

SÉMINAIRE ÉQUATIONS AUX DÉRIVÉES PARTIELLES – ÉCOLE POLYTECHNIQUE

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Séminaire Équations aux dérivées partielles (Polytechnique) (1991-1992), exp. n° 9, p. 1-9

http://www.numdam.org/item?id=SEDP_1991-1992__A9_0

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Séminaire 1991-1992

EQUATIONS AUX DERIVEES PARTIELLES

SOME GLOBALLY STABLE APPROXIMATIONS FOR THE NAVIER-STOKES EQUATIONS AND FOR SOME OTHER EQUATIONS OF VISCOUS INCOMPRESSIBLE FLUIDS

Olga LADYZHENSKAYA

We describe some approximations for the two-dimensional Navier-Stokes equations which are globally stable and have the minimal global B -attractors ($= MIGBA_s$) in ε -vicinities of the $MIGBA$ for the investigated problem. In the end of the lecture we point out some other equations for which we have analogous results.

In all our theorems Ω is any bounded domain (b.d.) in $\mathbf{R}^2$.

1 The Galerkin-Faedo (G.-F.) approximations.

Consider the problem

$$(1_1) \quad \partial_t v(t) - \nu \Delta v(t) + v(t) \cdot \nabla v(t) = -\nabla p(t) + f ,$$

$$(1_2) \quad \operatorname{div} v(t) = 0 , \quad v(t)|_{\partial\Omega} = 0 , \quad v(0) = \varphi ,$$

in a b.d. $\Omega \subset \mathbf{R}^2$ and $t \in \mathbf{R}^+ = [0, \infty)$. Here $v(t) : \Omega \rightarrow \mathbf{R}^2$ and $p(t) : \Omega \rightarrow \mathbf{R}^1$ are unknown functions, $v(t) \cdot \nabla v(t) = \sum_{k=1}^2 v_k(t) \partial_{x_k} v(t)$, $v_k(t)$ ($k = 1, 2$) are components of $v(t)$, ν is a positive constant, φ and f are known functions independent on t , $\partial_t v$ and $\partial_{x_k} v$ are partial derivatives of v in t and x_k .

Let H_0 be the closure in the norm $\|\cdot\|$ of $L^2 \equiv L^2(\Omega, \mathbf{R}^2)$ of the set

$$J^{\bullet\infty}(\Omega) = \{u | u \in C^\infty(\Omega, \mathbf{R}^2), \operatorname{div} u = 0, \operatorname{supp} u \text{ is a compact in } \Omega\} ,$$

H_1 be the closure of $J^{\bullet\infty}(\Omega)$ in the norm $\|\cdot\|_1$ of Dirichlet integral, i.e. in the norm

$$\|u\|_1 = \left[\int_{\Omega} \sum_{i,k=1}^2 (\partial_{x_k} u_i(x))^2 dx \right]^{1/2} = \|\partial_x u\| ,$$

and H_{-1} be the dual space to H_1 relative to H_0 with the standard norm $\|\cdot\|_{-1}$. We denote by (u, v) the inner product u and v in H_0 .

It is known ([1]-[3]) that for any $f \in H_{-1}$ the solution operators $V_t : \varphi \rightarrow v(t, \varphi)$ for the problem (1_k) exist in the whole H_0 and form the continuous bounded semi-group $\{V_t, t \in \mathbf{R}^+, H_0\}$. For it, the ball

$$B_R = \{u | u \in H_0, \|u\| \leq R\}, R > R_0^- = (\nu \sqrt{\lambda_1})^{-1} \|f\|_{-1},$$

is B -absorbing set and the intersection

$$\bigcap_{t \geq 0} V_t(B_R) \equiv \mathfrak{M}$$

is MIGBA. Here $-\lambda_1$ is the first eigenvalue of Stokes operator $\tilde{\Delta}$ with $\mathcal{D}(\tilde{\Delta}) \subset H_1$ ([4], ch.2). $\mathfrak{M}$ is a bounded subset of H_1 and there is a majorant ϕ_1 for

$$(2) \quad \sup_{\varphi \in \mathfrak{M}} \sup_{t \in \mathbf{R}^1} \{ \|\varphi\|_1, \|\partial_t v(t, \varphi)\|, \int_t^{t+1} \|\partial_{x\tau}^2 v(\tau, \varphi)\|^2 d\tau \} \leq \phi_1(\|f\|_{-1}, \nu^{-1}).$$

These facts and some other properties of $\mathfrak{M}$ (for $f \in H_0$) were proved by us in [1] (see also [2],[3]). But the method of proving the basic estimates given in [1] required a smoothness of $\partial\Omega$ and could be applied only to the Rothe approximations and to the G.-F. approximations with the eigenfunctions $\{\varphi_k\}_{k=1}^\infty$ of $\tilde{\Delta}$ as the coordinate functions in H_1 . In [5],[6] we have given an other method of estimating the solutions to the Navier-Stokes equations which can be applied directly to the G.-F. approximations with the arbitrary coordinate $\{\psi_k\}_{k=1}^\infty$ in H_1 .

Denote by $v^m(t, \varphi) = \sum_{k=1}^m c_k^m(t, \varphi) \psi_k$ the G.-F. approximations and by $V_t^m : \varphi \rightarrow v^m(t, \varphi)$ the solution operators for the G.-F. equations. The family $\{V_t^m, t \in \mathbf{R}^+, H_0^m\}$ for each $m = 1, 2, \dots$, is a continuous semi-group. Here $H_0^m = \text{span} \{\psi_1, \dots, \psi_m\}$ is considered as a subspace of H_0 . The following facts are true :

Theorem 1.— *Let Ω be a b.d. in $\mathbf{R}^2$ and $f \in H_{-1}$. The Galerkin-Faedo approximations with arbitrary coordinate functions in H_1 have MIGBAs $\mathfrak{M}^m (m = 1, 2, \dots)$ lying in H_0^m and having the properties :*

$$(3) \quad \sup_{\varphi \in \mathfrak{M}^m} \sup_{t \in \mathbf{R}^1} \{ \|\varphi\|_1, \|\partial_t v^m(t, \varphi)\|, \int_t^{t+1} \|\partial_{x\tau}^2 v^m(\tau, \varphi)\|^2 d\tau \} \leq \phi_1(\|f\|_{-1}, \nu^{-1})$$

with the same ϕ_1 as in (2). For any $\varepsilon > 0$ exists a number $m(\varepsilon) \in \mathbf{N}^+$ such that

$$(4) \quad \mathfrak{M}^m \subset 0_\varepsilon(\mathfrak{M}) \quad \text{for } m \geq m(\varepsilon).$$

Here $0_\varepsilon(\mathfrak{M})$ is the ε -vicinity of $\mathfrak{M}$ in H_0 ■

It is useful to bear in mind the following known fact :

Lemma 1.— *If $\{\varphi_k\}_{k=1}^\infty$ is a coordinate system in $\mathring{W}_2^2(\Omega)$ then $\{\psi_k = \nabla^\perp \varphi_k \equiv (-\partial_{x_2} \varphi_k, \partial_{x_1} \varphi_k)\}_{k=1}^\infty$ is the coordinate system in H_1 . The inverse statement is also true : if $\{\psi_k\}_{k=1}^\infty$ is a coordinate system in H_1 , then each ψ_k determines a function $\varphi_k \in \mathring{W}_2^2(\Omega)$ and $\{\varphi_k\}_{k=1}^\infty$ is a coordinate system in $\mathring{W}_2^2(\Omega)$ ■*

The proof of Theorem 1 is based on some a priori estimates for v^m . They are the same as for solutions $v(t)$ of problem (1_k) proved in [5] (see also [6]). These estimates are derived only from the inequalities :

$$(5_1) \quad \frac{1}{2} \frac{d}{dt} \|v(t)\|^2 + \nu \|\partial_x v(t)\|^2 = (f, v(t)) \leq \|f\|_{-1} \|\partial_x v(t)\|,$$

$$(5_2) \quad \|\partial_t v(t)\|^2 + \frac{\nu}{2} \frac{d}{dt} \|\partial_x v(t)\|^2 = -(v(t). \nabla v(t), \partial_t v(t)) + \\ + (f, \partial_t v(t)) \leq \|v(t)\|_{4,\Omega} \|\partial_x v(t)\| \|\partial_t v(t)\|_{4,\Omega} + \|f\|_{-1} \|\partial_{xt}^2 v(t)\| ,$$

$$(5_3) \quad \frac{1}{2} \frac{d}{dt} \|\partial_t v(t)\|^2 + \nu \|\partial_{xt}^2 v(t)\| = -(\partial_t v(t). \nabla v(t) + v(t). \nabla \partial_t v(t), \\ \partial_t v(t)) = -(\partial_t v(t). \nabla v(t), \partial_t v(t)) \leq \|\partial_t v(t)\|_{4,\Omega}^2 \|\partial_x v(t)\| .$$

Here $\|\cdot\|_{4,\Omega}$ is the standard norm in $L^4(\Omega; \mathbf{R}^2)$.

The assertion (4) can be proved by *reductio ad absurdum*. Suppose that (4) is not true. Then there is an $\varepsilon > 0$ and a sequence $a_{m_j} \in \mathfrak{M}^{m_j}$, $m_j \rightarrow \infty$, such that

$$(4') \quad \text{dist} \{a_{m_j}, \mathfrak{M}\} > \varepsilon .$$

Due to (3) the set $\bigcup_{m=1}^{\infty} \mathfrak{M}^m \cup \mathfrak{M}$ lies in the ball $B_{\phi_1}(H_1)$ of H_1 (with radius $\phi_1 = \phi_1(\|f\|_{-1}, \nu^{-1})$), is a precompact in H_0 and lies in a ball $B_{R_1}(H_0)$ of the space H_0 . In particular, $a_{m_j} \in B_{R_1}(H_0) \cap B_{\phi_1}(H_1)$. Choose $T \in \mathbf{R}^+$ such that

$$(4'') \quad V_T(B_{R_1}(H_0)) \subset 0_{\varepsilon/2}(\mathfrak{M}).$$

Each a_{m_j} determines a $\varphi_{m_j} \in \mathfrak{M}^{m_j}$ for which $V_T^{m_j}(\varphi_{m_j}) = a_{m_j}$. The set $\{\varphi_{m_j}\}$ also belongs to $B_{R_1}(H_0) \cap B_{\phi_1}(H_1)$. Therefore we can choose a subsequence $\{\varphi_{m'_j}\}$ converging to a φ in the space H_0 and $\varphi \in B_{R_1}(H_0) \cap B_{\phi_1}(H_1)$. We have for $\{V_t^{m'_j}(\varphi_{m'_j})\}$ the estimates (3). They permit to extract a subsequence m''_j for which $V_t^{m''_j}(\varphi_{m''_j})$ converge to a $v(t)$ uniformly in $t \in [0, T]$ in the norm of H_0 . In particular,

$$(4''') \quad \|V_T^{m''_j}(\varphi_{m''_j}) - v(T)\| < \frac{\varepsilon}{2} \quad \text{for} \quad m''_j \geq m_0 .$$

Following standard arguments we prove that $v(t)$ is the solution $V_t(\varphi)$ of the problem (1_k) with $\varphi \in B_{R_1}(H_0)$. Due to (4'') and (4''') $\text{dist} \{a_{m''_j}, \mathfrak{M}\} < \varepsilon$, but this contradicts to the hypothesis (4').

Remark : The MIGBAs which we have in this lecture are invariant compact connected sets in the phase spaces chosen by us. They have all properties of $\mathfrak{M}$ for the problem (1_k) proved in [1]-[3]. We can give for them common majorants for the number of determining modes and for their fractal dimensions.

2 A discretization of t .

For the study and computations of attractors some discretizations of t can be useful. One of them for the F.-G. approximations has the form

$$(6) \quad (v_t^m(t), \psi_k) + \nu(\nabla v^m(t), \nabla \psi_k) + (v^m(t - \tau). \nabla v^m(t), \psi_k) = \\ = (f, \psi_k), \quad k = 1, 2, \dots, m, \quad v^m(0) = \varphi^m,$$

for $t = \ell\tau, \ell \in \mathbb{N}^+, \tau = \text{const} > 0$; $v_t^m(t) = \tau^{-1}[v^m(t) - v^m(t - \tau)]$. The systems (6) determine successively in $t = \tau, 2\tau, \dots$, the velocity fields $v^m(t, \varphi^m, \tau)$. The solution operators $V_t^{m, \tau} : \varphi \rightarrow v^m(t, \varphi, \tau)$ form the discrete semi-group $\{V_{\ell\tau}^{m, \tau}, \ell \in \mathbb{N}^+, H_0^m\}$. It has MIGBA $\mathfrak{M}^{m, \tau}$. Let $\tau = \tau_k \rightarrow 0$ when $k \rightarrow \infty$.

Theorem 2.— *Let the conditions of Theorem 1 be fulfilled and $\mathfrak{M}^m$ be attractors from Theorem 1. For any $\delta > 0$ exists a number $n(\delta, m) \in \mathbb{N}^+$ such that*

$$\mathfrak{M}^{m, \tau_k} \subset 0_\delta(\mathfrak{M}^m) \quad \text{for } k \geq n(\delta, m).$$

Here $0_\delta(\mathfrak{M}^m)$ is δ -vicinity of $\mathfrak{M}^m$ in H_0^m and $\tau_k \rightarrow 0$. There is a common majorant for all $\mathfrak{M}^{m, \tau_k}$ in H_1 , i.e.

$$\sup_{\varphi \in \mathfrak{M}^{m, \tau_k}} \|\varphi\|_1 \leq \phi_2(\|f\|_{-1}, \nu^{-1}, \tau_0) \blacksquare$$

The proof of Theorem 2 is based on a priori estimates for $v^m(t), t = \ell\tau, \ell = 1, 2, \dots$, which we derived from the following relations :

$$(7_1) \quad \|v^m(t)\|^2 - \|v^m(t - \tau)\|^2 + \|v^m(t) - v^m(t - \tau)\|^2 + 2\tau\nu\|\partial_x v^m(t)\|^2 = \\ = 2\tau(f, v^m(t)) \leq 2\tau\|f\|_{-1}\|\partial_x v^m(t)\|, \\ 2\tau\|v_t^m(t)\|^2 + \nu\|\partial_x v^m(t)\|^2 - \nu\|\partial_x v^m(t - \tau)\|^2 + \nu\|\partial_x v^m(t) -$$

$$(7_2) \quad -\partial_x v^m(t - \tau)\|^2 = -2\tau(v^m(t - \tau). \nabla v^m(t), v_t^m(t)) + 2\tau(f, v_t^m(t)) \leq \\ \leq 2\tau\|v^m(t - \tau)\|_{4, \Omega}\|\partial_x v^m(t)\| \|v_t^m(t)\|_{4, \Omega} + 2\tau\|f\|_{-1}\|\partial_x v_t^m(t)\|,$$

and

$$(7_3) \quad \|v_t^m(t)\|^2 - \|v_t^m(t - \tau)\|^2 + \|v_t^m(t) - v_t^m(t - \tau)\|^2 + 2\tau\nu\|\partial_x v_t^m(t)\|^2 = \\ = -2\tau(v^m(t - 2\tau). \nabla v_t^m(t) + v_t^m(t - \tau). \nabla v^m(t), v_t^m(t)) = \\ = -2\tau(v_t^m(t - \tau). \nabla v^m(t), v_t^m(t)) \leq 2\tau\|v_t^m(t - \tau)\|_{4, \Omega} \cdot \\ \|\partial_x v^m(t)\| \|v_t^m(t)\|_{4, \Omega}, \quad t = \ell\tau.$$

They are corollaries of (6) and they are difference analogues of the relations (5_k).

3 An ε -approximation

Consider the ε -approximation

$$(8_1) \quad \partial_t v - \nu \Delta v - \varepsilon^{-1} \nabla \operatorname{div} v + v \cdot \nabla v + \frac{1}{2} v \operatorname{div} v = f ,$$

$$(8_2) \quad v|_{\partial\Omega} = 0 , \quad v|_{t=0} = \varphi , \quad \varepsilon \in (0, 1] ,$$

of the problem (1_k) . The following statement holds for (8_k) :

Theorem 3.— *Let Ω be a b.d. in $\mathbf{R}^2$ and $f \in \overset{\circ}{W}_2^{-1}(\Omega)$. The solution operators $V_t^\varepsilon : \varphi \rightarrow v^\varepsilon(t, \varphi)$ form the continuous semi-group $\{V_t^\varepsilon, t \in \mathbf{R}^+, L^2(\Omega, \mathbf{R}^2)\}$. It belongs to the class 1 and has a compact MIGBA $\mathfrak{M}^\varepsilon$. There is a common majorant ϕ_z for*

$$\sup_{\varepsilon \in (0, 1]} \sup_{\varphi \in \mathfrak{M}^\varepsilon} \sup_{t \in \mathbf{R}^1} \{ \|\varphi\|_1, \varepsilon^{-1/2} \|\operatorname{div} \varphi\|, \|\partial_t v^\varepsilon(t, \varphi)\| ,$$

$$\int_t^{t+1} (\varepsilon^{-1} \|\operatorname{div} \partial_\tau v^\varepsilon(\tau, \varphi)\|^2 + \|\partial_{x\tau}^2 v^\varepsilon(\tau, \varphi)\|^2) d\tau \} \leq \phi_3(\|f\|_{-1}, \nu^{-1}) .$$

For any $\delta > 0$ there is a $\varepsilon(\delta) > 0$ such that

$$\mathfrak{M}^\varepsilon \subset 0_\delta(\mathfrak{M}) \quad \text{for } \varepsilon \in (0, \varepsilon(\delta)] .$$

Here $0_\delta(\mathfrak{M})$ is δ -vicinity of $\mathfrak{M}$ in the space $L^2(\Omega, \mathbf{R}^2)$. For the solving of (8_k) , $k = 1, 2$, can be used the approximations of n.n.1 and 2 with arbitrary coordinates $\{\psi_k\}_{k=1}^\infty$ in the space $\overset{\circ}{W}_2^1(\Omega)$. ■

Write down the relations from which we derive a priori estimates for the solutions of (8_k) :

$$(9_1) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|v(t)\|^2 + \nu \|\partial_x v(t)\|^2 + \varepsilon^{-1} \|\operatorname{div} v(t)\|^2 = \\ = (f, v(t)) \leq \|f\|_{-1} \|\partial_x v(t)\| , \end{aligned}$$

$$(9_2) \quad \begin{aligned} \|\partial_t v(t)\|^2 + \frac{\nu}{2} \frac{d}{dt} \|\partial_x v(t)\|^2 + \frac{1}{2\varepsilon} \frac{d}{dt} \|\operatorname{div} v(t)\|^2 = \\ = -(v(t) \cdot \nabla v(t), \partial_t v(t)) - \frac{1}{2} (v(t) \operatorname{div}(t), \partial_t v(t)) + (f, \partial_t v(t)) \leq \\ \leq \|v(t)\|_{4, \Omega} \|\partial_x v(t)\| \|\partial_t v(t)\|_{4, \Omega} + \frac{1}{2} \|v(t)\|_{4, \Omega} \|\operatorname{div} v(t)\| \|\partial_t v(t)\|_{4, \Omega} + \end{aligned}$$

$$+\|f\|_{-1}\|\partial_{xt}^2 v(t)\| ,$$

$$\begin{aligned}
(9_3) \quad & \frac{1}{2} \frac{d}{dt} \|\partial_t v(t)\|^2 + \nu \|\partial_{xt}^2 v(t)\|^2 + \varepsilon^{-1} \|\operatorname{div} \partial_t v(t)\|^2 = \\
& = -(\partial_t v(t) \cdot \nabla v(t) + \frac{1}{2} v(t) \operatorname{div} \partial_t v(t), \partial_t v(t)) - (v(t) \cdot \nabla \partial_t v(t) + \\
& + \frac{1}{2} \partial_t v(t) \operatorname{div} v(t), \partial_t v(t)) = -(\partial_t v(t) \cdot \nabla v(t) + \frac{1}{2} v(t) \operatorname{div} \partial_t v(t), \\
& \partial_t v(t)) \leq \|\partial_t v(t)\|_{4,\Omega}^2 \|\partial_x v(t)\| + \frac{1}{2} \|v(t)\|_{4,\Omega} . \\
& \|\operatorname{div} \partial_t v(t)\| \|\partial_t v(t)\|_{4,\Omega}
\end{aligned}$$

4 Difference schemes

We have consider difference schemes suggested by us in 50th and 60th (see, for example, [7], [8], [4]) and have found that some of them are globally stable and have MIGBAs lying near $\mathfrak{M}$. Let us discribe here on of them. Take $h \in (0, h_0]$ and $\tau \in (0, \tau_0]$, with some h_0 and τ_0 , and the mesh $\mathbf{R}_h^2$: $x = (kh) = (k_1 h, k_2 h), (k_1, k_2) \in \mathbf{N} \times \mathbf{N}$ in $\mathbf{R}^2$. Let $\omega_{kh} = \{x = (x_1, x_2) \in \mathbf{R}^2 | x_j \in (k_j h, (k_j + 1)h), j = 1, 2\}$; $\bar{\Omega}_h$ - the set $\bigcup_{\omega_{kh} \subset \Omega} \bar{\omega}_{kh} \subset \mathbf{R}^2$; $\mathbf{S}_h = \partial \bar{\Omega}_h$ and $\Omega_h = \bar{\Omega}_h \setminus \mathbf{S}_h \subset \mathbf{R}^2$. We shall use the same notations $\bar{\Omega}_h, \mathbf{S}_h$ and Ω_h for the sets of points $x = (kh) \in \mathbf{R}_h^2$ belonging to $\bar{\Omega}_h, \mathbf{S}_h$ and Ω_h corresponding by. Introduce also notations :

$$v_{x_i}(x, t) = h^{-1}[v(x + h e^i, t) - v(x, t)], v_{\bar{x}_i}(x, t) = h^{-1}[v(x, t) - v(x - h e^i, t)] ,$$

$$v_{\bar{t}}(x, t) = \tau^{-1}[v(x, t) - v(x, t - \tau)], \bar{v}^{\pm i}(x, t) = v(x \pm h e^i, t), \bar{v}^{\pm 0}(x, t) = v(x, t \pm \tau),$$

where e^i is the ort along the axis x_i . Take the following difference scheme (see [4], p. 238):

$$(10_1) \quad v_{i\bar{t}} - \nu v_{ix_k \bar{x}_k} + \frac{1}{2} \overset{-0, +k}{v_k} v_{ix_k} + \frac{1}{2} \overset{-0}{v_k} v_{i\bar{x}_k} = -p_{\bar{x}_i} + f_i^h, i = 1, 2,$$

$$(10_2) \quad v_{kx_k} = 0,$$

$$(10_3) \quad v|_{\mathbf{S}_h} = 0,$$

$$(10_4) \quad v|_{t=0} = \varphi^h.$$

the equations (10₁) have to be fulfilled in the points (x, t) with $t = \ell\tau$, $\ell = 1, 2, \dots$, and $x \in \Omega_h$; the equations (10₂)- in (x, t) with $t = \ell\tau$, $\ell = 1, 2, \dots$, and $x = \Omega'_h \equiv \Omega_h \cup S'_h$, where S'_h is a part of S_h which we get replacing the points of Ω_h by vectors $-he^i$, $i = 1, 2$; the equations (10₃) - in (x, t) with $t = \ell\tau$, $\ell = 1, 2, \dots$ and $x \in S_h$, the equations (10₄) - in $(x, 0)$ with $x \in \bar{\Omega}_h$. In (10₁) f^h is a mesh-function on Ω_h and φ^h is a mesh-function on $\bar{\Omega}_h$ satisfying the equations

$$\varphi^h|_{S_h} = 0 \text{ and } \varphi_{kx_k}^h = 0 \text{ in } x \in \Omega'_h.$$

We add to (10_k) the equations

$$(10_5) \quad \sum_{\Omega'_k} p = 0 \text{ for } t = \ell\tau, \ell = 1, 2, \dots$$

It was proved in [8] (see also [4]) that the system (10_k), $k = 1, \dots, 5$, is uniquely solvable and its solutions $v^{h,\tau}$ converge when $h = \mu\tau \rightarrow 0$ (μ is a fixed positive number) to the solution v of the problem (1_k) on the finite intervals $[0, T]$ of t -axis if f, φ and $\partial\Omega$ are smooth enough and f^h, φ^h approximate f and φ in a properly way. Now we have proved that the scheme (10_k) is globally stable. More precisely : introduce the linear set of mesh-functions $u^h : H^h = \{u^h|_{\bar{\Omega}_h} | u^h|_{S_h} = 0, u_{kx_k}^h = 0 \text{ in the points of } \Omega'_h\}$ and consider H^h as Euclidian space with the norm

$$(11) \quad \|u^h\|_{\Omega_h} \equiv (h^2 \sum_{\bar{\Omega}_h} (u^h)^2)^{1/2}.$$

For a fixed $f^h \in H^h$ the solution operators $V_t^{h,\tau} : \varphi^h \rightarrow v^{h,\tau}(t, \varphi^h)$, $t = \ell\tau$, $\ell \in \mathbb{N}^+$, form a discrete semi-group in H^h . It has MIGBA $\mathfrak{M}^{h,\tau}$ and there is a common majorant ϕ_4 for all $\mathfrak{M}^{h,\tau}$ with $h = \mu\tau$, $\tau \in (0, \tau_0]$:

$$\sup_{\varphi^h \in \mathfrak{M}^{h,\tau}} \sup_{\ell \in \mathbb{N}} \{ \|\varphi_x^h\|_{\Omega_h}, \|v_t^{h,\tau}(\ell\tau, \varphi^h)\|_{\Omega_h} \} \leq \phi_4(\|f^h\|_{\Omega_h}, \nu^{-1}, \mu, \tau_0)$$

Denote by $\tilde{u}^h$ the piece-wise constant interpolation of $u^h \in H^h$, i.e. $\tilde{u}^h \in L^2(\Omega, \mathbf{R}^2)$, $\tilde{u}^h(x) = u^h(kh)$ for $x \in \omega_{kh}$, $\subset \Omega$, $\tilde{u}^h(x) = 0$ for $x \in \Omega \setminus \Omega_h$. Let, for example, $\tau = \tau_k = \tau_0 2^{-k}$, $k = 0, 1, 2, \dots$, $h = h_k = \mu\tau_k$, $\mathfrak{M}^{h_k, \tau_k} \equiv \mathfrak{M}_\mu^k$ and $\tilde{\mathfrak{M}}_\mu^k$ - the set of $\tilde{\varphi}^{h_k}$ for all $\varphi^{h_k} \in \mathfrak{M}_\mu^k$. The following statement is true :

Theorem 4.— *Let Ω be a.b.d. in $\mathbf{R}^2$ and $f \in L^2(\Omega; \mathbf{R}^2)$. For any $\varepsilon > 0$ exists a number $n = n(\varepsilon, \mu)$ such that*

$$\tilde{\mathfrak{M}}_\mu^k \subset 0_\varepsilon(\mathfrak{M}) \text{ for } k \geq n(\varepsilon, \mu).$$

Here $0_\varepsilon(\mathfrak{M})$ is ε -vicinity of $\mathfrak{M}$ in the space $L^2(\Omega, \mathbf{R}^2)$ ■.

Analogous results hold for the systems of $\mathcal{ODE}$:

$$(11_1) \quad \partial_t v_i(t) - \nu v_{ix_k \bar{x}_k}(t) + \frac{1}{2} v_k(t) v_{ix_k}(t) + \frac{1}{2} v_k(t) v_{i\bar{x}_k}(t) = -p_{\bar{x}_i}(t) + f_i, \quad i = 1, 2,$$

$$(11_2) \quad v_{kx_k}(t) = 0, \quad v(t)|_{S_h} = 0, \quad v(0) = \varphi^h, \quad t \in \mathbf{R}^+.$$

The basic relations which we use for the solutions of problem (10_k) resemble the relations (7_k) and for the solutions of problem (11_k) - the relations (5_k) .

The results analogous to results described above are true for :

1) the Navier-Stokes equations with the periodic or non homogeneous boundary conditions : $v|_{\partial\Omega} = a|_{\partial\Omega}$ with $a(x) = \text{rot } b(x)$, $b \in W_2^2(\Omega)$, if $\partial\Omega$ is a piece-smooth curve.

2) the thermo-convection and magneto-hydrodynamical systems for viscous incompressible fluids in $b.d. \Omega \subset \mathbf{R}^2$.

3) the modifications of the three dimensional Navier-Stokes equations in $b.d. \Omega \subset \mathbf{R}^3$ which were suggested by us in [9] [10] (see also addendum in [4], second russian edition).

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