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EQUATIONS AUX DERIVEES PARTIELLES

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0. Introduction

We give here an account of recent work [BR4] of the authors characterizing those real algebraic hypersurfaces in $\mathbb{C}^N$ between which all holomorphic mappings must be algebraic. Some applications of this work were given in joint work with X. Huang [BHR] to prove analyticity of sufficiently smooth CR mappings between such hypersurfaces. We outline here some of the proofs in [BR4], including a simplification of one part, as well as some other improvements.

A real hypersurface in $\mathbb{C}^N$ is *algebraic* if it is given by the vanishing of a real valued polynomial with nonvanishing gradient. A germ of a holomorphic function is *algebraic* if it is the root of a polynomial with holomorphic polynomial coefficients. Similarly, a germ of a holomorphic map is algebraic if its components are. We need to introduce the following definition. A real analytic hypersurface M in $\mathbb{C}^N$ is *holomorphically degenerate* at a point $p_0 \in M$ if there exists a nontrivial germ of a holomorphic vector field, with holomorphic coefficients, tangent to M in a neighborhood of p_0. (See also Stanton [S], where this definition was introduced.) We have the following.

Proposition 0.1. *If M is a connected real analytic hypersurface, then M is holomorphically degenerate at a given $p_0 \in M$ if and only if it is holomorphically degenerate at every point in M.*

If M is connected, we shall say that M is *holomorphically nondegenerate* if it is not holomorphically degenerate at any point in M, or, equivalently, by Proposition 0.1, it is not holomorphically degenerate at a given point p_0 in M. We can now state our main result.

Theorem. *Let M and M' be two algebraic hypersurfaces in $\mathbb{C}^N$ and $p_0 \in M$. If M is connected and holomorphically nondegenerate, and H is a germ at p_0 of a biholomorphism of $\mathbb{C}^N$ mapping M into M', then H is algebraic. Conversely, if M is algebraic and holomorphically degenerate at p_0, then there exists a germ of a biholomorphism of $\mathbb{C}^N$ at p_0, mapping M into itself and fixing p_0, which is not algebraic.*

The following is an easy consequence of the Theorem and of a result in [BR3].

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Corollary. *Let M and M' be as in the first part of the Theorem and H a holomorphic mapping defined in a neighborhood of M in $\mathbb{C}^N$ with $H(M) \subset M'$. Then H is algebraic if either the Jacobian determinant of H does not vanish identically or M' does not contain any nontrivial complex variety.*

We shall say that a property holds *generically* on M if it holds in an open, dense subset of M. It should be noted that if M is generically Levi nondegenerate, then M is holomorphically nondegenerate. The converse, however, holds only in $\mathbb{C}^2$. For instance, the hypersurface in $\mathbb{C}^3$ given by $(\Re Z_1)^2 + (\Re Z_2)^2 - (\Re Z_3)^2 = 0$ is Levi degenerate at every point (with $\Re Z \neq 0$), but holomorphically nondegenerate. In 1977 Webster [W1] proved the first part of the above theorem for the Levi nondegenerate case. Previous results were proved by Poincaré [P] and Tanaka [T] for pieces of spheres in $\mathbb{C}^N$. We note here that Webster's result has been extended in some cases to nondegenerate hypersurfaces of different dimensions. See e.g. Webster [W2], Forstnerič [Fo], Huang [H] and their references.

1. Normal coordinates and the Levi type of a hypersurface

Let $M \subset \mathbb{C}^N$ be a real analytic hypersurface, given by $\rho(Z, \bar{Z}) = 0$ near p_0 with $d\rho \neq 0$ and $N = n + 1$. We can find (see [CM], [BJT]) holomorphic coordinates (z, w), (called *normal coordinates*), $z \in \mathbb{C}^n, w \in \mathbb{C}$ vanishing at p_0 such that near p_0, M is given by

$$w = Q(z, \bar{z}, \bar{w}), \tag{1.1}$$

where $Q(z, \chi, \tau)$ is holomorphic in a neighborhood of 0 in $\mathbb{C}^{2n+1}$ and satisfies $Q(z, 0, \tau) \equiv Q(0, \chi, \tau) \equiv \tau$. We associate to M the complex hypersurface $\mathcal{M}$ in $\mathbb{C}^{2N}$ locally defined near $(p_0, \bar{p}_0)$ by

$$\mathcal{M} = \{(Z, \zeta) : \rho(Z, \zeta) = 0\}, \tag{1.2}$$

where $\rho(Z, \bar{Z})$ is the defining function for M near p_0 as above. We define the germ of an analytic subset $\mathcal{V}_{p_0} \subset \mathbb{C}^N$ through p_0 by

$$\mathcal{V}_{p_0} = \{Z : \rho(Z, \zeta) = 0 \text{ for all } \zeta \text{ near } \bar{p}_0 \text{ with } \rho(p_0, \zeta) = 0\}. \tag{1.3}$$

Note in fact that $\mathcal{V}_{p_0} \subset M$. Recall that M is called *essentially finite* at p_0 if $\mathcal{V}_{p_0} = \{p_0\}$. We also recall (see [BR2]) that the set of essentially finite points in each connected component of M is either empty or open and dense. In the sequel we assume $p_0 = 0$.

Let $L_1, \ldots, L_n$, $n = N - 1$, given by $L_j = \sum_{k=1}^N a_{jk}(Z, \bar{Z}) \partial / \partial \bar{Z}_k$ be a basis of the CR vector fields on M near 0 with the a_{jk} real analytic. We need to introduce the following vector-valued functions. For a multi-index α, let V_α be the real analytic function defined near 0 in $\mathbb{C}^N$ by

$$V_\alpha(Z, \bar{Z}) = L^\alpha \rho_Z(Z, \bar{Z}), \tag{1.4}$$

where ρ_Z denotes the gradient of ρ with respect to Z and $L^\alpha = L_1^{\alpha_1} \cdots L_n^{\alpha_n}$. We have the following lemma whose proof could be essentially found in [BHR].

Lemma 1.5. *Let M be a connected real analytic hypersurface in $\mathbb{C}^N$. The following conditions are equivalent:*

(i) *M is holomorphically nondegenerate.*
(ii) *$\{V_\alpha(Z, \bar{Z}), \alpha \in \mathbb{Z}_+^n\}$ span $\mathbb{C}^N$ generically in a neighborhood of p_0 in M.*
(iii) *There exists an integer k, with $1 \leq k \leq n$ so that $\{V_\alpha(Z, \bar{Z}), |\alpha| \leq k\}$ span $\mathbb{C}^N$ generically in a neighborhood of p_0 in M.*

We say that the hypersurface M is *k-holomorphically nondegenerate* at $Z \in M$ if $\{V_\alpha(Z, \bar{Z}), |\alpha| \leq k\}$ span $\mathbb{C}^N$, with k minimal. In particular, it is easy to see that M is 1-holomorphically nondegenerate at Z if and only if the Levi form of M is nondegenerate at Z. Note that if M is connected and holomorphically nondegenerate then there exists $\ell = \ell(M)$, $1 \leq \ell(M) \leq N - 1$, such that M is ℓ-holomorphically nondegenerate at every point in an open dense subset of M. We call $\ell(M)$ the *Levi type* of M. The Levi type of M is 1 if and only if M is generically Levi nondegenerate.

If M is given by (1.1), or equivalently by $\bar{w} = \bar{Q}(\bar{z}, z, w)$, and $Z = (z, w)$, then

$$V_\alpha(Z, \bar{Z}) = -\bar{Q}_{\bar{z}^\alpha, Z}(\bar{z}, z, w). \tag{1.6}$$

2. Proof of the first part of the Theorem

Assume M, M', p_0 and H are as in the assumptions of the Theorem. By Lemma 1.5 and the comments following, by slightly moving p_0, we may assume that M is ℓ-holomorphically nondegenerate at p_0, with $\ell = \ell(M)$, the Levi type of M as defined in §1. We choose normal coordinates (z, w) for M, vanishing at p_0, and normal coordinates (z', w') for M' vanishing at $H(p_0)$. We write the mapping $H = (f, g)$ with $z' = f(z, w)$ and $w' = g(z, w)$. We assume that M is given by (1.1) and M' is given by $w' = Q'(z', \bar{z}', \bar{w}')$. Since $H(M) \subset M'$, we have for $(z, w) \in M$ in a neighborhood of 0,

$$\bar{g}(\bar{z}, \bar{w}) = \bar{Q}'(\bar{f}(\bar{z}, \bar{w}), f(z, w), g(z, w)).$$

Since the manifold $\mathcal{M}$ introduced in (1.2) is given by $\tau = \bar{Q}(\chi, z, w)$ for $(z, \chi, w, \tau) \in \mathbb{C}^{2N}$, and a simliar equation for $\mathcal{M}'$, it follows from the above that we have for $(z, w, \chi, \tau) \in \mathcal{M}$

$$\bar{g}(\chi, \tau) = \bar{Q}'(\bar{f}(\chi, \tau), f(z, w), g(z, w)). \tag{2.1}$$

We now introduce the following holomorphic vector fields which are tangent to $\mathcal{M}$:

$$\mathcal{L}_j = \frac{\partial}{\partial \chi_j} + \bar{Q}_{\chi_j}(\chi, z, w) \frac{\partial}{\partial \tau}, \quad j = 1, \dots n, \tag{2.2}$$

Note that the $\mathcal{L}_j$ commute with each other. Since M and M' are algebraic, and the functions Q and Q' are obtained by the implicit function theorem, it follows that they are algebraic also.

Lemma 2.3. *For (z, w, χ, τ) in a neighborhood of 0 in $\mathcal{M}$ the following holds:*

$$f_j(z,w) = \Psi_j(\mathcal{L}^\gamma \bar{f}_p(\chi,\tau), \mathcal{L}^\beta \bar{g}(\chi,\tau)), \quad j = 1, \ldots, n, \tag{2.4}$$

with $|\gamma|, |\beta| \leq \ell$ and the Ψ_j holomorphic functions of their arguments.

Proof. By Lemma 1.5, identity (1.6), and the choice of p_0, we have

$$\text{span}\, \{\bar{Q}_{\bar{z}^\alpha, Z}(0,0,0) : |\alpha| \leq \ell\} = \mathbb{C}^N. \tag{2.5}$$

Since H is a biholomorphism at the origin, it follows from the choice of normal coordinates, that the matrix $(\mathcal{L}_j \bar{f}_k)(0)$ is invertible. Using this, applying repeatedly the $\mathcal{L}_j$ to (2.1) and using (2.5), we may obtain the lemma by the use of the implicit function theorem. □

We are now ready to prove the first part of the Theorem. We first prove the following preliminary result.

Lemma 2.6. *For every integer q the mapping $z \mapsto \frac{\partial^q}{\partial w^q} H(z, 0)$ is holomorphic algebraic in a neighborhood of 0 in $\mathbb{C}^n$.*

Proof. We begin with the identity (2.4). We note that for $z \in \mathbb{C}^n$ close to 0, the point $(z, w, \zeta, \tau) = (z, 0, 0, 0)$ is in $\mathcal{M}$, since $Q(z, 0, 0) \equiv 0$. Since the coefficients of $\mathcal{L}_j$ given by (2.2) are algebraic holomorphic, for any holomorphic function $J(\zeta, \tau)$, the functions $(z, w) \mapsto (\mathcal{L}^\gamma J(\zeta, \tau))|_{\zeta=0,\tau=0}$ are algebraic holomorphic. Evaluating (2.4) at $(z, 0, 0, 0)$, we obtain the conclusion of the lemma for $q = 0$ since $g(z, 0) \equiv 0$.

To prove the lemma for $q > 0$, we take $\chi = 0$ and $\tau = w$ in (2.4). Differentiating the resulting identity q times with respect to w and evaluating at $w = 0$ gives the desired result for the f_j. From (2.1) we have on $\mathcal{M}$

$$g(z,w) = Q'(f(z,w), \bar{f}(\chi,\tau), \bar{g}(\chi,\tau)). \tag{2.7}$$

As before, we take $\chi = 0$ and $\tau = w$ in (2.7), differentiate q times in w, and evaluate at $w = 0$. The conclusion of the lemma for g then follows from that for the f_j. □

To complete the proof of the first part of the Theorem, we use (2.4) in which we take $\tau = 0$ and substitute $Q(z, \chi, 0)$ for w to obtain

$$f_j(z, Q(z,\chi,0)) = \Psi_j(\mathcal{L}^\gamma \bar{f}_p(\chi,0), \mathcal{L}^\beta \bar{g}(\chi,0)), \quad j = 1, \ldots, n, \tag{2.8}$$

which holds as an identity in $(z, \chi) \in \mathbb{C}^{2n}$ near 0. Note that after this substitution the coefficients of the vector fields $\mathcal{L}_j$ are then algebraic holomorphic in (z, χ). Since M is ℓ-holomorphically nondegenerate at 0, and the coordinates are taken to be normal, we conclude that the vector function $Q_\chi(z, \chi, 0)$ does not vanish identically. Hence we may assume there is (z^0, χ^0) such that $Q_{\chi_1}(z^0, \chi^0, 0) \neq 0$. Note that (z^0, χ^0) can be chosen arbitrarily close to 0 in $\mathbb{C}^{2n}$. Put $w^0 = Q(z^0, \chi^0, 0)$. By the implicit function theorem , we can find an algebraic holomorphic function $\theta(z, w)$ defined near (z^0, w^0) and satisfying $\theta(z^0, w^0) = \chi_1^0$, such that the following identity holds for (z, w) near (z^0, w^0) in $\mathbb{C}^{n+1}$:

$$Q(z, \theta(z,w), \chi_2^0, \ldots, \chi_n^0, 0) \equiv w. \tag{2.9}$$

We now take $\chi = (\theta(z,w), \chi_2^0, \dots, \chi_n^0)$ in (2.8). After making this substitution, we consider that (z,w) are independent variables near (z^0, w^0). (Recall that τ has been set to 0 throughout this part of the proof.) After this substitution the functions

$$(z,w) \mapsto \Psi_j(\mathcal{L}^\gamma \bar{f}_p, \mathcal{L}^\beta \bar{g}) \tag{2.10}$$

are seen to be algebraic holomorphic, by using Lemma 2.6. This proves that the components $f_j(z,w)$ of H are algebraic. To prove that $g(z,w)$ is algebraic we again use (2.7) with the same substitution as above. The already proved algebraicity of the f_j gives that of g. This finishes the proof of the first part of the Theorem. □

3. Proof of the second part of the Theorem; Flow of holomorphic vector fields

We now give the proof of the second part of the Theorem. Let $p_0 \in M$ and assume that X is a nontrivial germ at p_0 of a holomorphic vector field tangent to M. To any such X, there is a holomorphic one parameter group of local biholomorphisms in $\mathbb{C}^N$ sending M into M defined by the complex flow of X i.e.

$$\dot{\phi}(t,Z) = X(\phi(t,Z)), \quad \phi(0,Z) = Z. \tag{3.1}$$

Then $\phi(t,Z)$ is holomorphic for $t \in \mathbb{C}, |t| < \epsilon$, and $Z \in V$, where V is an open neighbrhood of p_0 in $\mathbb{C}^N$. For fixed t, the map $Z \mapsto \phi(t,Z)$ is a local biholomorphism preserving M, and if $X(p_0) = 0$, then $\phi(t,p_0) \equiv p_0$.

The second part of the Theorem will be a consequence of (ii) of the following proposition.

Proposition 3.2. *Let M be a real algebraic hypersurface in $\mathbb{C}^N$, $p_0 \in M$, and X a germ at p_0 of a nontrivial holomorphic vector field tangent to M. Then the following hold.*

- (i) *The germ at 0 of the holomorphic complex curve $t \mapsto \phi(t,p_0)$, where $\phi(t,p_0)$ is the flow of X starting from p_0 given by (3.1), is contained in $\mathcal{V}_{p_0}$, as defined by (1.3).*
- (ii) *There exists f, a germ at p_0 of a holomorphic function and arbitrarily small t such that if $\psi(t,Z)$ is the flow of $Y = fX$, the mapping $Z \mapsto \psi(t,Z)$ is a nonalgebraic local biholomorphism mapping M into itself and fixing p_0.*

Proof. We show that the function $t \mapsto h(t) = \rho(\phi(t,p_0),\zeta)$ vanishes identically for $\zeta \in \mathbb{C}^N$ close to $\bar{p}_0$ with $\rho(p_0,\zeta) = 0$. If $X = \sum_{j=1}^N a_j(Z)\frac{\partial}{\partial Z_j}$, then $\frac{dh}{dt}(t) = \sum_{j=1}^N a_j(\phi(t,p_0))\frac{\partial\rho}{\partial Z_j}(\phi(t,p_0),\zeta)$, which must be a multiple of $h(t)$. Since $h(0) = 0$, by the uniqueness of the solution of differential equations, we conclude that $h(t) \equiv 0$, proving (i).

To prove (ii), by standard arguments using the local group property, we have

$$\sum_{j=1}^N a_j(Z)\frac{\partial \phi_k}{\partial Z_j}(t,Z) = a_k(\phi(t,Z)), \quad k = 1, \dots, N. \tag{3.3}$$

We may assume $X(p_0) = 0$. If for some arbitrarily small t the map $Z \mapsto \phi(t, Z)$ is not algebraic, we are done. Otherwise, we assume $a_1 \not\equiv 0$, and let $f(Z) = e^{Z_1}$ and $Y = e^{Z_1}X$. We denote by $\psi(t, Z)$ the holomorphic flow of Y. By (3.3) for the vector field Y instead of X, and taking $k = 1$, we have

$$\sum_{j=1}^{N} e^{Z_1} a_j(Z) \frac{\partial \psi_1}{\partial Z_j}(t, Z) = e^{\psi_1(t,Z)} a_1(\psi(t, Z)). \tag{3.4}$$

If $Z \mapsto \psi(t, Z)$ is algebraic for some fixed t, then since all the coefficients a_k are algebraic, it would follow from (3.4) that the function $Z \mapsto e^{Z_1 - \psi_1(t,Z)}$ is also algebraic. Note that $Z \mapsto Z_1 - \psi_1(t, Z)$ is algebraic and not constant (since $a_1 \not\equiv 0$). However, if $A(Z)$ is any nonconstant algebraic holomorphic function, then the function $Z \mapsto e^{A(Z)}$ cannot be algebraic. This contradiction proves (ii). $\square$

4. Remarks

Remark 4.1. It follows from Proposition 0.1, Proposition 3.2 (ii) and the openness of the set of essentially finite points that a connected real analytic hypersurface M is holomorphically nondegenerate if and only if M is essentially finite at some point $p_0 \in M$.

Remark 4.2. An algebraic holomorphic function $h(Z)$ is said to be of degree m if it satisfies a polynomial equation $P(Z, f(Z)) = 0$, where $P(Z, X)$ is an irreducible polynomial in $N + 1$ variables of total degree m. By the total degree of an algebraic hypersurface M we mean the total degree of its defining real polynomial. An inspection of the proof of the Theorem shows that the degrees of the components of H are bounded by a constant depending only on the dimension N and the total degrees of M and M'.

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