microlocal Sobolev theorem could be used to prove that the restriction exists for an operator with symbol $S^{-d-\epsilon,m}$. It would be also interesting to compare our conditions to the conditions of Delort and Lebeau [DL] in an analytic framework.

Our problem is also closely connected to transmission conditions introduced by Boutet de Monvel ([B1], [B2], th.18.2.15 in [H]). A pseudo-differential operator A on $\mathbb{R}^n$ satisfies the transmission condition with respect to $M = \{x_n \geq 0\}$ if it can be extended to a mapping from $C^\infty(M)$ into itself. When the symbol a of A has the polyhomogeneous expansion $a \sim \sum_{j\geq 0} a_{m-j}$, one constructs a symbol $\check{a}$ given by

$$\check{a}(x,\xi) \sim \sum_{j \geq 0} a_{m-j}(x, -\xi) e^{-i\pi(m-j)} . \tag{4.10}$$

The transmission property is proved equivalent to the vanishing of infinite order of $a - \check{a}$ on the interior conormal bundle of ∂M : this means that , for all α, β,

$$D_x^{\alpha} D_{\xi}^{\beta} a_{m-j}(x', 0, 0, -1) = e^{i\pi(m-j-|\beta|)} D_x^{\alpha} D_{\xi}^{\beta} a_{m-j}(x', 0, 0, +1) . \tag{4.11}$$

This is in fact a direct consequence of formulas (3.1-2) in lemma 3.1. One sees on (4.11) that this condition is always satisfied for differential operators and that the orientation of the boundary is irrelevant when m is an integer. Writing (3.2) for $m = -1$, one gets, modulo continuous functions,

$$a^{\natural}(x', \xi'; 0, z_n) = -\Big[a_{-1}(x', 0, 0, -1) + a_{-1}(x', 0, 0, +1)\Big] \ln|z_n| + \tfrac{i\pi}{2} \Big[a_{-1}(x', 0, 0, +1) - a_{-1}(x', 0, 0, -1)\Big] \operatorname{sign}(z_n). \tag{4.12}$$

The first transmission condition in (4.11), for $\alpha = \beta = 0$, is devised to get rid of the logarithmic term, whereas the signum term is harmless since it enjoys left and right continuity. Our averaging procedure, described in section 2, is more stringent since the function $\mathbf{sign}(\varepsilon x_n - \sigma y_n)$ has no limit in the distribution sense when ε and σ tend to zero. We thus require vanishing of a_{-1} on the whole conormal bundle. The same differences occur with theorem 18.2.17 in [H], studying the one-sided limit of $A\Gamma^*$, where Γ^* is the dual mapping of the restriction $(\Gamma^*(v)(x', x_n) = v(x') \otimes \delta(x_n))$.

Transmission conditions for the $S^m_{\rho,\delta}$ classes are studied in [GH], where a detailed analysis of the Poisson operators is provided. We have seen in section 2, that these operators could be zero without inducing the existence of the double limit in (2.12).

5. Appendix

The goal of this appendix is to give a simple derivation for higher order invariants for pseudo-differential operators on a manifold. We use the Weyl quantization rule and set, in a chart coordinate X,

$$Au(x) = \iint a(\frac{x+x'}{2}, \xi) \ \exp 2i\pi\langle\ x - x'\ ,\ \xi\ \rangle\ \ u(x')dx'd\xi \quad . \tag{5.1}$$

We assume that the operator A sends ρ densities to σ densities, so that its Schwartz kernel $K(x, x')$ is a σ density with respect to the first variable and a $(1-\rho)$ density with respect to the second variable. As a matter of fact, the most interesting case for us will be $\rho = 1, \sigma = 0$. Since we seek a local result, we shall assume that the kernel K is compactly supported in $X \times X$. Let κ be a C^∞ diffeomorphism $Y \longrightarrow X$. We have, following the rules of transformations of densities through the pullback U^*_κ of densities on X to densities on Y,

$$(U^*_\kappa(Au))(y) = (Au)(\kappa(y))\ |\kappa'(y)|^\sigma =$$

$$\iint a(\frac{\kappa(y)+\kappa(y')}{2}, \xi)\ \ |\kappa'(y')|^{1-\rho} |\kappa'(y)|^\sigma \exp 2i\pi\langle\ \kappa(y) - \kappa(y')\ ,\ \xi\ \rangle\ \ u(\kappa(y'))\ |\kappa'(y')|^\rho dy'd\xi \quad , \tag{5.2}$$

where $|\kappa'(y)| = |\det \kappa'(y)|$. The kernel $L(y, y')$ of $B = U^*_\kappa A U_\kappa$ is given by

$$L(y, y') = \int a(\frac{\kappa(y)+\kappa(y')}{2}, \xi)\ \ |\kappa'(y')|^{1-\rho} |\kappa'(y)|^\sigma \exp 2i\pi\langle\ \kappa(y) - \kappa(y')\ ,\ \xi\ \rangle\ d\xi,$$

so that its Weyl symbol $b(y, \eta)$ is

$$b(y, \eta) = \int e^{2i\pi s\cdot\eta} \quad L(y - \frac{s}{2}, y + \frac{s}{2})\ ds =$$

$$\iint a(\Omega(s)\ ,\ \tau + {}^t\Lambda(s)^{-1}\eta)\ |\kappa'(y + \frac{s}{2})|^{1-\rho} |\kappa'(y - \frac{s}{2})|^\sigma \exp 2i\pi\{s\cdot\eta - \Lambda(s)s\cdot(\tau + {}^t\Lambda(s)^{-1}\eta)\} ds d\tau \quad ,$$

where

$$\Omega(s) = \frac{1}{2}\kappa(y + \frac{s}{2}) + \frac{1}{2}\kappa(y - \frac{s}{2}) \qquad , \qquad \Lambda(s) = \int_{-1/2}^{1/2} \kappa'(y + \theta s)\ d\theta \quad . \tag{5.3}$$

Moreover, we set

$$t(s) = \Lambda(s)s = \kappa(y + \frac{s}{2}) - \kappa(y - \frac{s}{2}) \quad .$$

We get then, using the fact that $s \mapsto t(s)$ is a diffeomorphism of neighborhoods of zero,

$$b(y, \eta) = \iint a(\Omega(s(t))\ ,\ \tau + {}^t\Lambda(s(t))^{-1}\eta)\ \ J(t)\ \ e^{-2i\pi t\cdot\tau} dt d\tau \quad , \tag{5.4}$$

with

$$J(t) = \ |\kappa'(y + \frac{s(t)}{2})|^{1-\rho} |\kappa'(y - \frac{s(t)}{2})|^\sigma \left|\frac{1}{2}\kappa'(y + \frac{s(t)}{2}) + \frac{1}{2}\kappa'(y - \frac{s(t)}{2})\right|^{-1} \quad . \tag{5.5}$$

We notice that, since Ω is an even function of s and t an odd function of s, $t \mapsto \Omega(s(t))$ is even as well as $t \mapsto {}^t\Lambda(s(t))^{-1}$. On the other hand, $J(t)$ is even when $1-\rho=\sigma$.

Whenever A is a classical pseudo-differential operator it is a consequence of theorems 18.1.17 and 18.5.10 in [H] that B is also a classical pseudo-differential operator. Formula (5.4) implies readily for $a \sim \sum_{k\geq 0} a_{m-k}$ with $a_{m-k}(x,\xi)$ homogeneous of degree $m-k$ with respect to ξ, that $b \sim \sum_{k\geq 0} b_{m-k}$ with

$$
\begin{aligned}
b_{m-k}(y,\eta) = a_{m-k}(\kappa(y),{}^t\kappa'(y)^{-1}\eta) \quad |\kappa'(y)|^{\sigma}|\kappa'(y)^{-1}|^{\rho} \\
+ \sum_{\substack{|\alpha|+l=k \\ |\alpha|\geq 1}} \frac{1}{\alpha!}\partial_t^{\alpha}\left[D_\xi^{\alpha} a_{m-l}(\chi(t))\ J(t)\right]_{|\,t=\tau=0}
\end{aligned}
\tag{5.6}
$$

where $J(t)$ is given by (5.5) and

$$
\chi(t) = \left(\Omega(s(t))\ ,\ {}^t\Lambda(s(t))^{-1}\eta\right) \quad . \tag{5.7}
$$

We obtain

$$
b_{m-k} = \kappa^* a_{m-k} + \sum_{0\leq l\leq k-1}\ \sum_{|\beta|+|\gamma|=k-l} \frac{1}{\beta!}\partial_t^{\beta}\left[D_\xi^{\beta+\gamma} a_{m-l}(\chi(t))\right] \frac{1}{\gamma!}\ \partial_t^{\gamma} J(t)_{|\,t=\tau=0}\ , \tag{5.8}
$$

that is

$$
b_{m-k} = \kappa^* a_{m-k} + \sum_{0\leq l\leq k-1} \kappa^*\left(P_{k-l}(\kappa,D_x,D_\xi)(a_{m-l})\right)\ , \tag{5.9}
$$

where $P_{k-l}(\kappa,D_x,D_\xi)$ is a differential operator given by

$$
P_{k-l}(\kappa,D_x,D_\xi)(c) = \sum_{|\beta|+|\gamma|=k-l} \frac{1}{\gamma!}\ \partial_t^{\gamma} J(0) Q_{\beta\gamma}(\kappa,D_x,D_\xi)(c)\ J(0)^{-1}\quad , \tag{5.10}
$$

where the differential operator $Q_{\beta\gamma}(\kappa,D_x,D_\xi)$ is such that

$$
Q_{\beta\gamma}(\kappa,D_x,D_\xi)(c) = \sum_{\substack{(\alpha_1,\ldots,\alpha_j)\in\mathbb{N}^n\times\ldots\times\mathbb{N}^n \\ \alpha_1+\ldots+\alpha_j=\beta \\ \min_{1\leq r\leq j}|\alpha_r|\geq 1}} \frac{1}{j!}\left[D_\xi^{\beta+\gamma}c\right]^{(j)}(\chi(0))\ \frac{\chi^{(\alpha_1)}(0)}{\alpha_1!}\cdots\frac{\chi^{(\alpha_j)}(0)}{\alpha_j!}\quad . \tag{5.11}
$$

This last expression follows from Faà de Bruno's formula (see below in this appendix) ; note that here, we do not need an explicit expression for the coefficients. It is useful to notice that, since χ is an even function all the multi-indices $\alpha_1,\ldots\alpha_j$ should have an even length and in particular $\min_{1\leq r\leq j}|\alpha_r|\geq 2$ in (5.11). As a consequence, we get in the summation above $2j\leq|\beta|$. Since the order of $Q_{\beta\gamma}$ is less than $|\beta|+|\gamma|+j$ we obtain

$$
\text{order } Q_{\beta\gamma} \leq \frac{3|\beta|}{2}+|\gamma|\quad ,
$$

which implies that the order of P_{k-l} is less than $3(k-l)/2$. This implies that, in (5.8)

$$
\text{the number of derivatives falling on}\quad a_{m-l}\leq\frac{3}{2}(k-l)\ . \tag{5.12}
$$

We can check easily the standard facts on the principal and subprincipal symbols. Let us write down the three first formulas coming from (5.8-11). We get, with $\nu = \kappa^{-1}$,

$$
(5.13)\quad \begin{cases} k=0 : & \nu^*(b_m) = a_m. \\ k=1 : & \nu^*(b_{m-1}) = a_{m-1} + \sum_{|\alpha|=1} (D_\xi^\alpha a_m)\,(\partial_t^\alpha J)(0)J(0)^{-1}. \\ k=2 : & \nu^*(b_{m-2}) = a_{m-2} + \sum_{|\alpha|=1} (D_\xi^\alpha a_{m-1})\,(\partial_t^\alpha J)(0)J(0)^{-1} \\ & \qquad + \sum_{|\alpha|=2} \frac{1}{\alpha!}(D_\xi^\alpha a_m)(\partial_t^\alpha J)(0)J(0)^{-1} + \frac{1}{\alpha!}(D_\xi^\alpha a_m)^{(1)}\cdot(\partial_t^\alpha \chi)\ (0). \end{cases}
$$

The first line of (5.13) gives the invariance of the principal symbol, the second line the invariance of the subprincipal symbol on the double characteristic set of the principal symbol. Moreover, one should note that when J is even, in particular when $\rho = 1-\sigma$, the sum in the second line vanishes, so that we get a *refined principal symbol* for a classical pseudo-differential operator A on a smooth manifold M such that $A \in \psi^m(M;\Omega^\rho,\Omega^{1-\rho})$, using the notations of section 18.1 in [H] (that is A is a pseudo-differential operator of order m on M sending ρ densities to $(1-\rho)$ densities). This is a small variation on the theme of theorem 18.1.33 in [H] (see also the remark on page 161 of [H]). In particular, when $\rho = 1, \sigma = 0$, the symbol $a = a_m + a_{m-1}$ is actually invariant as a density with respect to ξ, which means that for each $x \in M$ $a(x,\cdot)$ is a density on the vector space $T_x^*(M)$. So a is defined as a smooth section of a density bundle over $T^*(M)$. From the third line of (5.13), one gets that the *sub²principal* symbol a_{m-2} is invariantly defined on the set

$$
(5.14)\qquad \partial^\alpha a_m = 0\ ,\ |\alpha| \le 3 \qquad , \qquad \partial^\alpha a_{m-1} = 0\ ,\ |\alpha| \le 1\ .
$$

Note that this set is correctly defined from the identities in the first two lines of (5.13), namely that its pullback through κ is actually defined by the same equalities as in (5.14), b replacing a. Before embarking upon the study of the general case, it may be useful to summarize the results on the first three invariants.

Theorem 5.1

Let M be a smooth manifold, and $A \in \psi^m(M;\Omega^\rho,\Omega^\sigma)$, the set of polyhomogeneous pseudo-differential operators of order m mapping ρ-densities to σ-densities. In each chart coordinate, the operator A has a Weyl symbol $a \sim \sum_{j\ge 0} a_{m-j}$ where $a_{m-j}(x,\xi)$ is homogeneous of degree $m-j$ in the ξ variable.

a. *The principal symbol a_m is invariantly defined as a (σ,ρ) density on $T^*(M)\backslash 0$.*
b. *The subprincipal symbol a_{m-1} is invariantly defined as a (σ,ρ) density on $\Xi_2(a_m) \equiv \{\partial^\alpha a_m = 0,\ |\alpha| \le 1\}$.*
c. *The sub²principal symbol a_{m-2} is invariantly defined as a (σ,ρ) density on*
$\Xi_4(a_m) \cap \Xi_2(a_{m-1}) \equiv \{\partial^\alpha a_m = 0\ ,\ |\alpha| \le 3 \quad , \quad \partial^\alpha a_{m-1} = 0\ ,\ |\alpha| \le 1\}$.

Moreover, when $\sigma + \rho = 1$,
d. *The refined principal symbol $a_m + a_{m-1}$ is invariantly defined as a (σ,ρ) density on $T^*(M)\backslash 0$.*
e. *The sub²principal symbol a_{m-2} is invariantly defined as a (σ,ρ) density on*
$\Xi_4(a_m) \equiv \{\partial^\alpha a_m = 0\ ,\ |\alpha| \le\ 3\ \}$.

The only point yet to be checked is the fifth one, which is a direct consequence of (5.13) and the fact that , for $\sigma + \rho = 1$, J is an even function (see (5.5)).□

We now move forward to the general case with the following statement.

Theorem 5.2

Let M be a smooth manifold, and A an operator satisfying the assumptions of Theorem 5.1. Let $k \ge 1$ be an integer.The sub^{k}principal symbol a_{m-k} is invariantly defined as a (σ,ρ) density on the set

$$
(5.15)\qquad \partial^\alpha a_{m-l} = 0\ ,\quad |\alpha| \le \frac{3(k-l)}{2} \quad , \quad 0 \le l < k\ .
$$

Proof. From (5.8) and the remark in (5.9), it follows that on the set (5.15), using the notations above, $b_{m-k} = \kappa^* a_{m-k}$. Moreover, this set is correctly defined, namely, if a point $(x,\xi) = (\kappa(y), {}^t\kappa'(y)^{-1}\eta)$ satisfies (5.15) then (y,η) satisfies (5.15) for b : in fact from (5.8) and (5.12)

$$b_{m-l} = \sum_{0\le r\le l} \quad \sum_{|\alpha_r|\le\frac{3}{2}(l-r)} c(\kappa,\alpha_r)\ \partial^{\alpha_r}(a_{m-r}).$$

This implies readily that, for $|\alpha| \le \frac{3(k-l)}{2}$, $0 \le l < k$,

$$\partial^\alpha b_{m-l} = \sum_{0\le r\le l} \quad \sum_{|\alpha_r|\le\frac{3}{2}(l-r+k-l)=\frac{3}{2}(k-r)} d(\kappa,\alpha_r)\ \partial^{\alpha_r}(a_{m-r}) = 0 \ ,$$

if (5.15) is satisfied.The proof of theorem 5.2 is complete. □

Note that, in the Weyl quantization,the expression of $\kappa^*(a_{m-k})$ involves $\frac{3k}{2}$ derivatives of a_m.

Remark 5.3. Let us note also that, since the expression of b_{m-k} involves $\frac{3(k-l)}{2}$ derivatives of a_{m-l} for $l \le k$ and a_{m-l} involves $2(l-p)$ derivatives of $\tilde{a}_{m-p}$, $p \le l$, the invariants of order k (b_{m-k}) are linear combination of $\frac{3(k-l)}{2} + 2(l-p) = \frac{3k}{2} + \frac{l}{2} - 2p \le 2(k-p)$ derivatives of $\tilde{a}_{m-p}$. Eventually, one gets that in the standard quantization,the expression of $\kappa^*(\tilde{a}_{m-k})$ involves $2k$ derivatives of a_m.

The last part of this appendix is devoted to the formula known as Faà de Bruno's* , dealing with the iterated chain rule. We write here all the coefficients explicitly. Although the derivation of this formula is elementary, it is not easy to find a precise reference in the recent literature for the exact expression of the coefficients.

Theorem 5.4

Let $k \ge 1$ be an integer and U, V, W open sets in Banach spaces. Let a and b be k times differentiable fonctions $b : U \longrightarrow V$ and $a : V \longrightarrow W$. Then the k- multilinear symmetric mapping $(a\circ b)^{(k)}$ is given by ($\mathbb{N}^* = \mathbb{N}\setminus\{0\}$)

$$(5.16)\qquad (a\circ b)^{(k)} = \sum_{\substack{1\le j\le k\\ (k_1,\dots,k_j)\in\mathbb{N}^{*j}\\ k_1+\dots+k_j=k}} \frac{a^{(j)}\circ b}{j!}\ b^{(k_1)}\dots b^{(k_j)}\frac{k!}{k_1!\dots k_j!} \ .$$

To prove theorem 5.4, it is convenient to write the Taylor expansion of a and Fourier inversion formula to express powers of b. It is also easy to derive the following

Corollary 5.5 *Let a and b be functions satisfying the assumptions of theorem 5.4 so that $U \subset \mathbb{R}^m_x$, $V \subset \mathbb{R}^n_y$, $W \subset \mathbb{R}$ and α is a multi-index $\in \mathbb{N}^m$. Then (using the standard notation for a multi-index $\beta \in \mathbb{N}^n$, $a^{(\beta)} = \partial^\beta_y a$ and if $\gamma \in \mathbb{N}^l$, $\gamma! = \gamma_1!\dots\gamma_l!$) we get*

$$(5.17)\qquad \partial^\alpha_x(a\circ b) = \sum_{\substack{1\le|\beta|=j\le|\alpha|\ ,\ \beta\in\mathbb{N}^n\\ (\alpha_1,\dots,\alpha_j)\in\mathbb{N}^m\times\dots\times\mathbb{N}^m=\mathbb{N}^{mj}\\ \alpha_1+\dots+\alpha_j=\alpha\\ \min_{1\le r\le j}|\alpha_j|\ge1}} \frac{a^{(\beta)}\circ b}{\beta!}\ b^{(\alpha_1)}\dots b^{(\alpha_j)}\frac{\alpha!}{\alpha_1!\dots\alpha_j!} \ .$$

---

* One could find a version of theorem 5.4 on pages 69-70 of the thesis of " Chevalier François FAÀ DE BRUNO, Capitaine honoraire d'État-Major dans l'armée Sarde". This thesis was defended in 1856, in the Faculté des Sciences de Paris in front of the following jury : Cauchy (chair) , Lamé and Delaunay.

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