Séminaire Équations aux dérivées partielles - École Polytechnique

L. Stoyanov

Regularity properties of the generalized hamiltonian flow

Séminaire Équations aux dérivées partielles (Polytechnique) (1992-1993), exp. n° 6, p. 1-10

<http://www.numdam.org/item?id=SEDP_1992-1993____A6_0>

CENTRE
DE
MATHEMATIQUES

Unité de Recherche Associée D 0169

ECOLE POLYTECHNIQUE

F-91128 PALAISEAU Cedex (FRANCE)
Tél. (1) 69 33 40 91
Fax (1) 69 33 30 19 ; Télex 601.596 F

Séminaire 1992-1993

EQUATIONS AUX DERIVEES PARTIELLES

REGULARITY PROPERTIES OF THE GENERALIZED HAMILTONIAN FLOW

L. STOYANOV

Exposé n° VI

17 novembre 1992

1 Introduction

Let S be a symplectic manifold with boundary ∂S and let $p : S \longrightarrow \mathbf{R}$ be a smooth (C^∞) function with $dp_{|\partial S} \neq 0$. Following [MS] (see also sec. 24.3 in [H]), one defines the generalized Hamiltonian flow of p as follows.

Let $\varphi \in C^\infty(S)$ be a defining function of ∂S, i.e. $\varphi > 0$ in $S \setminus \partial S$ and $\varphi = 0$ on ∂S (φ might be only locally defined around ∂S). Assume that

$$\{\varphi, \{\varphi, p\}\} \neq 0.$$

We are going to define the flow of p on the zero level set

$$\Sigma = p^{-1}(0).$$

Consider the folowing subsets of Σ:

$$G = \{\sigma \in \Sigma : \varphi(\sigma) = H_p\varphi(\sigma) = 0\} \quad \text{(glancing set)},$$

$$G_d = \{\sigma \in G : H_p^2\varphi(\sigma) > 0\} \quad \text{(diffractive set)},$$

$$G_g = \{\sigma \in G : H_p^2\varphi(\sigma) < 0\} \quad \text{(gliding set)},$$

$$G^k = \{\sigma \in G : H_p^j\varphi(\sigma) = 0 \ \ \forall \ \ j = 0, 1, \dots, k-1\},$$

$$G^\infty = \bigcap_{k=2}^{\infty} G^k.$$

The *gliding vector field* H_p^G on G is defined by

$$H_p^G = H_p + \frac{H_p^2\varphi}{H_\varphi^2 p} H_\varphi.$$

Definition ([MS]). Let $I \subset \mathbf{R}$ be an interval. A curve $\gamma : I \longrightarrow \Sigma$ is called a *generalized integral curve (bicharacteristic)* of p if there exists a discrete subset B of I such that:

(i) if $t \in I \setminus B$ and $\gamma(t) \in (S \setminus \partial S) \cup G_d$, then there exists

$$\gamma'(t) = H_p(\gamma(t));$$

(ii) if $t \in I \setminus B$ and $\gamma(t) \in G \setminus G_d$, then there exists

$$\gamma'(t) = H_p^G(\gamma(t));$$

(iii) for each $t \in B$, $\gamma(t+s) \in S \setminus \partial S$ for all small $s \neq 0$ and there exist the limits $\gamma(t-0) \neq \gamma(t+0)$ which are points of one and the same integral curve of φ on ∂S.

Clearly, such a curve γ has discontinuities at the points of B. To get a continuous curve we have to identify some pairs of points on ∂S. Consider the following equivalence relation on Σ: $x \sim y$ iff either $x = y$ or $x \in \Sigma \cap \partial S$, $y \in \Sigma \cap \partial S$ and x and y lie on one and the same integral curve of φ on ∂S. The quotient space $\tilde{\Sigma} = \Sigma / \sim$, which carries a natural structure of a manifold with boundary, is called *compressed characteristic set* and the projection $\tilde{\gamma}$ of a generalized integral curve γ on $\tilde{\Sigma}$ is a continuous curve called *compressed integral curve* of p.

In what follows **we assume that**

$$G^{\infty} = \emptyset.$$

In this case one can define a flow

$$F_t : \tilde{\Sigma} \longrightarrow \tilde{\Sigma} \quad , \ t \in \mathbf{R},$$

such that $\{F_t : t \in \mathbf{R}\}$ is a compressed integral curve of p for each $\sigma \in \tilde{\Sigma}$ (cf. [MS]). It was shown in [MS] that the maps F_t are continuous.

Remark. It is clear from the definition that the maps F_t depend on φ. In general φ is only locally defined and so in such cases $\{F_t\}$ is a local flow defined for small $|t|$. However, the integral curves of p, disregarding their parametrization, are globally defined and do not depend on φ. To avoid the inconvenience caused by the change of the parameter along integral curves, one may consider maps between cross-sections of a given integral curve (the same definition as that of a Poincaré map). Since the problem we deal with below is of local nature, and locally the maps between cross-sections and F_t have equivalent behaviour, we consider the maps F_t as if they were globally defined.

Note that in general the maps F_t are not smooth. This is easily seen for

$$S = T^{*}(\Omega \times \mathbf{R}), \tag{1}$$

Ω being a domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, and p given by

$$p(x, \xi) = \sum_{i=1}^{n} \xi_i^2 - \xi_{n+1}^2. \tag{2}$$

An elementary argument shows that if Ω is the interior or the exterior of a ball in $\mathbf{R}^n$, then the maps F_t are Hölder continuous with Hölder exponent $\frac{1}{2}$, and $\frac{1}{2}$ is the maximal number with this property.

It is natural to ask if the maps F_t are Hölder continuous in the general case. In the present talk we consider some partial results in this direction.

Given S and p as in the beginning of this section, fix an arbitrary metric d on $\tilde{\Sigma}$ generating its topology.

Theorem 1. *Let $\rho_0 \in \tilde{\Sigma}$ and $T_0 > 0$ be fixed. There exist constants $C > 0$ and $\alpha > 0$ such that*

$$d(F_t\rho_0, F_t\rho) \leq C(d(\rho_0, \rho))^\alpha \tag{3}$$

for every $\rho \in \tilde{\Sigma}$ and every t with $|t| \leq T_0$.

For $k = 2, 3, \dots$ denote

$$G_+^k = \{\sigma \in G^k : H_p^k \varphi(\sigma) > 0\}.$$

Theorem 2. *Let K be a compact subset of Σ and $T_0 > 0$ be such that*

$$F_t(K) \subset G_g \cup \cup_{k=2}^{\infty} G_+^k \quad \forall t \in [0, T_0]. \tag{4}$$

Denote by $\tilde{K}$ the projection of K in $\tilde{\Sigma}$. Then there exist constants $C > 0$ and $\alpha > 0$ such that

$$d(F_t\sigma, F_t\rho) \leq C(d(\sigma, \rho))^\alpha$$

for all $\sigma, \rho \in \tilde{K}$ and $t \in [0, T_0]$.

It is natural to expect that the assertion of Theorem 2 remains true without assuming (4). Actually the proof of Theorem 2 is much easier than that of Theorem 1. That is why below we restrict our attention to Theorem 1. A scheme of its proof is given in section 3.

2 Motivation

In this section we briefly discuss a problem comming from the scattering theory, which indicates that regularity properties of the generalized Hamiltonian flow might be useful.

Let Ω be a domain in $\mathbf{R}^n, n \geq 3, n$ odd, with C^∞ boundary $\partial\Omega$ such that

$$K = \overline{\mathbf{R}^n \setminus \Omega}$$

is compact. Define S and p by (1) and (2), respectively.

The *scattering operator* related to the wave equation in $\mathbf{R} \times \Omega$ with Dirichlet boundary conditions on $\mathbf{R} \times \partial\Omega$ can be represented as an unitary operator

$$S : L^2(\mathbf{R} \times S^{n-1}) \longrightarrow L^2(\mathbf{R} \times S^{n-1})$$

(see [LP1]). The kernel of $S-\mathrm{Id}$, which can be considered as a distribution

$$s_K(t,\theta,\omega) \in \mathcal{D}'(\mathbf{R} \times S^{n-1} \times S^{n-1}),$$

is called the *scattering kernel.*

The following **problem** arises: is it true that there exists subset R of full Lebesgue measure in $S^{n-1} \times S^{n-1}$ such that

$$\operatorname{sing\,supp} s_K(t,\theta,\omega) = \{-T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}\} \tag{5}$$

for all $(\omega,\theta) \in R$? Here $\mathcal{L}_{\omega,\theta}$ is the set of all (ω,θ)*-rays* in Ω, i.e. infinite continuous curves in Ω with incoming direction ω and outgoing direction θ which are projections of generalized integral curves of p in S. By T_γ we denote the sojourn time of $\gamma \in \mathcal{L}_{\omega,\theta}$ (see [PS1] or ch. 1 in [PS2] for the precise definitions). There is no doubt that the right-hand side of (5) contains certain geometric information about the obstacle K, and so if (5) holds for sufficiently many pairs (ω,θ), one could get this information knowing the singularities of the scattering kernel for the same pairs (ω,θ). It is already known that this can be done for a special class of obstacles K. More precisely, the answer to the above question is affirmative, provided K is a finite union of disjoint convex bodies ([PS3]). Using this fact, it was shown in [S] that if K and L are two obstacles, each of them being a finite disjoint union of convex bodies, satisfying an additional condition (H) of M. Ikawa [I], and if

$$\operatorname{sing\,supp} s_K(t,\theta,\omega) = \operatorname{sing\,supp} s_L(t,\theta,\omega)$$

for almost all $(\omega,\theta) \in S^{n-1} \times S^{n-1}$, then $K = L$. For convex obstacles K and L such a result was established by Majda [Ma] (see also Lax and Phillips [LP2]).

Turning back to the question posed above, let us consider one possible way to deal with it. In fact, it follows from the results in [PS3] that to give an affirmative answer, it is sufficient to establish the existence of a set R of full Lebesgue measure in $S^{n-1} \times S^{n-1}$ such that for $(\omega,\theta) \in R$ there are no (ω,θ)-rays of mixed type in Ω, i.e. (ω,θ)-rays having non-trivial segments lying on $\partial\Omega$. To do so consider a fixed (ω,θ)-ray γ of mixed type in Ω and take a point (z,ζ) contained in a gliding segment of γ (lying entirely on $\partial\Omega$). Denote by G_t the generalized geodesic flow on $S^*(\Omega)$ generated by the flow F_t, $S^*(\Omega)$ being the cosphere bundle of Ω. Then taking a sufficiently large rational number $q > 0$, we have

$$\omega = \mathrm{pr}_2 G_{-q}(z,\zeta) \quad , \quad \theta = \mathrm{pr}_2 G_q(z,\zeta),$$

where $\mathrm{pr}_2(y,\eta) = \eta$. This can be written as

$$(\omega,\theta) = W_q(z,\zeta),$$

$W_q : S^*_{\partial\Omega}(\Omega) \longrightarrow S^{n-1} \times S^{n-1}$ being defined by

$$W_q(y,\eta) = (\mathrm{pr}_2 G_{-q}(y,\eta), \mathrm{pr}_2 G_q(y,\eta)).$$

The choice of (z,ζ) now shows that

$$(\omega,\theta) \in W_q(S^*(\partial\Omega)). \tag{6}$$

More generally, it is clear that the existence of a (ω,θ)-ray of mixed type is equivalent to the existence of a rational $q > 0$ with (6). Consequntly, the set of those pairs (ω,θ) for which there exist (ω,θ)-rays of mixed type is contained in

$$R_0 = \bigcup_{q\in\mathbf{Q}, q>0} W_q(S^*(\partial\Omega)).$$

Since

$$\dim S^*_{\partial\Omega}(\Omega) = \dim S^{n-1} \times S^{n-1} = 2(n-1),$$

and $\dim S^*(\partial\Omega) = 2n-3$, it is natural to expect that $W_q(S^*(\partial\Omega))$ has Lebesgue measure zero in $S^{n-1} \times S^{n-1}$. This will be so provided W_q has some "good" regularity properties, which could be eventually derived from corresponding properties of the flows G_t and F_t. Unfortunately, our Theorems 1 and 2 do not provide such properties.

3 Sketch of the proof of Theorem 1

A standard compactness argument shows that the assertion of Theorem 1 is a consequence of the following (local) lemma.

Lemma 1. *Let $\rho_0 \in \tilde{\Sigma}$ be fixed. There exist a neighbourhood U_0 of ρ_0 in $\tilde{\Sigma}$ and constants $T > 0, C > 0, \alpha > 0$ such that* (3) *holds for all $\rho \in U_0$ and $t \in [0,T]$.*

Denote again by ρ_0 an element of Σ the projection of which in $\tilde{\Sigma}$ coincides with ρ_0. It follows by [MS] (cf. also sec. 24.3 in [H]) that there exist local coordinates

$$(x,\xi) = (x_1,\ldots,x_n;\xi_1,\ldots,\xi_n)$$

around $\rho_0 = (0,0)$ in S such that $\varphi = x_1$, i.e. locally

$$S = \{(x,\xi) : x_1 \geq 0\} \quad , \quad \partial S = \{(x,\xi) : x_1 = 0\},$$

and

$$p(x,\xi) = \xi_1^2 - r(x,\xi'),$$

r being a smooth function. Throughout we use the notation

$$x' = (x_2, \dots, x_n) \quad , \quad \xi' = (\xi_2, \dots, \xi_n).$$

Define the metric d by

$$d((x,\xi),(y,\eta)) = \max_{1 \le i \le n} \max\{|x_i - y_i|, |\xi_i - \eta_i|\},$$

and set

$$F_t(x,\xi) = (x(t), \xi(t)).$$

There are several cases for ρ_0.

case 1. $\rho_0 \in S \setminus \partial S$. In this case locally around ρ_0 the generalized integral curves of p coincide with the integral curves of the Hamiltonian vector field H_p, so the assertion follows trivially with $\alpha = 1$.

case 2. $\rho_0 \in G_d$. This means that $\frac{\partial r}{\partial x_1}(\rho_0) > 0$. Then there exists a neighbourhood V_0 of ρ_0 in S and a constant $c > 0$ with $\frac{\partial r}{\partial x_1}(\rho) \ge c$ for all $\rho \in V_0$. Choose a neighbourhood U_0 of ρ_0 and $T > 0$ such that $F_t(U_0) \subset V_0$ for all $t \in [0,T]$. It then follows by Lemma 24.3.4 in [H] that for each $\rho \in U_0$ the generalized integral curve $\{F_t\rho : t \in [0,T]\}$ has at most one reflection. Using this one can easily derive that the assertion of the lemma holds with $\alpha = \frac{1}{2}$.

case 3. $\rho_0 \in G_g$. As in the previous case, we find neighbourhoods $U_0 \subset V_0$ of ρ_0 and $c > 0$ such that $\frac{\partial r}{\partial x_1}(\rho) \le -c$ for each $\rho \in V_0$. Using Lemma 24.3.5 from [H] we find a constant $C' > 0$ such that if $\{F_t\rho : t \in [0,T]\}$ is a reflecting bicharacteristic (in this case it is equivalent to say that the bicharacteristic is not entirely contained in G_g), then we have

$$\eta_1^2(t) + y_1(t) \le C'(\eta_1^2(0) + y_1(0))$$

for all $t \in [0,T]$, where

$$F_t(\rho) = (y(t); \eta(t)).$$

From this the assertion of the lemma follows easily with $\alpha = \frac{1}{2}$.

case 4. $\rho_0 \in G^k \setminus G^{k+1}, k \ge 3$. Let $(\tilde{x}'(t), \tilde{\xi}'(t))$ be the integral curve of the vector field H_p^G on G with initial conditions $\tilde{x}'(0) = x'(0)$, $\tilde{\xi}'(0) = \xi'(0)$. Set

$$e(t) = \frac{\partial r}{\partial x_1}(0, \tilde{x}'(t), \tilde{\xi}'(t)),$$

$$f(t) = |x'(t) - \tilde{x}'(t)| + |\xi'(t) - \tilde{\xi}'(t)|.$$

Given $\rho \in \Sigma$, define $e_\rho(t)$ and $f_\rho(t)$ as $e(t)$ and $f(t)$, respectively, replacing ρ_0 with ρ.

Choose neighbourhoods $U_0 \subset V_0$ of ρ_0 and $T > 0$ so small that H_p^k has a constant sign in V_0 and $F_t U_0 \subset V_0$ for all $t \in [0, T]$. Later we will have to eventually take smaller U_0 and T.

In the case under consideration we have

$$e(t) = at^{k-2} + \lambda(t)t^{k-1}$$

for some constant $a \neq 0$ and some smooth function $\lambda(t)$ (cf. [MS] or [H]). Fix $L > 0$ with

$$|\lambda(t)| \leq \frac{L}{2} \quad , \quad |\lambda'(t)| \leq \frac{L}{2} \quad \forall t \in [0, T].$$

Using standard facts from the theory of differential equations, it follows that if U_0 is small enough, then there exists a constant $c > 0$ such that for every $\rho \in U_0$ we have the representation

$$e_\rho(t) = a_0 + a_1 t + \ldots + a_{k-2}t^{k-2} + at^{k-2} + \mu(t)t^{k-1} \tag{7}$$

with

$$|a_i| \leq c\delta \quad \forall i = 0, 1, \ldots, k-2\,;\; |\mu(t)| \leq L\,,\; |\mu'(t)| \leq L \quad \forall t \in [0, T], \tag{8}$$

where

$$\delta = d(\rho_0, \rho). \tag{9}$$

We may assume that $T \leq \frac{1}{2}$, then (7) and (8) imply

$$at^{k-2} - 2c\delta - Lt^{k-1} \leq e_\rho(t) \leq at^{k-2} + 2c\delta + Lt^{k-1} \quad , \quad t \in [0, T]. \tag{10}$$

Next, we distinguish two subcases.

Subcase 4.1. $a < 0$. Fix an arbitrary $\beta > 0$. The assertion of Lemma 1 follows immediately from the following

Lemma 2. *U_0 and $T > 0$ can be chosen so small that there exists a constant $A > 0$ with*

$$d(F_t\rho_0, F_t\rho) \leq A(d(\rho_0, \rho))^{\frac{1-\beta}{2}} \quad \forall \rho \in U_0 \ , \ \forall t \in [0, T].$$

Proof of Lemma 2. Set

$$\epsilon = \frac{\beta(k-2)}{1 + (1+\beta)(k-2)}$$

and choose $T > 0$ such that

$$T \leq \frac{\epsilon|a|}{2(k+1)L}.$$

Take $\rho \in U_0$ and set $\delta = d(\rho_0, \rho)$. Then $y_1(0) \leq \delta, |\eta_1(0)| \leq \delta$. The choice of T yields

$$(1 - \frac{\epsilon}{2})at^{k-2} - 2c\delta \leq e_\rho(t) \leq (1 + \frac{\epsilon}{2})at^{k-2} + 2c\delta$$

for all $t \in [0, T]$. Using the inequalities (24.3.7) in [H], it is not hard to see that there exists a constant $C_1 > 0$ such that

$$f_\rho(t) \leq C_1(\delta + |a|t^{k+1}) \quad , \quad y_1(t) \leq C_1(\delta + |a|t^k) \quad , t \in [0, T].$$

Set

$$h(t) = \frac{\partial r}{\partial x_1}(y(t), \eta'(t)),$$

and note that there exists a constant $C_2 > 0$, which does not depend on ρ, with

$$|h(t) - e_\rho(t)| = |\frac{\partial r}{\partial x_1}(y(t), \eta'(t)) - \frac{\partial r}{\partial x_1}(0, \tilde{y}'(t), \tilde{\eta}'(t))| \leq C_2(f_\rho(t) + y_1(t))$$

for $t \in [0, T]$ (cf. p. 436 in [H]). Consequently, one finds a constant $C_0 > 0$ with

$$(1 - \frac{\epsilon}{2})at^{k-2} - C_0\delta \leq h(t) \leq (1 + \frac{\epsilon}{2})at^{k-2} + C_0\delta \tag{11}$$

for $t \in [0, T]$.

As in [H], we see that

$$|h'(t) - e'_\rho(t)| \leq \text{const}(f_\rho(t) + y_1(t) + |\eta_1(t)|)$$

for all $t \in [0, T]$ for which $h'(t)$ exists. Using an argument similar to that above, we find a constant $C_0 > 0$ (we may assume this is the same constant as in (11)) such that

$$(k - 2 - \frac{\epsilon}{2})at^{k-3} - C_0\delta \leq h'(t) \leq (k - 2 + \frac{\epsilon}{2})at^{k-3} + C_0\delta \tag{12}$$

for all $t \in [0, T]$.

Consider the function

$$g(t) = \eta_1^2(t) - y_1(t)h(t).$$

It is clearly continuous and $g'(t)$ exists almost everywhere in $[0, T]$. Set

$$t_\delta = (\frac{2C_0\delta}{\epsilon|a|})^{\frac{1}{k-2}} = \text{const}\delta^{\frac{1}{k-2}}.$$

For those $t \in [t_\delta, T]$ for which $g'(t)$ exists, (11) and (12) imply

$$\frac{g'(t)}{g(t)} \leq \frac{k-2+\epsilon}{(1-\epsilon)t},$$

and integrating the latter inequality gives

$$g(t) \leq \text{const}\frac{g(t_\delta)}{\delta^{\frac{k-2+\epsilon}{(1-\epsilon)(k-2)}}} = \text{const}\frac{g(t_\delta)}{\delta^{1+\beta}}$$

for $t \in [t_\delta, T]$.

On the other hand, it follows easily by the definition of t_δ that $g(t) \leq \text{const}\delta^2$ for $t \in [0, t_\delta]$. Therefore $g(t) \leq \text{const}\delta^{1-\beta}$ for all $t \in [0, T]$. Consequently, $y_1(t) \leq \text{const}\delta^{1-\beta}$ and $|\eta_1(t)| \leq \text{const}\delta^{\frac{1-\beta}{2}}$ in $[0, T]$. Applying a standard argument from the theory of differential equations to the rest of coordinate functions, one gets

$$d(F_t\rho_0, F_t\rho) \leq \text{const}\delta^{\frac{1-\beta}{2}}$$

for all $t \in [0, T]$. This completes the proof of Lemma 2.

Subcase 4.2. $a > 0$. This case is easier than the previous one. One can define t_δ in a similar way and show that the integral curve $F_t\rho$ has no reflections for $t \in [t_\delta, T]$, provided U_0 is small enough and $\rho \in U_0$. In this way we find

$$d(F_t\rho_0, F_t\rho) \leq \text{const}\delta^{\frac{1}{k-2}}$$

for all $t \in [0, T]$.

References

[I] M. Ikawa: *Decay of solutions of the wave equation in the exterior of several strictly convex bodies.* Ann. Inst. Fourier **38** (1988), 113-146.

[H] L. Hörmander: *The Analysis of Linear Partial Differential Operators*, vol. III, Springer-Verlag, Berlin, 1985.

[LP1] P.Lax, R.Phillips: *Scattering Theory.* Academic Press, New York, 1967.

[LP2] P.Lax, R.Phillips: *The scattering of sound waves by an obstacle.* Comm. Pure Appl. Math. **30** (1977), 195-233.

[Ma] A.Majda: *A representation formula for the scattering operator and the inverse problem for arbitrary bodies.* Comm. Pure Appl. Math. **30** (1977), 165-194.

[MS] R. Melrose, J. Sjöstrand: *Singularities in boundary value problems*, I. Comm. Pure Appl. Math. **31** (1978), 593-617.

[PS1] V. Petkov, L. Stoyanov: *Singularities of the scattering kernel and scattering invariants for several strictly convex obstacles.* Trans. Amer. Math. Soc. **312**, (1989), 203-235.

[PS2] V. Petkov, L. Stoyanov: *Geometry of Reflecting Rays and Inverse Spectral Problems.* John Wiley & Sons, Chichester, 1992.

[PS3] V. Petkov, L. Stoyanov: *Sojourn times of trapping rays and the behaviour of the modified resolvent of the Laplacian.* University of Bordeaux I – C.N.R.S., Preprint 9209, 1992.

[S] L. Stoyanov: *An inverse scattering result for several convex bodies.* Preprint, 1992.

Institute of Mathematics,
1090 Sofia, Bulgaria

Current address:
Department of Mathematics,
University of Western Australia,
Nedlands, Perth 6009,
Western Australia
stoyanov@maths.uwa.edu.au