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ON A LAGRANGE PROBLEM AND ITS GENERALIZATIONS

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On a Lagrange problem and its generalizations

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Introduction

The Lagrange problem consists in finding the plane curve which by its revolution about an axis in its plane determines the column of greatest efficiency. For column of unit length and volume, efficiency denotes the structure's resistance to buckling under axial compression. When λ is the magnitude of the axial load and u the resulting transverse deplacement, the potential energy is

$$\int_0^l EI|u''|^2 dx - \lambda \int_0^l |u'|^2 dx$$

with the two terms measuring bending and elongation respectively. Here I is the second moment of area of the column's cross section and E is its Young's modulus. For sufficiently small λ, the minimum of the potential energy over all admissible displacements is zero. That means that

$$\lambda_1 = \inf \int EI|u''|^2 dx / \int |u'|^2 dx,$$

where inf is taken in the class $H_0^2(0,1)$. The first order necessary conditions require then that the function u, on which the infimum is attained, satisfies the equation

$$(EIu'')'' + \lambda_1 u'' = 0,\ u(0) = u'(0) = u(1) = u'(1) = 0.$$

For the Lagrange problem the Young's module is assumed to be constant and, as the column is a solid of revolution, each cross section's second moment of area is simply a constant multiple of the square of its area A, i.e.

$I(x) = cA^2(x)$. (If the column is hollow, then $I(x) = cA^3(x)$.) We assume that $A(x)dx = 1$, i.e. the volume is unit.

The problem was attacked at first in 1962 by Keller and Tadjbakhsh. They received a necessary condition $A^4|u''|^2 = A^3$ and proved that $16\pi^2/3$ is the buckling load of the resulting column. In 1977 Olhoff and Rasmussen noted that the proof of Keller and Tadjbakhsh is not neat, because the first eigenvalue can be multiple and the least eigenvalue does not vary smoothly with A at points, where the multiplicity exceeds one. Cox and Overton attemted to prove that the solution of Keller and Tadjbakhsh is not optimal.

It is worth to remark that the article of Keller and Tadjbakhsh was very useful and the final result is correct in spite of some errors in the proofs. We have used other methods, independent of the multiplicity of the eigenvalues. We study just the mathematical problem of optimisation of the first eigenvalue under the general condition $I(x) = cA^q(x)$ with an arbitrary real q and find the solution. It is interesting to note that we need for this the Sobolev's type spaces, in which first or second derivatives are summable in some real (sometimes negative) power.

We will consider here also other close problems related with estimates of the eigen-values of an elliptic operator.

Close results for other boundary value problems were obtained in our works [10]-[13].

I. The Lagrange problem

The considered Lagrange problem consists in the finding of extremal values of the functional

$$L[Q,y] = \frac{\int_0^1 Q(x)y''(x)^2dx}{\int_0^1 y'(x)^2dx},$$

under the conditions

$$\int_0^1 Q(x)^\alpha dx = 1,\ Q(x) \geq 0, \tag{1}$$

$$y(0) = 0,\ y'(0) = 0,\ y(1) = 0,\ y'(1) = 0, \tag{2}$$

where α is a real, $\alpha \neq 0$. It is easy to see that this problem is equivalent to the variational problem: to find the extremum of the functional

$$F[Q,y] = \frac{\int_0^1 Q(x)y'(x)^2dx}{\int_0^1 y(x)^2dx},$$

under the conditions (1) and

$$y(0) = 0,\ y(1) = 0,\ \int_0^1 y(x)dx = 0. \qquad (3)$$

The Euler-Lagrange equation for the functional F has the form

$$(Q(x)y')' + \mu y = C,\ y(0) = 0,\ y(1) = 0.$$

Let $\alpha \in \mathbf{R}\setminus 0$ and K be the set of functions Q satisfying the conditions (1). Let

$$m_\alpha = \inf_{Q\in K}\inf_{y\in H} L[Q,y],\ M_\alpha = \sup_{Q\in K}\inf_{y\in H} L[Q,y].$$

Our aim is to find the values of m_α and M_α.

Our main result is following.

Theorem 1. *M_α is finite for $\alpha > -1/2$, $\alpha \neq 0$ and $M_\alpha = \infty$ for $\alpha \leq -1/2$; $m_\alpha > 0$ for $\alpha \leq -1$ and $m_\alpha = 0$ for $\alpha > -1$.*

We can prove also that the extremal values are attained on certain functions Q and y.

Theorem 2. *If $\alpha < -1$ then there exist a function $y \in H$ and a function Q satisfying (1), such that $y''^2(x) = Q(x)^{\alpha-1}$ and*

$$L[Q,y] = m_\alpha = \frac{4(2\alpha+1)}{\alpha}\Big(\frac{\alpha+1}{2\alpha+1}\Big)^{1-1/\alpha} B\Big(\frac{1}{2}, \frac{1}{2} + \frac{1}{2\alpha}\Big)^2,$$

where B is the Euler function.

If $1 > \alpha > -1/2$, $\alpha \neq 0$, then there exist a function $y_0 \in H$ and a function Q satisfying (1), such that $y_0''^2(x) = Q(x)^{\alpha-1}$ and

$$\inf_y L[Q,y] = L[Q,y_0] = M_\alpha.$$

If $\alpha \geq 1$, then

$$M_\alpha \leq \frac{4(2\alpha+1)}{\alpha}\Big(\frac{1+\alpha}{1+2\alpha}\Big)^{1-1/\alpha} B\Big(\frac{1}{2}, \frac{1}{2} + \frac{1}{2\alpha}\Big)^2.$$

If $0 < \alpha < 1$, *then*

$$M_\alpha = \frac{4(2\alpha+1)}{\alpha}\Big(\frac{1+\alpha}{1+2\alpha}\Big)^{1-1/\alpha} B\Big(\frac{1}{2}, \frac{1}{2} + \frac{1}{2\alpha}\Big)^2.$$

If $-1/2 < \alpha < 0$, *then*

$$M_\alpha = \frac{4(2\alpha+1)}{|\alpha|}\Big(\frac{1+\alpha}{1+2\alpha}\Big)^{1-1/\alpha}\Big(\int_0^\infty \frac{dt}{(1+t^2)^{1/2-1/2\alpha}}\Big)^2.$$

Corollary 1. *If* $\alpha = -1$, *then* $m = 16$, *but the extremal function* $Q(x)$ *does not exist.*

Corollary 2. *If* $\alpha = 1$, *then* $m = \lim_{\alpha \to 1} m(\alpha) = 48$. *The estimate is realized by the function* y_1 *such that* $y(x) = x$ *for* $0 < x < 1/4$, $y(x) = 1/2 - x$ *for* $1/4 < x < 3/4$ *and* $y(x) = x - 1$ *for* $3/4 < x < 1$.

Our proof uses, in particular, the following Lemma.

Lemma. *Let* $0 < p < 2/3$ *and* K *be the class of absolutely continuous non-negative in* $[0,h]$ *functions* y *of the space* $W_p^1(0,h)$ *such that* $y(0) = 0$, $y'(x) \geq 0$. *Let*

$$G[y] = \frac{(\int_0^h y'(x)^p dx)^{2/p}}{\int_0^h y(x)^2 dx}$$

and $m = \sup_{y \in K} G[y]$. *Then*

$$m = \Big(\frac{2-2p}{2-3p}\Big)^{2/p}\Big(\frac{2}{p} - 3\Big)h^{2/p-3}\Big(\int_0^\infty \frac{dt}{(1+t^2)^{1/p}}\Big)^2$$

and there exists an absolutely continuous monotone function $y_0(x)$ *such that* $y_0(0) = 0$ *and* $G[y_0] = m$.

Let $p < 0$ *and* K *the class of non-negative functions* y *of the space* $W_2^1(0,h)$ *such that* $y(0) = 0$, $y'(x) \geq 0$. *Let*

$$G[y] = \frac{(\int_0^h y'(x)^p dx)^{2/p}}{\int_0^h y(x)^2 dx}.$$

Let $m = \sup_{y \in K} G[y]$. *Then*

$$m = \frac{1}{4}\Big(\frac{2-2p}{2-3p}\Big)^{2/p}\Big(3 - \frac{2}{p}\Big)h^{2/p-3} B(1/2, 1 - 1/p)^2$$

and there exists an absolutely continuous monotone concave on $[0,1]$ *function* y_0 *such that* $y_0(0) = 0$ *and* $G[y_0] = m$.

Besides, we need to know the behavior of the solutions of the equations of the form

$$Q(x)y''(x) + my(x) = 0$$

in the points where Q vanishes.

In particular, in the classical Lagrange problem with $\alpha = 1/2$ the solution has the following form:

$$x = (2t + \sin 2t)/4\pi, \ Q(x) = 16\cos^4 t/9, \text{ where } 0 \le t \le 2\pi.$$

The optimal value $M = 16\pi^2/3$ was indicated by Keller-Tadjbaksh. The optimal column has two points, at which $Q(x)$ vanishes.

2. The Sturm-Liouville problems

Consider the problem: to find the extremal values of the functional

$$J[Q,y] = \frac{\int_0^1 y'(x)^2 dx}{\int_0^1 P(x)y(x)^2 dx}$$

in the class $K = H_0^1(0,1)$ of functions y and in the class L_α of positive functions P such that

$$\int_0^1 P(x)^\alpha dx = 1.$$

Our main result is the following theorem.

Theorem 3. *Let* λ_k *be the* $k-th$ *eigen-value of the Sturm-Liouville problem*

$$y'' + \lambda P(x)y = 0, \ y(0) = 0, \ y(1) = 0,$$

$k = 1, 2, \ldots, \ \lambda_1 \le \lambda_2 \le \ldots.$

If $\alpha > 1$, *then*

$$\inf_{P\in L} \lambda_1 \equiv m_\alpha = 4\left|\frac{\alpha}{2\alpha - 1}\right|^{1/\alpha}\Big(\int_0^1 \frac{dt}{\sqrt{1 - t^{2\alpha/(\alpha-1)}}}\Big)^2, \ \lambda_k \ge m_\alpha k^2.$$

If $\alpha = 1$, *then* $\inf_{P\in L} \lambda_1 = 4, \ \lambda_k \ge 4k^2$.
If $\alpha \le 1$, *then* $\inf_{P\in L} \lambda_k = 0$.

If $\alpha < 1/2$, *then*

$$\sup_{P\in L} \lambda_1 \equiv M_\alpha = 4\Big|\frac{\alpha}{2\alpha-1}\Big|^{1/\alpha}\Big(\int_0^1 \frac{dt}{\sqrt{1+t^{2\alpha/(\alpha-1)}}}\Big)^2, \ \lambda_k \le M_\alpha k^2.$$

If $\alpha \ge 1/2$, *then* $\sup_{P\in L} \lambda_1 = \infty$.

The close problem of finding extremal values of the functional

$$J_1[y] = \frac{(\int_0^1 p_1(x)|y^{(n)}(x)|^r dx)^{1/r}}{(\int_0^1 p_2(x)|y(x)|^s dx)^{1/s}},$$

where p_1, p_2 are given positive functions, $r > 1, s > 1$, was studied by Buslaev and Tikhomirov [14]. They proved that the extremal values of the functional J_1 coincide with Kolmogorov's diameter of the weighted Sobolev class W^n_{r,p_1} in the metrics of L_{s,p_2}.

A more general problem is to find the extremal values of the eigen-values of the problem:

$$(Q(x)y')' + \lambda P(x)y = 0, \ y(0) = 0, \ y(1) = 0,$$

if $P \in L_\alpha$, $Q \in L_\beta$. The class L_γ is defined as the set of positive in $[0,1]$ functions R such that

$$\int_0^1 R(x)^\gamma dx = 1.$$

The problem is to find

$$m_{\alpha,\beta} = \inf_{P\in L_\alpha, Q\in L_\beta} \lambda_1, \ M_{\alpha,\beta} = \sup_{P\in L_\alpha, Q\in L_\beta} \lambda_1.$$

Here we have obtained the following sharp result.

Theorem 4. *If* $\alpha < -1$ *and* $\beta > 1$ *then* $m_{\alpha,\beta} > 0$. *For all other values of the parameters* α, β *we have* $m_{\alpha,\beta} = 0$.

If $\alpha > 0$ *and* $1/\beta \le 1/\alpha + 2$ *or* $-1/2 < \alpha < 0$ *and* $1/\beta \ge 1/\alpha + 2$, *then* $M_{\alpha,\beta} < \infty$. *For all other values of the parameters* α, β *we have* $M_{\alpha,\beta} = \infty$.

Let us consider the problem:

$$-y'' + P(x)y = \lambda y, \ y(0) = y(1) = 0,$$

if $P \in L_\alpha$.

Let λ_1 be the first eigen-value, $\lambda_1 \le \lambda_2 \le \ldots$. Our aim is to find

$$m_\alpha = \inf_{P \in L_\alpha} \lambda_1, \; M_\alpha = \sup_{P \in L_\alpha} \lambda_1.$$

Theorem 5. *If $\alpha \ge 1$, then $M_\alpha = h_\alpha < \infty$. If $\alpha < 1$, then $M_\alpha = \infty$. If $\alpha > 0$, then $m_\alpha = \pi^2$. If $\alpha < 0$, then $m_\alpha = h_\alpha > \pi^2$.*
Here h_α is the least eigen-value of the following non-linear problem:

$$-y'' + y^{(\alpha+1)/(\alpha-1)} = \mu y, \; y(0) = y(1) = 0; \; \int_0^1 |y(x)|^{2\alpha/(\alpha-1)} dx = 1.$$

One can consider the same problem for the equation

$$-y'' - P(x)y = \lambda y.$$

Then we can prove the following statement.

Theorem 6. *If $\alpha \ge 1$, then $m_\alpha > -\infty$. If $\alpha < 1$, then $m_\alpha = -\infty$. For all real α, we have $\lambda_1 \le \pi^2$.*

One can also consider the following problem. Let us find the extremal values of the eigen-values in the problem:

$$y^{(m)} + \lambda P(x) y = 0, \; y(a_i) = y'(a_i) = \ldots = y^{(r_i - 1)}(a_i) = 0,$$

where r_i are odd for $1 \le i \le s$, $0 = a_1 < a_2 < \ldots < a_s = 1, r_1 + r_2 + \ldots + r_s = m, (-1)^{r_s+1} > 0$.

We are looking for the values of

$$m_\alpha = \inf_{P \in L_\alpha} \lambda_1, \; M_\alpha = \sup_{P \in L_\alpha} \lambda_1.$$

This problem is in general not self-adjoint. M.Krein has proved that if r_i are odd for all i, then the eigen-values of the problem are real, $0 < \lambda_1 < \lambda_2 \ldots$, and $\lambda_k \to +\infty$ as $k \to +\infty$.

Here we have the following result.

Theorem 7. *If $\alpha \ge 1$, then $M_\alpha = \infty$, $m_\alpha > 0$. If $\alpha < 1/m$, then $M_\alpha < \infty$, $m_\alpha = 0$. If $1/m \le \alpha < 1$, then $m_\alpha = 0, M_\alpha = \infty$.*

3. Differential operators of higher orders

Consider the boundary problem:

$$(-1)^m y^{(2m)} + \lambda P(x) y = 0,$$

$$y(0) = y'(0) = \ldots = y^{(m-1)}(0) = 0, \; y(1) = y'(1) = \ldots = y^{(m-1)}(1) = 0,$$

if $P \in L_\alpha$, i.e. $\int_0^1 P(x)^\alpha dx = 1$.

The problem is to find

$$m_{\alpha,k} = \inf_{P \in L_\alpha} \lambda_k, \; M_{\alpha,k} = \sup_{P \in L_\alpha} \lambda_k,$$

where λ_k is the $k-$th eigenvalue.

We have succeded to prove the following result.

Theorem 8. *If $\alpha \geq 1$, then*

$$\lambda_k \geq C_\alpha k^m, \; M_{\alpha,1} = \infty.$$

If $\alpha < 1/m$, then

$$\lambda_k \leq C_\alpha k^m, \; m_{\alpha,k} = 0.$$

If $1/m < \alpha < 1$, then

$$m_{\alpha,k} = 0, \; M_{\alpha,k} = \infty, \; k = 1, 2, \ldots$$

The close non-linear problem: to find the extremal values of the eigen-values in the following boundary problem:

$$(-1)^m y^{(2m)} + \mu y^{(\alpha+1)/(\alpha-1)} = 0, \; \alpha > 1,$$

$$y(0) = y'(0) = \ldots = y^{(m-1)}(0) = 0, \; y(1) = y'(1) = \ldots = y^{(m-1)}(1) = 0,$$

has been considered by Buslaev and Tikhomirov (see [14]).

The spectrum of the latter problem is continuous. However it is discrete, if one adds the condition

$$\int_0^1 |y(x)|^{2\alpha/(\alpha-1)} dx = 1.$$

The asymptotic properties of the eigen-values were studied in [14].

4. Multidimensional problems

Let Ω be a bounded domain in $\mathbf{R}^n$ and

$$(-\Delta)^m u = \lambda P(x)u, \; u \in W^m_{2,0}(\Omega),$$

where Δ is the Laplace operator, $m \geq 1$, the function P is positive and belongs to L_α, i.e. $\int_\Omega P(x)^\alpha dx = 1$.

The problem is to find

$$m_\alpha = \inf_{P \in L_\alpha} \lambda_1, \; M_\alpha = \sup_{P \in L_\alpha} \lambda_1.$$

Here we have the following result (see [10]).

Theorem 9. *If $n > 2m$, $\alpha > n/2m$, then $M_\alpha = \infty$, $m_\alpha > 0$.*
If $n > 2m$, $\alpha \leq n/2m$, then $m_\alpha = 0$.
If $n \leq 2m$, $\alpha \geq 1$, then $m_\alpha > 0$.

The following theorem shows that the regularity of the boundary is essential in this problem.

Let us recall that the Minkovsky dimension of a set A is defined as

$$\overline{\lim}_{\rho \to 0} \frac{\ln mes A_\rho}{\ln \rho},$$

where A_ρ is the ρ-neighbourhood of A.

Theorem 10. *If the boundary $\partial\Omega$ is smooth enough (for example, satisfies the Lipschitz condition, if $m = 1$) and $\alpha \leq 1/2m$, then $M_\alpha < \infty$.*

If $m = 1$, $\alpha \leq 1/2$, then $m_\alpha \geq \lambda$, where λ is the least eigen-value in the following problem:

$$\Delta u + \lambda u^{(\alpha+1)/(\alpha-1)} = 0, \; u|_{\partial K} = 0, \; \int_K |u(x)|^{2\alpha/(\alpha-1)} dx = 1$$

in the ball K whose measure is equal to the measure of Ω.

If $m = 1, n \geq 2$, the Minkovsky dimension of $\partial\Omega$ is μ and $\mu \geq 2\alpha$, then $M_\alpha < \infty$. However, for any μ such that $\mu < 2\alpha$ there exists a domain Ω with the Minkovsky dimension of $\partial\Omega$ equal to μ and such that $M_\alpha = \infty$.

Proof. If $\alpha < 0$, then

$$\int_\Omega u(x)^p dx \leq (\int_\Omega P(x)|u(x)|^2 dx)^{p/2} (\int_\Omega P(x)^\alpha dx)^{1/(1-\alpha)},$$

where $p = 2\alpha/(\alpha - 1) > 0$. Therefore

$$M_\alpha \leq \frac{\int_\Omega |D^m u_0|^2 dx}{(\int_\Omega |u_0|^p dx)^{2/p}},$$

where u_0 is a function whose integrals in the right-hand size of the latter inequality are finite. For instance, if $u_0 = d(x)^\gamma$ in some neighbourhood of the boundary, where $d(x)$ is the distance of x from the boundary, then it is true when

$$\gamma > m - 1/2,\ \gamma > -1/p.$$

If $0 < \alpha < 1$, then $p < 0$ and

$$\int_\Omega P(x)^\alpha dx \leq (\int_\Omega P(x) u(x)^2 dx)^\alpha (\int_\Omega u(x)^p dx)^{1-\alpha}.$$

In this case the function u_0 satisfies the conditions of convergence if

$$-1/p > \gamma > m - 1/2.$$

Therefore such a γ exists, if $1 > 2m\alpha$.

Now let $m = 1$. If the boundary of Ω is irregular and its Minkovsky content is $\mu < 1$, then the function u_0 satisfies the required conditions when

$$-\mu/p \geq \gamma \geq 1 - \mu/2,$$

i.e. when $\mu > 2\alpha$.

On the other hand, if $\mu < 2\alpha$, let us take as Ω the domain in the plane of x, y, contained in the square $0 < x < 1,\ 0 < y < 1$, and obtained from the square by removing the segments $x = A_n, 1/3 < y < 2/3$, where $A_n = k_0 \sum_{j=1}^n j^{-s}$, $s > 1$, $n = 1, 2, \ldots$ and k_0 is such that $k_0 \sum_{j=1}^\infty j^{-s} = 1$. Let us put $a_n = k_0 n^{-s}$, $Q(x, y) = h(y) b_n$, for $A_n < x < A_{n+1}$, where $h \in C_0^\infty(1/3, 2/3)$, $h(y) = 0$ for $y < 1/3$ and for $y > 2/3$, and the constants b_n are chosen so that

$$b_n a_n^2 \to 0 \text{ as } n \to \infty,\ \sum_{n=1}^\infty b_n^\alpha a_n = \infty.$$

Given $\varepsilon > 0$ one can take $Q_\varepsilon = 0$ for $x < A_m$ and $x > A_k$, where m is such that $b_m a_m^2 < \varepsilon$ and k is such that

$$\int_{A_m}^{A_k} \int_0^1 Q_\varepsilon(x, y)^\alpha dx dy \sim 1.$$

Then
$$\int_0^1\int_0^1 Q_\varepsilon(x)u(x)^2dxdy \le \varepsilon^2\int_0^1\int_0^1 |\nabla u(x)^2dxdy,$$
for all $u \in W_2^2(\Omega)$ and therefore $M_\alpha = \infty$. It is easy to see that the Minkovsky dimension μ of the boundary is equal to $1 - 1/s$. So we can put $b_n = n^{(s-1)/\alpha}$ and all conditions can be satisfied, if $\mu < 2\alpha$. □

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