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EQUATIONS AUX DERIVEES PARTIELLES

CONSTRUCTIBLE SHEAVES, WHITNEY FUNCTIONS AND SCHWARTZ'S DISTRIBUTIONS

P. SCHAPIRA

(joint work with Masaki Kashiwara)

Constructible sheaves, Whitney functions and Schwartz's distributions

PIERRE SCHAPIRA (JOINT WORK WITH MASAKI KASHIWARA)

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1 Introduction

Let f be a proper morphism of real analytic manifolds. It is a natural problem to characterize the space of the integrals along f of all $\mathcal{C}^\infty$ or distribution densities. Such problems occur in particular when studying correspondences:

$$\begin{array}{ccc} & S & \\ f \swarrow & & \searrow g \\ X & & Y, \end{array}$$

as, for example, the Penrose correspondance or else the Radon transform (see [2]). The simple case of $X = \mathbb{C}(\simeq \mathbb{R}^2)$ and $f : X \longrightarrow X$ is the map $z \mapsto z^2$ shows that the constructible sheaf $f_*\mathbb{C}_X$, the direct image of the constant sheaf $\mathbb{C}_X$ on X by f , plays an essential role in this description.

In [5], Kashiwara has introduced the functor $TH_X(\cdot)$ of moderate cohomology. It is defined on the category of $\mathbb{R}$ -constructible sheaves on the real analytic manifold X , and is characterized by the fact that it is exact and its value on $\mathbb{C}_Z$ for a closed subanalytic subset Z of X is the sheaf $\Gamma_Z(\mathcal{D}b_X)$ of distributions supported by Z . Here, we shall give another construction of $TH_X(\cdot)$ (that we prefer to denote by $\mathcal{T}hom(\cdot, \mathcal{D}b_X)$) and at the same time, we construct a new functor, dual to the preceding one, the Whitney functor $\cdot \overset{w}{\otimes} \mathcal{C}_X^\infty$. This is again an exact functor characterized by the fact that its value on $\mathbb{C}_U$, for U an open subanalytic subset of X , is the subsheaf $\mathcal{I}_{X \setminus U, X}^\infty$ of $\mathcal{C}^\infty$ -functions which vanish to infinite order on $X \setminus U$. If now X is a complex manifold, taking the Dolbeault complexes of the preceding functors, we get the functors $\mathcal{T}hom(\cdot, \mathcal{O}_X)$ and $\cdot \overset{w}{\otimes} \mathcal{O}_X$. The main result of this paper will be the adjunction formulas in section 6.

Section 1 to 6 of this paper are extracted from [7], and section 7 is extracted from [2]. The redaction is due to PS, and does not involve the responsibility of other authors.

2 Construction of functors on $\mathbb{R} - C(X)$

Let X be a real analytic manifold and denote by $\text{Mod}(\mathbb{C}_X)$ the abelian category of sheaves of $\mathbb{C}$ -vector spaces on X , by $\mathbb{R} - C(X)$ the abelian subcategory of $\mathbb{R}$ -constructible sheaves and by $\mathbb{R} - C_c(X)$ the subcategory of $\mathbb{R} - C(X)$ of sheaves with compact support. Denote by $\mathcal{S}_X$ the category whose objects are the open subanalytic relatively compact subsets of X , the only morphisms being the inclusions $U \subset V$. Then $U \mapsto \mathbb{C}_U$ gives a faithful functor $\mathcal{S}_X \longrightarrow \mathbb{R} - C(X)$. Let $\mathcal{A}$ be an abelian category over $\mathbb{C}$. This means that $\text{Hom}_{\mathcal{A}}(M, N)$ has a structure of $\mathbb{C}$ -vector space for $M, N \in \mathcal{A}$, and the composition of morphisms is $\mathbb{C}$ -bilinear. Let $\psi : \mathcal{S}_X \longrightarrow \mathcal{A}$ be a functor, and consider the conditions:

$$\psi(\emptyset) = 0. \quad (2.1)$$

$$\begin{cases} \text{for any } U, V \text{ in } \mathcal{S}_X, \text{ the sequence} \\ \psi(U \cap V) \longrightarrow \psi(U) \oplus \psi(V) \longrightarrow \psi(U \cup V) \longrightarrow 0 \\ \text{is exact.} \end{cases} \quad (2.2)$$

$$\text{for any open inclusion } U \subset V \text{ in } \mathcal{S}_X, \psi(U) \longrightarrow \psi(V) \text{ is a monomorphism} \quad (2.3)$$

Theorem 2.1 (i) Assume (2.1) and (2.2). Then there is a right exact functor, unique up to isomorphism,

$$\Psi : \mathbb{R} - C_c(X) \longrightarrow \mathcal{A}$$

such that $\Psi(\mathbb{C}_U) \simeq \psi(U)$ functorially in $U \in \mathbb{R} - C_c(X)$.

(ii) Assume (2.1), (2.2) and (2.3). Then Ψ is exact.

(iii) Let ψ_1 and ψ_2 be two functors from $\mathcal{S}_X$ to $\mathcal{A}$ both satisfying (2.1) and (2.2), and let Ψ_1 and Ψ_2 be the corresponding functors given in (i). Let $\theta : \psi_1 \longrightarrow \psi_2$ be a morphism of functors. Then θ extends uniquely to a morphism of functors

$$\Theta : \Psi_1 \longrightarrow \Psi_2.$$

(iv) In the situation of (i), assume that $\mathcal{A}$ is a subcategory of the category $\text{Mod}(\mathbb{C}_X)$ of sheaves of $\mathbb{C}$ -vector spaces on X , and that $\mathcal{A}$ is local, that is: an object F of $\text{Mod}(\mathbb{C}_X)$ belongs to $\mathcal{A}$ if for any relatively compact open U there exists F' in $\mathcal{A}$ such that $F|_U \simeq F'|_U$. Assume further that ψ is local, that is: $\text{supp}(\psi(U)) \subset \bar{U}$ for any $U \in \mathcal{S}_X$.

Then ψ extends uniquely to $\mathbb{R} - C(X)$ as a right exact functor Ψ which is local, that is, $\Psi(F)|_U \simeq \Psi(F_U)|_U$ for any $F \in \mathbb{R} - C(X)$ and $U \in \mathcal{S}_X$. Moreover the assertion (ii) remains valid, as well as (iii), provided that both ψ_1 and ψ_2 are local.

3 The functors $\cdot \overset{w}{\otimes} \mathcal{C}_X^\infty$ and $\mathcal{Thom}(\cdot, \mathcal{D}b_X)$

On a real analytic manifold X , we denote respectively by $\mathcal{A}_X, \mathcal{C}_X^\infty, \mathcal{D}b_X, \mathcal{B}_X$ the sheaves of real analytic functions, $\mathcal{C}^\infty$ -functions, Schwartz's distributions and Sato's hyperfunctions. We denote by $\mathcal{A}_X^\vee$ the sheaf of real analytic densities and if $\mathcal{F}$ is an $\mathcal{A}_X$ -module, we set $\mathcal{F}^\vee = \mathcal{F} \otimes_{\mathcal{A}_X} \mathcal{A}_X^\vee$. We denote by $\mathcal{D}_X$ the sheaf of finite order differential operators with coefficients in $\mathcal{A}_X$.

Theorem 3.1 *There exist exact functors:*

$$\begin{aligned} \cdot \overset{w}{\otimes} \mathcal{C}_X^\infty : \mathbb{R} - C(X) &\longrightarrow \text{Mod}(\mathcal{D}_X) \\ \mathcal{Thom}(\cdot, \mathcal{D}b_X) : \mathbb{R} - C(X)^{op} &\longrightarrow (\text{Mod}(\mathcal{D}_X)), \end{aligned}$$

such that for any U (resp. Z) open (resp. closed) subanalytic subset of X , one has:

$$\begin{aligned} \mathbb{C}_U \overset{w}{\otimes} \mathcal{C}_X^\infty &\simeq \mathcal{I}_{X \setminus U, X}^\infty, \\ \mathcal{Thom}(\mathbb{C}_Z, \mathcal{D}b_X) &\simeq \Gamma_Z(\mathcal{D}b_X). \end{aligned}$$

(Recall that for Z a closed subset of X , $\mathcal{I}_{Z, X}^\infty$ denotes the subsheaf of $\mathcal{C}_X^\infty$ of functions vanishing to infinite order on Z .)

Proof: This follows immediately from Theorem 2.1 and the Lojasiewicz's theorem (see [9]) which asserts that if Z_1 and Z_2 are closed subanalytic subsets of X , then the two sequences:

$$\begin{aligned} 0 &\longrightarrow \mathcal{I}_{Z_1 \cup Z_2, X}^\infty \longrightarrow \mathcal{I}_{Z_1, X}^\infty \oplus \mathcal{I}_{Z_2, X}^\infty \longrightarrow \mathcal{I}_{Z_1 \cap Z_2, X}^\infty \longrightarrow 0, \\ 0 &\longrightarrow \Gamma_{Z_1 \cap Z_2} \mathcal{D}b_X \longrightarrow \Gamma_{Z_1} \mathcal{D}b_X \oplus \Gamma_{Z_2} \mathcal{D}b_X \longrightarrow \Gamma_{Z_1 \cup Z_2} \mathcal{D}b_X \longrightarrow 0 \end{aligned}$$

are exact.

q.e.d.

Remark 3.2 (i) The functor $\mathcal{Thom}(\cdot, \mathcal{D}b_X)$ has been defined in [5], without using Theorem 2.1.

(ii) The sheaves $F \overset{w}{\otimes} \mathcal{C}_X^\infty$ and $\mathcal{Thom}(F, \mathcal{D}b_X)$ are sheaves of $\mathcal{C}_X^\infty$ -modules, hence are soft.

(iii) The vector space $\Gamma(X; F \overset{w}{\otimes} \mathcal{C}_X^\infty)$ may naturally be endowed with a topology of type FN (Fréchet nuclear), the vector space $\Gamma_c(X; \mathcal{Thom}(F, \mathcal{D}b_X^\vee))$ with a topology of type DFN (dual of Fréchet nuclear) and this two spaces are dual to each other. This is proved by reducing to the case where $F = \mathbb{C}_Z$, for Z a closed subanalytic subset of X .

- (iv) The functors $\cdot \overset{w}{\otimes} \mathcal{C}_X^\infty$ and $\mathcal{Thom}(\cdot, \mathcal{D}b_X)$ being exact, they extend to the derived categories. Hence we obtain functors:

$$\begin{aligned} \cdot \overset{w}{\otimes} \mathcal{C}_X^\infty &: \mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X) \longrightarrow \mathbf{D}^b(\mathcal{D}_X) \\ \mathcal{Thom}(\cdot, \mathcal{D}b_X) &: \mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X)^{op} \longrightarrow \mathbf{D}^b(\mathcal{D}_X). \end{aligned}$$

- (v) Let $F \in \mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X)$ and denote by $D'F$ its dual, $D'F = R\mathcal{H}om(F, \mathbb{C}_X)$. then we have a commutative diagram:

$$\begin{array}{ccccc} D'F \otimes \mathcal{A}_X & \xrightarrow{\hspace{2cm}} & & R\mathcal{H}om(F, \mathcal{A}_X) & \\ \downarrow & & & \downarrow & \\ D'F \otimes \mathcal{C}_X^\infty & \xrightarrow{\hspace{1cm}} & D'F \overset{w}{\otimes} \mathcal{C}_X^\infty & \xrightarrow{\hspace{1cm}} & R\mathcal{H}om(F, \mathcal{C}_X^\infty) \\ \downarrow & & \downarrow & & \downarrow \\ D'F \otimes \mathcal{D}b_X & \xrightarrow{\hspace{1cm}} & \mathcal{Thom}(F, \mathcal{D}b_X) & \xrightarrow{\hspace{1cm}} & R\mathcal{H}om(F, \mathcal{D}b_X) \\ \downarrow & & & & \downarrow \\ D'F \otimes \mathcal{B}_X & \xrightarrow{\hspace{2cm}} & & R\mathcal{H}om(F, \mathcal{B}_X) & \end{array}$$

4 Operations on $\cdot \overset{w}{\otimes} \mathcal{C}_X^\infty$ and $\mathcal{Thom}(\cdot, \mathcal{D}b_X)$

Let $f : Y \longrightarrow X$ be a morphism of real analytic manifolds. As usual, one denotes by $\mathcal{D}_{Y \rightarrow X}$ (resp. $\mathcal{D}_{X \leftarrow Y}$) the sheaf $\mathcal{A}_Y \otimes_{f^{-1}\mathcal{A}_X} f^{-1}\mathcal{D}_X$ (resp. $\mathcal{A}_Y^\vee \otimes_{f^{-1}\mathcal{A}_X} f^{-1}(\mathcal{D}_X \otimes_{\mathcal{A}_X} \mathcal{A}^{\vee \otimes -1})$) endowed with its structure of a $(\mathcal{D}_Y, f^{-1}\mathcal{D}_X)$ -module (resp. $(f^{-1}\mathcal{D}_X, \mathcal{D}_Y)$ -module). Let $F \in \mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X)$ and let $G \in \mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_Y)$, with f proper on $\text{supp } G$. There are natural (iso-)morphisms:

$$\mathcal{D}_{Y \rightarrow X} \otimes_{f^{-1}\mathcal{D}_X}^L f^{-1}(F \overset{w}{\otimes} \mathcal{C}_X^\infty) \longrightarrow f^{-1}F \overset{w}{\otimes} \mathcal{C}_X^\infty \quad (4.1)$$

$$f^{-1}(F \overset{w}{\otimes} \mathcal{C}_X^\infty) \longrightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y \rightarrow X}, f^{-1}F \overset{w}{\otimes} \mathcal{C}_X^\infty) \quad (4.2)$$

$$f_*G \overset{w}{\otimes} \mathcal{C}_X^\infty \simeq f_*R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y \rightarrow X}, G \overset{w}{\otimes} \mathcal{C}_Y^\infty) \quad (4.3)$$

Morphism (4.2) is deduced from (4.1) by adjunction. Morphism (4.1) is constructed by using Theorem 2.1 which allows to reduce to the case where $F = \mathbb{C}_U$, U an open subanalytic subset of X . Then, setting $Z = X \setminus U$, one has the natural morphism:

$$\mathcal{A}_Y \otimes_{f^{-1}\mathcal{A}_X} f^{-1}\mathcal{I}_{Z,X}^\infty \longrightarrow \mathcal{I}_{f^{-1}(Z),Y}^\infty.$$

When f is a closed embreding, (4.1) is an isomorphism and when f is smooth, (4.2) is an isomorphism. Morphism (4.3) is deduced from (4.2) by adjunction. To

prove it is an isomorphism, one can treat separately the case of a closed embedding and that of a smooth map. In the first case, one is reduced to the case where Y is defined by an equation $\{t = 0\}$ and $G = \mathbb{C}_U$ for an open subanalytic subset U of Y . Then (4.3) follows from the isomorphism $\mathcal{I}_{X \setminus U, X}^\infty \simeq \mathcal{I}_{Y \setminus U, Y}^\infty[[t]]$. If f is smooth, one reduces to the case where $Y = X \times \mathbb{R}$, f is the projection and $G = \mathbb{C}_Z$ where Z is a closed subanalytic subset and the fibers of f on Z are closed intervals. Then one has to check that the sequence:

$$0 \longrightarrow \mathcal{I}_{f(Z), X}^\infty \longrightarrow f_* \mathcal{I}_{Z, Y}^\infty \xrightarrow{\partial/\partial t} f_* \mathcal{I}_{Z, Y}^\infty \longrightarrow 0$$

is exact (see [5]).

Similarly, one has a natural isomorphism, assuming f is proper on $\text{supp } G$:

$$f_!(\mathcal{H}om(G, \mathcal{D}b_Y^\vee) \otimes_{\mathcal{D}_Y}^L \mathcal{D}_{Y \rightarrow X}) \xrightarrow{\sim} \mathcal{H}om(Rf_! G, \mathcal{D}b_X^\vee), \quad (4.4)$$

a natural isomorphism, assuming f is smooth:

$$R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y \rightarrow X}, \mathcal{H}om(f^{-1}F, \mathcal{D}b_Y)) \xrightarrow{\sim} f^{-1}\mathcal{H}om(F, \mathcal{D}b_X), \quad (4.5)$$

and a natural isomorphism, assuming f is a closed embedding:

$$\mathcal{H}om(f^{-1}F, \mathcal{D}b_Y) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X \leftarrow Y}, \mathcal{H}om(F, \mathcal{D}b_X)). \quad (4.6)$$

5 The functors ${}^w\!\otimes \mathcal{O}_X$ and $\mathcal{H}om(\cdot, \mathcal{O}_X)$

Now assume X is a complex manifold, denote as usual by $\overline{X}$ the anti-holomorphic associated complex manifold, and identify X to the diagonal of $X \times \overline{X}$.

Definition 5.1 Let $F \in \mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X)$. One sets:

$$\begin{aligned} F \overset{w}{\otimes} \mathcal{O}_X &= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{w}{\otimes} \mathcal{C}_X^\infty) \\ \mathcal{H}om(F, \mathcal{O}_X) &= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{H}om(F, \mathcal{D}b_X)) \end{aligned}$$

Example 5.2 (i) Let M be a real analytic manifold, $i : M \hookrightarrow X$ a complexification. Then:

$$i_* F \overset{w}{\otimes} \mathcal{O}_X \simeq i_*(F \overset{w}{\otimes} \mathcal{C}_M^\infty) \quad (5.1)$$

$$\mathcal{H}om(i_* F, \Omega_X[d_X]) \simeq i_* \mathcal{H}om(F, \mathcal{D}b_M^\vee) \quad (5.2)$$

Note that isomorphism (5.2) is a result of Andronikof [1], which extends a previous theorem of Martineau (who treated the case when $F = \mathbb{C}_M$).

(ii) Let Z be a closed complex analytic subset of X . Then:

$$\mathbb{C}_Z \overset{w}{\otimes} \mathcal{O}_X \simeq \mathcal{O}_X \hat{\mid}_Z \quad (5.3)$$

$$\mathcal{T}hom(\mathbb{C}_Z, \mathcal{O}_Z) \simeq R\Gamma_{[Z]} \mathcal{O}_X \quad (5.4)$$

Here, $\mathcal{O}_X \hat{\mid}_Z$ denotes as usual the formal completion of $\mathcal{O}_X$ along Z , and $R\Gamma_{[Z]} \mathcal{O}_X$ the algebraic cohomology of $\mathcal{O}_X$ supported by Z (see [4]).

In order to recall the main operations on these functors, we shall follow the notations of [6] for $\mathcal{D}$ -modules. In particular if $f : Y \rightarrow X$ is a morphism of complex manifolds and if $\mathcal{M}$ (resp. $\mathcal{N}$) is a $\mathcal{D}_X$ (resp. $\mathcal{D}_Y$)-module, one sets:

$$\begin{aligned} \underline{f}^{-1} \mathcal{M} &= \mathcal{D}_{Y \rightarrow X} \otimes_{\underline{f}^{-1} \mathcal{D}_X}^L \underline{f}^{-1} \mathcal{M}, \\ \underline{f}_! \mathcal{N} &= Rf_! (\mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{D}_Y}^L \mathcal{N}). \end{aligned}$$

Then, using isomorphisms (4.1)-(4.6), we get the following results.

Assuming that f is smooth, we have natural isomorphisms:

$$\underline{f}^{-1}(F \overset{w}{\otimes} \mathcal{O}_X) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y \rightarrow X}, \underline{f}^{-1} F \overset{w}{\otimes} \mathcal{O}_Y), \quad (5.5)$$

$$R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y \rightarrow X}, \mathcal{T}hom(\underline{f}^{-1} F, \mathcal{O}_Y)) \simeq \underline{f}^{-1} \mathcal{T}hom(F, \mathcal{O}_X). \quad (5.6)$$

Assuming that f is a closed embedding, we have natural isomorphisms:

$$\underline{f}^{-1}(F \overset{w}{\otimes} \mathcal{O}_X) \simeq \underline{f}^{-1} F \overset{w}{\otimes} \mathcal{O}_Y, \quad (5.7)$$

$$\mathcal{T}hom(\underline{f}^{-1} F, \mathcal{O}_Y) \simeq \underline{f}^{-1} \mathcal{T}hom(F, \mathcal{O}_X). \quad (5.8)$$

Assuming that f is proper on $\text{supp } G$, we have natural isomorphisms:

$$f_! R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y \rightarrow X}, G \overset{w}{\otimes} \mathcal{O}_Y) \simeq f_! G \overset{w}{\otimes} \mathcal{O}_X, \quad (5.9)$$

$$\underline{f}_! \mathcal{T}hom(G, \Omega_Y[d_Y]) \simeq \mathcal{T}hom(Rf_! G, \Omega_X[d_X]). \quad (5.10)$$

Remark 5.3 The functor $\mathcal{T}hom(\cdot, \mathcal{O}_X)$ has been microlocalized by Andronikof ([1]). The specialization of the functor $\cdot \overset{w}{\otimes} \mathcal{O}_X$ is related to the notion of asymptotic expansions. This will be developed elsewhere.

6 Adjunction formulas

Consider a correspondence of complex manifolds:

$$\begin{array}{ccc} & S & \\ f \swarrow & & \searrow g \\ X & & Y \end{array}$$

let $\mathcal{M} \in \mathbf{D}_{good}^b(\mathcal{D}_X)$, (the triangulated subcategory of $\mathcal{M} \in \mathbf{D}_{good}^b(\mathcal{D}_X)$ generated by the objects whose cohomology groups are coherent and may be endowed with a good filtration on each compact subset of X), and let $G \in \mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_Y)$. Assume:

$$f \text{ is non characteristic for } \mathcal{M}, \quad (6.1)$$

$$f \text{ is proper over } g^{-1}(\text{supp } G), \quad (6.2)$$

$$g \text{ is proper over } f^{-1}(\text{supp } \mathcal{M}). \quad (6.3)$$

Introduce the notations:

$$\phi_S(G) = Rf_! g^{-1} G [d_S - d_X]$$

$$\underline{\phi}_S(\mathcal{M}) = \underline{g}_* \underline{f}^{-1} \mathcal{M}.$$

Theorem 6.1 Assume (6.1)-(6.3). Then there are natural isomorphisms:

$$\begin{aligned} R\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \phi_S(G) \overset{w}{\otimes} \mathcal{O}_X)) [d_X] &\simeq \\ R\Gamma(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\phi}_S(\mathcal{M}), G \overset{w}{\otimes} \mathcal{O}_Y)) [d_Y] & \\ R\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(\phi_S(G), \mathcal{O}_X))) [d_X] &\simeq \\ R\Gamma(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\phi}_S(\mathcal{M}), \mathcal{T}hom(G, \mathcal{O}_Y))) [d_Y]. & \end{aligned}$$

Sketch of proof

It is enough to treat the case where $S = Y$ and the case where $S = X$. Using the isomorphisms (5.5)-(5.10), the remaining problem is to prove that if $f : Y \rightarrow X$ is a morphism of complex manifolds, if $\mathcal{N} \in \mathbf{D}_{good}^b(\mathcal{D}_Y)$, f is proper over $\text{supp } \mathcal{N}$, and $F \in \mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_X)$, then there are natural isomorphisms:

$$\begin{aligned} Rf_! R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \underline{f}^{-1}(F \overset{w}{\otimes} \mathcal{O}_X)) [d_Y] &\simeq Rf_! R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_! \mathcal{N}, F \overset{w}{\otimes} \mathcal{O}_Y) [d_X], \\ Rf_! \mathcal{T}hom(f^{-1} F, \Omega_X) \otimes_{\mathcal{O}_Y}^L \mathcal{N} &\simeq \mathcal{T}hom(F, \Omega_X) \otimes_{\mathcal{O}_X}^L \underline{f}_! \mathcal{N}. \end{aligned}$$

The second isomorphism is deduced from the first one by a duality argument. The first isomorphism is equivalent to:

$$Rf_! R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \underline{f}^{-1}(F \overset{w}{\otimes} \mathcal{O}_X)) \simeq Rf_! R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, f^{-1}(F) \overset{w}{\otimes} \mathcal{O}_Y).$$

Then the proof follows the main lines of [11]. One reduces to the case where $Y = Z \times X$ and f is the projection, $\mathcal{N} = \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{G}$, for $\mathcal{G}$ a coherent $\mathcal{O}_Y$ -module, f being propre on $\text{supp } \mathcal{G}$. Then one has to prove the isomorphism:

$$Rf_! R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{G}, \mathcal{O}_Y) \otimes_{\mathcal{O}_X}^L (F \overset{w}{\otimes} \mathcal{O}_X) \simeq Rf_! R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{G}, f^{-1} F \overset{w}{\otimes} \mathcal{O}_Y).$$

Since one may represent $Rf_! R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{G}, \mathcal{O}_Y)$ by a complex of FN-free $\mathcal{O}_X$ -modules and $F \overset{w}{\otimes} \mathcal{O}_X$ is an $\mathcal{O}_X$ -module of type FN, one may apply Proposition 3.13 of [11], a variant of a theorem of Ramis-Ruget [10].

7 Application to the Radon transform

As an application of Theorem 6.1, we recall some results of [2].

Let $\mathbf{P}$ be a complex n -dimensional projective space, $\mathbf{P}^*$ the dual projective space, and $\mathbf{A}$ the hypersurface of $\mathbf{P} \times \mathbf{P}^*$ given by the incidence relation. If $[\xi] = [\xi_0, \dots, \xi_n]$ is a homogeneous coordinate system on $\mathbf{P}$, $[\eta] = [\eta_0, \dots, \eta_n]$ the dual system on $\mathbf{P}^*$, then:

$$\mathbf{A} = \{(\xi, \eta) \in \mathbf{P} \times \mathbf{P}^*; \langle \xi, \eta \rangle = 0\}.$$

Let us consider the correspondence:

$$\begin{array}{ccc} & \mathbf{A} & \\ f \swarrow & & \searrow g \\ \mathbf{P} & & \mathbf{P}^*, \end{array} \quad (7.1)$$

and denote by q_1 and q_2 the projections from $\mathbf{P} \times \mathbf{P}^*$ to $\mathbf{P}$ and $\mathbf{P}^*$, respectively.

For $k \in \mathbb{Z}$, we denote by $\mathcal{O}_{\mathbf{P}}(k)$ the $-k$ -th power of the tautological line bundle $\mathcal{O}_{\mathbf{P}}(-1)$, and we set:

$$\mathcal{D}_{\mathbf{P}}(k) = \mathcal{D}_{\mathbf{P}} \otimes_{\mathcal{O}_{\mathbf{P}}} \mathcal{O}_{\mathbf{P}}(k).$$

For $(k, k') \in \mathbb{Z} \times \mathbb{Z}$, we set $\mathcal{O}_{\mathbf{P} \times \mathbf{P}^*}(k, k') = \mathcal{O}_{\mathbf{P} \times \mathbf{P}^*} \otimes_{\mathcal{O}_{\mathbf{P}}} \mathcal{O}_{\mathbf{P}}(k) \otimes_{\mathcal{O}_{\mathbf{P}^*}} \mathcal{O}_{\mathbf{P}^*}(k')$. To $k \in \mathbb{Z}$, we associate:

$$k^* = -n - 1 - k.$$

In [8], Leray introduced:

$$\begin{aligned} \omega^*(\xi) &= \sum_{i=0}^n (-1)^i \xi_i d\xi_0 \wedge \dots \wedge \widehat{d\xi_i} \wedge \dots \wedge d\xi_n, \\ s_k &= s_k(\xi, \eta) = \frac{\omega^*(\xi)}{\langle \xi, \eta \rangle^{n+1+k}}. \end{aligned}$$

Then s_k is a well defined section of $\mathcal{O}_{\mathbf{P} \times \mathbf{P}^*}^{(n,0)}(-k, k^*)$ on $\mathbf{P} \times \mathbf{P}^* \setminus \mathbf{A}$. If $n+1+k > 0$ (i.e. if $k^* < 0$), s_k has meromorphic singularities on $\mathbf{A}$, and its image via the natural morphism

$$\Gamma(\mathbf{P} \times \mathbf{P}^* \setminus \mathbf{A}; \mathcal{O}_{\mathbf{P} \times \mathbf{P}^*}^{(n,0)}(-k, k^*)) \longrightarrow H_{\mathbf{A}}^1(\mathbf{P} \times \mathbf{P}^*; \mathcal{O}_{\mathbf{P} \times \mathbf{P}^*}^{(n,0)}(-k, k^*))$$

defines a section (that we denote by the same symbol):

$$s_k \in \Gamma(\mathbf{P} \times \mathbf{P}^*; \mathcal{B}_{\mathbf{A}}^{(n,0)}(-k, k^*)),$$

where $\mathcal{B}_{\mathbf{A}}^{(n,0)}(-k, k^*)$ is the sheaf $\mathcal{B}_{\mathbf{A}} = H_{[\mathbf{A}]}^1(\mathcal{O}_{\mathbf{P} \times \mathbf{P}^*})$ twisted by $\mathcal{O}_{\mathbf{P} \times \mathbf{P}^*}^{(n,0)}(-k, k^*)$. One sees easily that the Leray section s_k defines a $\mathcal{D}_{\mathbf{P}^*}$ -linear morphism

$$\alpha(s_k) : \mathcal{D}_{\mathbf{P}^*}(-k^*) \longrightarrow \phi_{\mathbf{A}}(\mathcal{D}_{\mathbf{P}}(-k)) \quad (7.2)$$

The main result of [2] is that if $-n - 1 < k < 0$, then $\alpha(s_k)$ is an isomorphism. Applying Theorem 6.1, we find for $F \in \mathbf{D}_{\mathbb{R}-c}^b(\mathbb{C}_{\mathbf{P}})$, the isomorphisms:

$$R\Gamma(\mathbf{P}; F \overset{\mathbf{w}}{\otimes} \mathcal{O}_{\mathbf{P}}(k)) \simeq \quad (7.3)$$

$$\begin{aligned} & R\Gamma(\mathbf{P}^*; \phi_{\mathbf{A}}(F) \overset{\mathbf{w}}{\otimes} \mathcal{O}_{\mathbf{P}^*}(k^*)) \\ R\Gamma(\mathbf{P}; \mathcal{T}hom(F, \mathcal{O}_{\mathbf{P}}(k))) & \simeq \quad (7.4) \\ & R\Gamma(\mathbf{P}^*; (\phi_{\mathcal{S}}(\mathcal{M}), \mathcal{T}hom(\phi_{\mathbf{A}}(F), \mathcal{O}_{\mathbf{P}^*}(k^*))). \end{aligned}$$

Denote by P and P^* a real projective space of dimension $n > 1$ and its dual, and consider $\mathbf{P}$ and $\mathbf{P}^*$ as complexifications of P and P^* . Let $k \in \mathbb{Z}$, and let $\varepsilon, \bar{\varepsilon} \in \{0, 1\}$ have different parity. We denote by $\mathcal{C}_P^\infty(k, \varepsilon)$ the locally constant sheaf of rank one over $\mathcal{C}_P^\infty$ whose global sections are represented by those functions f on $\mathbb{R}^{n+1} \setminus \{0\}$ satisfying the homogeneity condition:

$$f(\lambda x) = (\text{sgn } \lambda)^\varepsilon \lambda^k f(x), \quad \text{for } \lambda \neq 0.$$

Using explicit integral formulas, Gelfand et al. [3] proved the isomorphisms for $-n - 1 < k < 0$:

$$\Gamma(P; \mathcal{C}_P^\infty(k, \varepsilon)) \simeq \begin{cases} \Gamma(P^*; \mathcal{C}_{P^*}^\infty(k^*, \bar{\varepsilon})) & \text{for } n \text{ even,} \\ \Gamma(P^*; \mathcal{C}_{P^*}^\infty(k^*, \varepsilon)) & \text{for } n \text{ odd.} \end{cases}$$

We recover here these isomorphisms (and the similar ones with $\mathcal{C}^\infty$ replaced by $\mathcal{D}b$) by applying (7.3) to the case where either $F = \mathbb{C}_P$ or $F = K_P$, the canonical line bundle on P . In fact, $\mathcal{C}_P^\infty(k, 0) \simeq \mathbb{C}_P \otimes \mathcal{O}_{\mathbf{P}}(k)$ and $\mathcal{C}_P^\infty(k, 1) \simeq K_P \overset{\mathbf{w}}{\otimes} \mathcal{O}_{\mathbf{P}}(k)$.

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