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EQUATIONS AUX DERIVEES PARTIELLES

ON THE SINGULAR SPECTRUM OF DISCRETE SCHRÖDINGER OPERATOR

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Abstract. The problem under investigation is the structure of the singular spectrum of Discrete Schrödinger Operator with decaying potential. We will consider both cases of slowly decaying and quickly potentials. In the last case the potential is supposed to be separable one and so we actually study the Friedrichs model operators.

Keywords. Discrete Schrödinger Operator, singular spectrum, point spectrum, Hausdorff dimension, Friedrichs model.

Introduction. In Hilbert space $\ell^2(\mathbf{Z})$ of all square summable sequences $\mathcal{U} = \{\mathcal{U}_n\}_{n=-\infty}^{\infty}, \|\mathcal{U}\|^2 = \sum_n |\mathcal{U}_n|^2$, we will consider the bounded Discrete Schrödinger Operator L

$$(L\mathcal{U})_n := \mathcal{U}_{n+1} + \mathcal{U}_{n-1} + q_n\mathcal{U}_n \ , \ n \in \mathbf{Z} \ . \tag{1}$$

Here the potential $q = \{q_n\}_{n=-\infty}^{\infty}$ gives rise to the diagonal operator Q in $\ell^2(\mathbf{Z})$: $(Qu)_n = q_n\mathcal{U}_n$. In what follows we will suppose that Q is a compact operator ($q_n \underset{n\to\infty}{\longrightarrow} 0$) and in general selfadjoint one ($q_n \in \mathbf{R}$, $n \in \mathbf{Z}$), although the case of nonselfadjoint Q is also possible. By the well-known Weyl theorem [1] on the compact perturbation we have that the essential spectrum of $L(\mathfrak{S}_c(L))$ coincide with the spectrum of the unperturbed operator $L_0 : (L_0\mathcal{U})_n = \mathcal{U}_{n+1} + \mathcal{U}_{n-1}, n \in \mathbf{Z}$. Therefore $\mathfrak{S}_c(L) = [-2,2]$. Let us prove this elementary fact with the only purpose of introducing some notations and discuss the connection between the Discrete Schrödinger Operator and the Friedrichs model.

A straightforward calculation shows that under Fourier transform

$$\{\mathcal{U}_n\} \in \ell^2(\mathbf{Z}) \mapsto \mathcal{U}(t) := \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \mathcal{U}_n \ e^{int} \in L_2(0, 2\pi) \ ,$$

the operator (1) goes into the operator $\widetilde{L}$ in $L_2(0, 2\pi)$ of the following form

$$(\widetilde{L}u)(t) = (2\cos t) \ u(t) + \int_0^{2\pi} K(t,x) \ u(x)dx \tag{2}$$

with the kernel $K(t,x) = q(t-x)$. Here the function $q(t)$ is 2π-periodic in t and represents the Fourier series with Fourier coefficients $q_n : q(t) := \sum_{n=-\infty}^{\infty} q_n e^{int}$. Of course the operators L and $\widetilde{L}$ are unitary equivalent. Formula (2) provides us an example of so-called generalized Friedrichs model [2] : the perturbation of the operator of multiplication by some function of independent variable t by an integral operator, which of course is not necessary of convolution in general. Obviously the continuous spectrum of $\widetilde{L}$ coincides with the continuous spectrum of the operator of multiplication by the function 2cost in $L_2(0.2\pi)$ and hence $\mathfrak{S}_c(L) = [-2,2]$. In the case that the sequence $\{q_n\}$ is real we have that the kernel $K(t,x)$ is Hermitian $(K(t,x) = \overline{K(x,t)})$ and the operator $\widetilde{L}$ selfadjoint. In what follows we frequently will additionally suppose that the according integral operator is a positive selfadjoint operator in Hilbert space. In the case of Discrete Schrödinger Operator this means that the sequence q_n is positive. Moreover it will be interesting to extend the class of operator under investigation, including the so-called separable potentials [3] :

$$K(X,t) = \sum_K \tilde{q}_k \varphi_k(x)\overline{\varphi_k(t)}\ ,\ \varphi_k \in H := L_2(0,2\pi)$$

That means that the operator of perturbation in (2) is not convolution operator in H but finite rank operator of the general type. Of course the belonging of Q in (1) to the class of finite rank operator leads to the fact that the sequense q_n is finite and don't give us any interesting examples. So the extension of the class of perturbations Q is of immediate interest to the singular spectrum perturbation theory. Then there is well-studied connection between the decaying property of the potential $\{q_n\}$ and the smoothness of Fourier series $q(t)$ with Fourier coefficients $\{q_n\}$ [4]. In general, the more rapidly decreasing potential we have, the more smooth is the function $q(t)$ and therefore the kernel $K(t,x)$ by both variables on the Cartesian product of two unit circles $\mathbf{T}$ rather than on $[0,2\pi]\times[0,2\pi]$. Moreover we will speak about the decaying property of the potential even in the case of the separable potential. In the last case this means that the finite rank kernel $K(t,x)$ in (2) is "sufficiently" smooth function by both variables on $\mathbf{T}\times\mathbf{T}$. The correspondence between the decaying properties of q_n and the smoothness of $q(t)$ was investigated by numerous mathematicians [4].

Now the main questions under consideration in the paper are the following ones. What is the singular spectrum structure of the Discrete Schrödinger Operator L on the continuous one ? How does it depends on the decaying properties of $\{q_n\}$, or the smoothness properties of the kernel $K(t,x)$ in the representation (2). There are many papers devoted to the investigation of the singular spectrum of L from the different viewpoints. For example B. Simon proved in [5] that if the potential q_n satisfies the estimate $q_n = 0(n^{\varepsilon-1/2})$ with some $\varepsilon > 0$ than in typical situation the spectrum of L is pure point (with probability 1 in some rigorous sense) and therefore the point spectrum is dense on the interval [-2,2]. But the approach of the paper [5] is not constructive. On the contrary D. Pearson [6] in the case of Schrödinger Operator in $L_2(\mathbf{R})$ gave a constructive example of Schrödinger Operator with decaying potential (but very slowly) and pure singular continuous spectrum. See also the paper [7] where have been investigated D.S.O. without absolutely continuous component of their spectrum but with unbounded potentials q_n.

Let us consider two rather different classes of decaying potentials $q_n \underset{n\to\infty}{\longrightarrow} 0$. The first one consists of quickly decaying potentials q_n such that

$$\sum_n |q_n| < \infty \tag{3}$$

In that case the operator Q belongs to the trace class γ_1 and by according to Kato-Rosenblum theorem [8] an absolutely continuous spectrum L $\mathfrak{S}_{a.c.}(L)$ coinsides with [-2,2]. Moreover under condition (3) there are no eigenvalues inside the open interval

$(-2,2)$ $(\mathfrak{S}_p(L) \cap (-2,2)) = \phi$ and the spectrum is free of the singular continuous component. Since this class is not interesting for the purposes of the investigation of the singular spectrum on the continuous one, we will substitute the diagonal operator Q in formula (1) by the selfadjoint finite rank (or even trace class) operator of the general type, i.e. we will speak about the "decaying potential", keeping in mind that the decaying" , properties means here the appropriate smoothness of Hermitian nonnegative kernel $K(t,x)$ in (2).

The second class consists of so-called slowly decaying potential while the sequence q_n does not belong to the class ℓ^1. Here the spectrum transform under perturbation of this class is very complicated, so we will bound ourself by the class of diagonal perturbation Q in (1). In what follows we will represent a survey of some results, mainly from the paper [10], [11], concerning Discrete Schrödinger Operator with potentials from both classes.

1. Quickly decaying potentials.

Consider the operator $\widetilde{L}$ in Fourier representation $H := L_2(0,2\pi)$

$$(\widetilde{L}u)(t) = (2\cos t)u(t) + (Vu)(t)$$

with separable potential V. The operator V is supposed to be selfadjoint, nonnegative and belonging to the trace class γ_1. So we will not restrict ourself by the class of finite rank operator V. The condition $V \geq 0$ is not very essential here but convenient for simplicity of the statements. Therefore the operator V is an integral operator with kernel $K(t,x)$:

$$(Vu)(t) = \int_0^{2\pi} K(t,x)u(x)dx \ .$$

Let us assume that the function $K(t,x)$ is Hermitian $(K(t,x) = \overline{K(x,t)})$, periodic both in t and x with period 2π. The condition $V \geq 0$ means that

$$\sum_{i,k=1}^{N} K(x_i,x_k)\xi_i\xi_k \geq 0$$

for arbitrary finite sequence of complex numbers $\{\xi_i\}_{i=1}^N$ and arbitrary set $\{x_i\}_{i=1}^N \subset (0,2\pi)$ (see [12]). This operator have been studied by L.D. Faddeev [14] and B. Pavlov, S. Petras [13] where in particular the following facts were established. If $K(t,x)$ belongs to the class Lip α with $\alpha > 1/2$ by both variables then the singular continuous component is absent in the spectrum of $\widetilde{L}$ and the point spectrum $\mathfrak{S}_p(L)$ is finite. But for $\alpha < 1/2$ the nontrivial singular continuous spectrum and infinite point spectrum may occur [13]. In the case of rank one perturbation $(K(t,x) = \mathcal{S}(t)\overline{\mathcal{S}(x)}, \mathcal{S} \in H)$ the structure of the singular spectrum was completely investigated in [13]. The answer for the problem of studing of the singular spectrum for rank $V \geq 2$ (Faddeev-Pavlov problem [14]) is contained in the following theorem [10].

Theorem 1.— *Let $V \in \gamma_1, V \geq 0$, and suppose that the kernel of perturbation $K(t,x)$ is periodic in t and x with period 2π and satisfies the condition ($\alpha < 1/2$)*

$$K(x+h, x+h) + K(x,x) - K(x+h, x) - K(x, x+h) \leq C|h|^{2\alpha}, x, h \in \mathbf{R} \; . \quad (4)$$

Then for arbitrary $\varepsilon > 0$ the singular spectrum $\mathfrak{S}_s(\widetilde{L})$ satisfies the estimate

$$mes\{\mathfrak{S}_s(\widetilde{L}) \cap (-2+\varepsilon, 2-\varepsilon)\}^{\delta} = 0(\delta^{2\alpha}), \delta \to 0 \; , \quad (5)$$

where mes Λ^{δ} *being Lebesgue measure on* $\mathbf{R}$ *of the* δ*-neighbourhood of the set* $\Lambda \subset \mathbf{R}$.

Give some comments to the conditions (4), (5).

1. The first one means that the kernel $K(t,x)$ satisfies Lipschitz condition of index α. Actually if we define by $K_{1/2}(t,x)$ the kernel of integral operator $V^{1/2} \geq 0$ square root of the operator V, then

$$\int_0^{2\pi} |K_{1/2}(t+h, x) - K_{1/2}(t,x)|^2 dx \leq C^{1/2}|h|^{\alpha} \; ; \; t, h \in \mathbf{R}$$

with the same constant C as in (4). Of course if rank $V < \infty$ then the last condition is equivalent to belonging $K(t,x)$ to the class Lip α by both variables.

2. The condition (5) leads to the following estimate for Hausdorff dimension [16] of the set $\mathfrak{S}_s(L)$ on $\mathbf{R}$

$$H - \dim \mathfrak{S}_s(L) \leq 1 - 2\alpha \; ,$$

depending on the smoothness index $\alpha < 1/2$. Here we see that only interval $\alpha \in [0, 1/2]$ is interesting for the styding of the singular spectrum. If $\alpha \geq 1/2$ then the singular spectrum coincides with the finite set of eigenvalues of finite multiplicity [19].

3. Moreover if we have a sequence λ_n of eigenvalues of $\widetilde{L}$ such that $\lambda_n = \text{const}/n^{\gamma}, n \in \mathbf{N}$, then $\gamma \geq \frac{2\alpha}{1-2\alpha}$ and counterexamples show that this estimate is sharp.

4. We will touch briefly of the singular spectrum structure near the points $-2, 2$, i.e. near the boundary of the continuous spectrum. This situation has been investigated by S. Yakovlev [17]. We will mention only one result emphasizing the difference between the structures of the singular spectrum near the boundary of the continuous one and inside of the interval $[-2.2]$. Let the condition (4) be statisfied and $\lambda_n = 2 - c/n^{\gamma}, c > 0$, be the sequence of eigenvalues of $\widetilde{L}$ accumulating at the boundary of the interval $[-2, 2]$, then $\gamma \geq 4\alpha/1 - 2\alpha$ and this estimate is sharp and can not be improved even for rank one perturbation V. So we have the doubling of the critical degree for the sequence of eigenvalues accumulating to the boundary of

the continuous spectrum. Of course this effect is closely connected with the following fact : $(2\cos t)' = 0 \Leftrightarrow 2\cos t = \pm 2$.

The proof of Theorem 1 is founded on the detailed investigation of the root set of some operator-valued analytic function $M(\lambda)$ on the upper half plane $\lambda \in \mathbf{C}_+$. Here $M(\lambda)$ is defined by

$$M(\lambda) = I + V^{1/2}\left(\frac{1}{2\cos t - \lambda}\right)V^{1/2}, \lambda \in \mathbf{C} \setminus [-2,2] ,$$

$$M(\lambda) : H \to H \quad \text{for every} \quad \lambda \in \mathbf{C} \setminus [-2,2] .$$

Moreover $M(\lambda)$ is a so-called operator-valued R-function (or Nevanlinna class function) [10], i.e. $M(\lambda)$ is analytic by λ in $\mathbf{C}_+$ and $\mathrm{Im}\, M(\lambda) \equiv (M^*(\lambda) - M(\lambda))/2i \geq 0$, $\lambda \in \mathbf{C}_+$. In the case of $V \in \gamma_1$ we have that $(M(\lambda) - I) \in \gamma_1, \lambda \in \mathbf{C}_+$ and the smoothness condition (4) leads [10] to Lipshitz continuity in the nuclear norm

$$\|M(\lambda^{-1}) - M(\lambda)\|_{\gamma_1} \leq \mathrm{const}|\lambda^1 - \lambda|^\alpha, \mathrm{Im}\ \lambda\ ,\ \lambda^1 \geq 0\ .$$

Therefore it is possible to introduce the set of all roots of the function $M(\lambda)$:

$$\Lambda := \{k \in \mathbf{R} : \exists e \in H\ , e \neq 0\ ,\ M(k)e = 0\}$$

Here $M(k) := s\lim_{\varepsilon \to +0} M(k + i\varepsilon)$. It is very easy to chek that

$$\Lambda \supset \mathfrak{S}_s(L)$$

so our study of the singular spectrum structure can be reduced to the investigation of the root set Λ for fixed operator-valued R-function M. The main instruments for that investigation are the elaborated theory of the boundary behaviour for the arbitrary γ_1-valued R-function [10] and the sharp estimates of the norm $M^{-1}(\lambda)$ in the neighbourhood of the fixed root $k \in \Lambda$. For $k \in \Lambda \cap (-2+\varepsilon, 2+\varepsilon), \varepsilon > 0$, we have :

$$\|M^{-1}(\lambda)\|^{-1} = 0(|\lambda - k|^{2\alpha})\ ,\ \lambda \in \mathbf{C}_+ \tag{6}$$

It is necessary to remark that it is very easy to prove the estimate

$$\|M^{-1}(\lambda)\|^{-1} = 0(|\lambda - k|^\alpha)\ ,$$

but the doubling of degree $\alpha \longmapsto 2\alpha$ here needs very elaborated techniques. The bound (6) enables us to estimate the Hausdorff dimension for the set $\Lambda \cap (-2+\varepsilon, 2-\varepsilon)$ and therefore for $\mathfrak{S}_s(L) \cap (-2+\varepsilon, 2-\varepsilon)$ too. At first such estimate was proved in paper [13] for V, rank $V = 1$, when $M(\lambda)$ is actually a scalar function. But there are many difficulties even for rank $V = 2$ when $M(\lambda)$ could be reduced to 2×2 matrix function.

2. Slowly decaying potential.

Now consider the alternative situation of a slowly decaying potential $\sum_n |q_n| = \infty$. In what follows we will study the point spectrum of L (with diagonal operator Q) in the continuous one. It is well-known [18] that if the potential q_n decreasing at infinity more quickly than Coulomb potential ($q_n = 0(1/n)$) then there are no eigenvalues in the interval $(-2,2)$. The situation at boundary points $-2,2$ require a separate consideration. But in the case of Coulomb like potential, decreasing at infinity as $0(\frac{1}{n}), n \to \infty$, it is possible to give a constructive example of L with arbitrary prescribed finite set of eigenvalues in $(-2,2)$. For the continuous analog-Schrödinger operator in $L_2(\mathbf{R})$ we can refer to the paper [9]. As it was mentioned earlier for slowly decreasing potential the spectrum of L could be pure point [5]. What will be with spectrum of L if $q_n \underset{n\to\infty}{\longrightarrow} 0$ slowly than Coulomb-like potential ? Specifically let $q_n n$ tends to infinity arbitrarily slowly. Can the point spectrum $\mathfrak{S}_p(L)$ be dense on the interval $[-2,2]$? The following theorem [11] provides us the positive answer on the previous question.

Theorem 2.1.— *For arbitrary positive sequence $A_n \underset{n\to\infty}{\longrightarrow} \infty$ (arbitrary slowly) there exists a no negative potential $\{q_n\}$, such that*

$$q_n \leq A_n/|n| \ , \ n \in \mathbf{Z} \tag{7}$$

and Discrete Schrödinger Operator L with the potential q_n has dense point spectrum on $[-2,2]$.

2) Actually we can construct the examples of the potential satisfing the condition (7), whose point spectrum included an arbitrary infinite set of eigenvalues $\{\lambda_1, \lambda_2, \cdots\} \subset (-2,2)$ with only condition having technical character : for arbitrary $n \in \mathbf{N}$ the set $\{\pi, \arccos \lambda_{1/2}, \cdots, \arccos \lambda_{n/2}\}$ is rationally independent.

Remind that a set of real numbers $\{a_1, \cdots, a_n\}$ is rationally independent if and only if the equality $\sum_{k=1}^n p_k a_k = 0$ with rational coefficients $p_k \in Q \ , k = 1, \cdots, n$ leads to that $p_k = 0 \ , k = 1, \cdots, n$.

The proof is connected with some ideas of the dynamical system theory, ergodic theory and group theory. For example the milestone of the proof is the property of ergodicity of the winding of the N-dimensional torus with $N \to \infty$. So we have in general that for the potential decreasing at infinity slowler than Coulomb like potential its point spectrum could densely cover the interval $[-2,2]$: $\mathfrak{S}_c(L) = \overline{\mathfrak{S}_p(L)}$. Therefore we see that there exists a sharp jump in the behavior of the singular spectrum (its point component) when the decreasing of q_n at infinity goes through Coulomb boundary. Actually there are no eigenvalues on $(-2,2)$ if $n \ q_n \underset{n\to\infty}{\longrightarrow} 0$ but it can occur the dense point spectrum for the potential with $nq_n \underset{n\to\infty}{\longrightarrow} \infty$ arbitrarily slowly.

Finally we briefly argue the intermediate situation of Coulomb like potentials : $q_n = 0(\frac{1}{n}), n \to \infty$. We will mention a few elementary facts concerning the problem of existing of an eigenvalue of L at some point $\lambda \in (-2, 2)$ depending on two factors : decreasing property of the potential and the distance between λ and the boundary of the continuous spectrum ± 2.

Theorem 3.—

1) Let $0 < \lambda, 2$ and the potential $\{q_n\}$ satisfies the condition

$$\sum_n \exp\{-\frac{1}{\sqrt{1-\lambda^{2/4}}} \sum_{k=1}^{n} |q_k|\} = \infty ,$$

then $(-\lambda\, \lambda) \cap \mathfrak{S}_p(L) = \phi$.

2) On the other hand if

$$\sum_n \exp\{-\frac{1}{\sqrt{1-\lambda^{2/4}}} \sum_{k=1}^{n} |q_k|\} < \infty$$

then for every $\mu \in (-2, -\lambda) \cup (\lambda, 2)$ such that the set $\{\pi$, $\arccos \mu/2\}$ is rationally independent (of course this additional condition is not essential and looks like technical one) there exists some potential $\{q'_n\}$, equivalent to $\{q_n\}$, so that Discrete Schrödinger Operator L' with the potential $\{q'_n\}$ has μ its eigenvalue.

Above we have mentioned the notion of the equivalence between two potentials. By the last one we understand that

$$\sum_{k=1}^{n} |q_k| / \sum_{k=1}^{n} |q'_k| \underset{n\to\infty}{\longrightarrow} 1 \tag{8}$$

Actually for constractive examples in section 2 of the theorem 3 we have resemblance between q_n and q'_n even more close than it was reflected in the condition (8).

Corollary.— *As the direct consequence of the theorem 3 the following criteria of the absence of point spectrum on $(-2, 2)$ holds. If for every $C > 0$*

$$\sum_{n=1}^{\infty} \exp\{-C \sum_{k=1}^{n} |q_k|\} = \infty \tag{9}$$

then $\mathfrak{S}_p(L) \cap (-2, 2) = \phi$. But if the condition (9) is failed for some $C > 0$ than there exists a potential q'_n, equivalent to q_n, such that Discrete Schrödinger Operator L' with the potential q'_n has an eigenvalue in the interval $(-2, 2)$.

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